Torus knots and polynomial invariants for a class of soliton equations

Renzo L. Ricca
Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom

(Received 5 August 1992; accepted for publication 8 December 1992)

In this paper is shown how to interpret the nonlinear dynamics of a class of one-dimensional physical systems exhibiting soliton behavior in terms of Killing fields for the associated dynamical laws acting as generators of torus knots. Soliton equations are related to dynamical laws associated with the intrinsic kinematics of space curves and torus knots are obtained as traveling wave solutions to the soliton equations. For the sake of illustration a full calculation is carried out by considering the Killing field that is associated with the nonlinear Schrödinger equation. Torus knot solutions are obtained explicitly in cylindrical polar coordinates via perturbation techniques from the circular solution. Using the Hasimoto map, the soliton conserved quantities are interpreted in terms of global geometric quantities and it is shown how to express these quantities as polynomial invariants for torus knots. The techniques here employed are of general interest and lead us to make some conjectures on natural links between the nonlinear dynamics of one-dimensional extended objects and the topological classification of knots.

I. INTRODUCTION

Since a series of seminal papers \(^1\) appeared in the literature, considerable progress has been made in understanding the underlying geometric structure of one-dimensional soliton equations. Eventually, this has led to a much deeper understanding of previous results giving whole hierarchies of soliton equations a comprehensive geometric framework in which these equations have been found to have their natural setting. Moreover, new links between the mathematics and the physics of the nonlinear systems that these equations aim to model have been disclosed. \(^4\)–\(^10\) Intrinsic geometric methods, in particular, appear to be very powerful in the discovery and interpretation of new aspects of nonlinear dynamics \(^11\),\(^12\) as well as potentially important in the ultimate problem of ordering and classifying different classes of soliton equations.

It is well-known \(^13\),\(^14\) that several different one-dimensional physical systems such as vortex filaments in perfect fluids, classical continuum Heisenberg spin systems, current density lines in type II superconductors, elastic strings and even polymers, can exhibit soliton behavior. The recent results obtained in Refs. 15 and 16 show that a wide class of solitons moving on one-dimensional extended objects—the support of which is mathematically modeled by a space curve that behaves as a waveguide—is generated by a family of dynamical laws which act as Killing fields on space curves.

In general, Killing fields are vector fields that generate a one-parameter group of rigid motions; hence rotational motions, translational motions, and screw motions are produced by flows due to the action of Killing fields. Each of these flows acting on a space curve can thus be seen as a particular generator of kink waves traveling without change of shape on that curve. \(^6\) By the action of one of these flows, a material non-self-intersecting closed curve can therefore be induced to reshape itself so as to support periodic waves (as solitary waves) distributed all along its length. For a given steady state configuration it is possible to obtain traveling wave solutions in form of torus knots, where the knottedness of the curve, produced by its periodic twisting, is generated by the action of a Killing field.

Therefore it is natural to consider nonlinear dynamical flows as generators of torus knots. This picture leads us to an intriguing description of knot geometry via soliton dynamics, where global geometric quantities may play a fundamental rôle. These nonlinear dynamics, purely local in their character, indeed generate global constraints which are given by the conserved quantities associated with soliton dynamics. By interpreting these quantities in terms of intrinsic geometry, we are able to give torus knots global geometric invariants as polynomial invariants.

In this paper we present a new approach to finding torus knot solutions and associated polynomial invariants for a class of soliton equations. In Sec. II we present the general intrinsic equations as a geometric counterpart of the soliton equations for an associated dynamical law and by introducing the concept of Killing fields we illustrate how soliton dynamics can be generated by the action of Killing fields on one-dimensional extended objects. Since the class of soliton equations considered in this paper admits traveling wave solutions, we consider a class of simple closed curves (isotopy equivalent to a circle) and show how to obtain torus knots—via perturbations of the circle (which in our case is a “zero-soliton” solution)—as traveling wave solutions to the soliton equations. Killing fields will be then interpreted as torus knot generators by their action on the circle. For the sake of simplicity, and for illustration, in Sec. III we give an example considering the simplest possible Killing field (the first of a given list); the technique here employed, however, is completely general. The Killing field considered is given by the so-called localized induction approximation (LIA) and generates the nonlinear Schrödinger equation (NLS) for a wide class of physical systems. \(^17\)–\(^21\)
By rewriting LIA in cylindrical polar coordinates, we find new torus knot solutions to NLS as perturbations of the circular solution. These solutions are automatically written in explicit form and therefore can be used to calculate geometric quantities. In Sec. IV, related soliton invariants are interpreted by the Hasimoto map as global geometric quantities expressed in terms of polynomial invariants, and it is shown how to calculate these invariants for the torus knot solutions. Finally, in Sec. V we conclude with some general remarks on the global geometry of nonlinear systems and make some conjectures on possible relations between the nonlinear dynamics of one-dimensional extended objects and the topological classification of knots.

II. SOLITON EQUATIONS AND KILLING FIELDS AS TORUS KNOT GENERATORS

Physical systems such as vortex filaments in perfect fluids, one-dimensional classical continuum Heisenberg spin systems and elastic strings can be thought of as one-dimensional extended objects, the support of which, their centerline X, may be mathematically modeled by a generally twisted space curve \( \mathcal{L} \) in \( \mathbb{R}^3 \) (or its compactification). Regardless of the physical properties that actually characterize the dynamics of such systems, for the moment let us take into consideration the pure kinematics of curves as an idealization of the general evolution of these systems.

A. Kinematics of space curves

In what follows we shall assume everything to be \( C^\infty \) for simplicity, and make use of the intrinsic description through curvature \( c = \mathcal{R}^{-1} \) (where \( \mathcal{R} \) is the radius of curvature), torsion \( \tau \) and the local intrinsic reference frame (the so-called Frenet frame) given by tangent \( t \), normal \( n \), and binormal \( b \) (Fig. 1). All these quantities are, in general, regular functions of arc length \( s \) and time \( t \).

Let us consider a fixed time \( t \) the space \( \Lambda \) of smooth, simple, arc-parametrized curves with nonvanishing curvature, that is (by identifying the mapping with its image):

**Definition**: The space \( \Lambda \) is defined as

\[
\Lambda = \{ \mathcal{L} : [a,b] \to S^3 : c \neq 0 \},
\]

\[
X = X(s) \text{ single-valued } \forall s \in [a,b] \subseteq \mathbb{R},
\]

where \( S^3 \) is the compactification of \( \mathbb{R}^3 \).

This is equivalent to saying that the curve is the image of a differentiable periodic function without singular points or that \( \mathcal{L} [t_0, t] \) can be seen as a one-to-one onto map free from inflexion points and self-intersections.

Space curves can in fact be always reduced to arc-length parametrized curves in \( \Lambda \) by a suitable reparametrization, according to the following:

**Definition**: \( U \) and \( T \) are two distinct compact sets in \( S^1 \), where \( T = [a,b] = \{ \Phi : U \to S^1 \} \) is the domain of \( \mathcal{L} \). Let \( \Phi \) be a differentiable (real-valued) function such that \( \Phi : U \to T \). Then the composite function \( \Psi = \mathcal{L}^*(\Phi) : U \to S^3 \) is called the reparametrization of \( \mathcal{L} \) by \( \Phi \).

In what follows we shall use the fundamental existence theorem for space curves with the Frenet–Serret equations as structure equations for \( \mathcal{L} \).

**Theorem**: Let \((t,n,b)\) be the (intrinsic, positive, orthonormal reference) Frenet frame on \( \mathcal{L} \). Let \( c \) and \( \tau \) be regular functions of \( s \). By definition \( c > 0 \), \( \forall s \). Then there exists a curve \( \mathcal{L} \) in \( S^3 \) for which the following structure equations (so-called Frenet–Serret equations) hold:

\[
\begin{pmatrix}
0 & c & 0 \\
-c & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}
= \begin{pmatrix}
t' \\
n' \\
b'
\end{pmatrix},
\]

where primes denote derivative with respect to \( s \). Such a curve is uniquely determined up to rigid motions in \( S^3 \).

Now let us take time variations of \( \mathcal{L} \) into consideration. For any time \( \tau \in [t_0,t_f] \), \( \mathcal{L} = \mathcal{L}(s,t) \). The evolution of \( \mathcal{L}(s,t) \) is given by the variation vector field acting along \( \mathcal{L} \) and defined as the velocity \( V(s,t) = \partial X(s,t) / \partial t \) with \( V(s) = V(t_0,s) \), that can be written in intrinsic components as

\[
V = v_t t + v_n n + v_b b,
\]

where \( v_t, v_n, v_b \) are regular functions of \( s \) and \( t \). The intrinsic kinematics of \( \mathcal{L} \) is given by considering the time variation of curvature and torsion determined by the action due to \( V = (v_t v_n v_b) \). The intrinsic equations of motion for an arc-parametrized curve are given by the following lemma:

**Lemma**: \( \mathcal{L} = \mathcal{L}(s,t) : [a,b] \times [t_0,t_f] \to S^3 \) is a one-parameter family of arc-length parametrized space curves. If \( V(s,t_f) = \partial X / \partial t |_{t=t_f} \) is the variation vector field along \( \mathcal{L} \) and \( \mathcal{L} \) has curvature \( c \) and torsion \( \tau \), then \( c \) and \( \tau \) vary according to the following set of equations:

\[
v_t' = v_c c,
\]

\[
v_n' = v_c c v_c - v_n \tau,
\]

\[
v_b' = (v_n' v_c - v_b \tau) \frac{1}{c} + (v_b' v_n - v_n' v_b) \frac{1}{c},
\]

where overdots and primes denote partial derivatives with respect to \( t \) and \( s \), respectively.

The proof of the lemma relies on the commutativity of partial derivatives with respect to \( s \) and \( t \) (for which we can write, for example, \( V' = X' = 1 \)) and on the SO(3) structure of the Frenet–Serret equations. Equations (2)–(4)
were derived by Germano and generalized by Ricca\textsuperscript{15} to the case of a multidimensional ambient manifold.

The relationship between $v'$ and $u_n$ expressed by Eq. (2) is necessary and sufficient condition for inextensibility of $\mathcal{L}$ and can be regarded as a congruence condition for material points of the curve. This can be shown by considering two neighboring points $X_i$ and $X_i=dX$ on $\mathcal{L}$: since $dX=ds$, we can write $dX=V\cdot ds$; henceforth, by Eq. (2), we have

$$\frac{\partial}{\partial t}(ds)^2=2\mathbf{a} \cdot \mathbf{dX}=2t \cdot V'(ds)^2=2(u'_n-c\mathbf{V})(ds)^2=0.$$  

Condition (2), however, does not restrict the study of possible dynamics in $\mathcal{L}$, provided that a suitable reparametrization is made and this, as we have seen, can always be done by introducing a linear operator that "reparametrizes" an arbitrary vector field along $\mathcal{L}$. Hence, for a given variation field $\mathbf{V}$ (by prescribing its intrinsic components), Eqs. (3) and (4) determine the time evolution of curvature and torsion of $\mathcal{L}$.

The skew-symmetry property of the structure equations, then, allows us to relate the kinematics of a nonlinear string to the motion of an arbitrary rigid body attached to the curve by interpreting the time evolution of the Frenet frame as a motion of a rigid body. Fundamental connections between soliton theory and rigid body motion have been discovered: it is now known,\textsuperscript{3,4} for example, that the Lax's operator of the soliton theory is connected to the equations of motion of a rigid body and that both the rigid body equations and the Frenet–Serret equations can be related, ultimately, to the AKNS scheme and to the inverse scattering formalism (IST). These and other ideas lead us to the concept of Killing fields as variation vector fields acting on $\mathcal{L}$.

**B. Soliton equations and Killing fields**

The fact that the intrinsic equations governing the evolution of generally twisted space curves with soliton behavior are intimately related to the inverse scattering transform (IST) and the Bäcklund transformation methods is well-known.\textsuperscript{3} The interpretation of soliton dynamics (with their associated group structures, pseudopotentials and prolongation structures) in terms of rigid body motion along space curves has led to a unified interpretation of a class of soliton equations in a very natural and simple way.\textsuperscript{4} It has been recently clarified\textsuperscript{16} that a whole hierarchy of soliton equations can be straightforwardly related to a hierarchy of dynamical laws (i.e., the flows $\mathbf{V}$ acting along $\mathcal{L} \in \mathbf{X}$) of increasing complexity that describe rigid body motion along $\mathcal{L}$ and it is to this particular class of dynamical laws that we shall devote our attention below.

Let us introduce the following:

\textbf{Definition:} A vector field $\mathbf{V}$ along $\mathcal{L}$ is called Killing if it is the restriction to $\mathcal{L}$ of an infinitesimal rigid body motion in $\mathcal{S}$\textsuperscript{3}.\hfill \Box

The condition for $\mathbf{V}$ to be a Killing field is thus equivalent to say that the infinitesimal variation of $\mathcal{L}$ by $\mathbf{V}$ should not change parametrization, curvature or torsion of $\mathcal{L}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{First four Killing fields generating soliton dynamics along $\mathcal{L}$.}
\end{figure}

An interesting study about the rôle played by a hierarchy of Killing fields for a class of soliton equations (that include the nonlinear Schrödinger equation, the modified Korteweg–de Vries equation and other “higher-order” soliton equations) has been carried out by Langer and Perline.\textsuperscript{16} According to their results, the hierarchy of Killing fields, interpreted as Hamiltonian flows generating the soliton equations, has a particular Poisson structure: a family of Killing fields can therefore be generated recursively by a functional operator, the recursive operator $\mathcal{H}$, such that $\mathcal{V}_{m+1}=\mathcal{H}(\mathcal{V}_m)$. Briefly

$$\mathcal{V}_{m+1} = \mathcal{H}(\mathcal{V}_m) = -\mathcal{N}(t \times \mathcal{V}_m),$$

where

$$\mathcal{N}(\mathbf{V}) = \left[ \int c_n(u,t)du \right]t + v_0n + v_0b$$

is a linear "normalization operator" on $\mathbf{V}$. The first four dynamical laws acting as Killing fields on $\mathcal{L}$ and generating soliton equations are listed in the table of Fig. 2.

By construction all these flows satisfy the congruence condition (2). By these flows Eqs. (3) and (4) represent, in the intrinsic description, the counterpart of the corresponding soliton equations and the Killing fields can be regarded as soliton generators. It is well-known, for example, that in the case of $\mathbf{V} = \mathbf{v}_1 = \mathbf{cb}$ Eqs. (3) and (4) are reduced to the so-called Da Rios–Betchov equations\textsuperscript{20}

$$\begin{align*}
\dot{c} &= -(c\dot{t})' - c't, \\
\dot{t} &= \left(\frac{c'' - c't^2}{t} \right)' + c'tc
\end{align*}$$

that can be transformed via the Hasimoto map $h$: $X(s,t) \rightarrow \psi(s,t)$, defined by

$$\psi(s,t) = c(s,t)e^{\frac{1}{2}r(s,t)du}, \quad \psi(s,t) \in C,$$

\text{to the nonlinear Schrödinger equation (NLS).}^2 In physical terms $\mathbf{V}_1$ gives the law governing the motion of a thin vortex filament in the context of the localized induction approximation (LIA) and can be recognized also as the flow that generates solitons on classical one-dimensional Heisenberg spin systems (via the tangent indicatrix $X \rightarrow X' = t$). It also generates solitons on elastic curves (see, for example, Refs. 18, 19, and 23).

Similarly, the second Killing field given by $\mathbf{V} = \mathbf{V}_2$ yields to the set of intrinsic equations

\text{CHAOS, Vol. 3, No. 1, 1993}
\[ \dot{e} = (c'' + \frac{1}{c^3} - c^2)\tau - (2c' + c\tau + c') \tau, \]
\[ \dot{\tau} = \frac{(c'' + \frac{1}{c^3} - c^2)\tau + (2c' + c\tau + c') \tau}{c} + (2c' + c\tau + c')c \]

(7)

that can be also reduced by the Hasimoto map to the modified Korteweg–de Vries equation (MKdV).\(^5\) Examples of physical contexts in which Killing fields of higher-order generate "higher-order" soliton equations (such as the Hirota equation) can be easily found in the literature (see, for example, Refs. 24 and 25).

The Hasimoto map establishes a one-to-one correspondence between the configuration space \( \Lambda \) under the \( V \) action and the corresponding soliton dynamics. Therefore, many geometric properties of soliton equations can be investigated by studying the corresponding intrinsic equations. For example, by a straightforward manipulation of Eqs. (7) we can show that for strings evolving under MKdV the following set of equations hold:

\[ \frac{1}{2} \frac{\partial (c^2)}{\partial t} = \frac{3}{2} \frac{\partial}{\partial s} \left[ \frac{c^4}{4} - c^2 - c^2 + \frac{2}{3} (c\tau)' \right], \]
\[ \frac{\partial \tau}{\partial t} = \frac{3}{2} \frac{\partial}{\partial s} \left[ \frac{c^2}{4} - c^2 - c^2 + \frac{2}{3} (c\tau)' \right] \]

and we write them in this way to emphasize that not only the total squared curvature and the total torsion are constants of the motion (as we already have in the case of NLS), but even the sum of their densities is conserved in time, i.e.,

\[ \frac{1}{2} c^2 + \tau = \text{constant}, \]

(8)

a local property that must be related to the geometry of the cnoidal wave solutions.\(^{26}\) We shall come back to the study and the geometric interpretation of certain conserved quantities in Sec. IV.

**C. Killing fields as torus knot generators**

Since the action of Killing fields \( V \) along a curve is to produce solitons that propagate on the curve, it is natural to consider these fields as generators of periodically spaced solitary waves on \( \mathcal{L} \subset \Lambda \) that travel without change of shape along the curve. If the curve is isotopically equivalent to a circle, then it is possible to obtain solutions to the soliton equation associated with the action of the Killing field in the form of waves traveling without change of shape along the curve so as to form torus knots. Alternatively, it is always possible to regard torus knots as topologically distinct objects that can be viewed in their standard form as generated by periodically spaced solitary waves on a torus, produced by the action of a rigid motion (i.e., a screw motion) of a traveling wave along a closed curve.

In order to illustrate this point let us introduce some basic concepts. Let \( S^1 \times S^1 \) be a boundary of a two-dimensional disc and \( T = S^1 \times S^1 \) be a standard unknotted torus of radii \( R \) and \( r \), with \( R > r \), and corresponding angular variables \( \alpha \) and \( \beta \). Note that the solid torus, given by \( D^2 \times S^1 \), is topologically equivalent to the circle (via a homotopy map) by a standard retraction \( \rho: D^2 \times S^1 \rightarrow S^1 \).

**Definition:** Let \( X = D^2 \times S^1 \) be a topological space, \( A = D^2 \subset X \). \( A \) is called a retract of \( X \) if there is a retraction \( \rho: X \rightarrow A \) that is a continuous map with \( \rho|_A = \text{Id}_A \). \( \square \)

A curve inscribed on the surface of a standard torus (so to be embedded in \( T \)) has a winding number defined as follows:

**Definition:** The winding number of a curve in \( T \) given by \( \alpha = \alpha(s) \) and \( \beta = \beta(s) \) is the number \( \bar{W} = \lim_{s \rightarrow \alpha(s)} \beta(s)/\alpha(s) \). \( \square \)

In other words, the winding number is given by the average number of wraps of the curve around the small radius of the torus (in the meridian plane) per number of wraps around the large radius (in the longitudinal plane). The winding number exists and is unique if the curve is not self-intersecting, and is a topological invariant of the curve, that means that \( W \) is unchanged under isotopic deformation of the curve. According to Massey,\(^{28}\) we can state the following:

**Theorem:** A closed non-self-intersecting curve embedded in \( T \) that cuts a meridian in \( p \) points and a longitude in \( q \) points with winding number \( W = q/p \), where \( q > 1 \) and \( p > 1 \) are coprime, is a nontrivial knot \( \mathcal{T}_{pq} \).

Note that \( \mathcal{T}_{1,q} \) and \( \mathcal{T}_{p,1} \) are trivial knots, while the type of \( \mathcal{T}_{pq} \) is unchanged by changing the sign of \( p \) or \( q \), or by interchanging \( p \) and \( q \).\(^{29}\) Examples of torus knots are the trefoil knot \( \mathcal{T}_{2,1} \) with \( W = 3/2 \) (which wraps around \( T \) in the meridian direction three times and in the longitudinal direction two times) and the Solomon's seal knot \( \mathcal{T}_{2,5} \) with \( W = 5/2 \). In Fig. 4 are sketched these two knots in their standard projection. It should be noted there that torus knots (in their standard projection) are essentially generated under the action of a rotational symmetry group by a \( q \)-times rigid rotation of the fundamental sector of \( \mathcal{T}_{pq} \). Therefore they can be viewed as periodically spaced waves rotating without change of shape around an axis of symmetry along a closed curve with prescribed winding number.

As we pointed out previously, torus knots can in fact
be generated by standard perturbations from the circle $\mathcal{S}_0$ under the action of one of the Killing fields listed in Fig. 2. To see this let us consider an unknotted simple closed curve $\mathcal{L}_0$ in $\mathbb{R}^2$. If $\mathcal{L}_0$ is a curve reducible to a circle $\mathcal{S}_0$ (the trivial knot), we can conceive of a mechanism to generate torus knots by Killing "actions" on $\mathcal{S}_0$. Let $\mathcal{L}_0$ be isotopically reducible to a circle by a map $\phi_i: \mathcal{L}_0 \rightarrow \mathcal{S}_0$ [Fig. 5(a)]. The circle $\mathcal{S}_0$ can be thought of as a zero-soliton solution for one of the above Killing fields (for the class of Killing fields and soliton equations here considered we know that this is true). Our goal is to obtain torus knots from the wave structure of the dynamical equations. In order to do this, let us introduce the class of traveling wave solutions of Kida type. Here, for illustration, we take for simplicity $V_1$, but similar conclusions can be reached for other choices of Killing fields. Let $V_1$ act along $\mathcal{S}_0$. We have the following definition:

**Definition:** Let $f(s-at)$ and $g(s-at)$ be a positive-valued function and a real-valued function on $\mathbb{R}$, respectively, and let $a$ be a real constant. A solution to the NLS (for the associated Killing field $V_1$) of the form

$$\psi(s,t) = f(s-at)e^{ig(s-at)+bi+ct}, \quad \psi(s,t) \in \mathbb{C} \quad (9)$$

with $b$ and $c$ real constants, is called *traveling wave*. □

The existence of traveling wave solutions for the class of soliton equations generated by Killing fields and here investigated can be shown in the framework of the inverse method in the so-called quasiperiodic case. Existence of traveling wave solutions to NLS has been proved in Ref. 30. The domain of the class of traveling waves under the Hasimoto map is also known as the *Kida class* of solutions.31

The class of torus knots and the Kida class of solutions can be made to coincide by putting their elements in a one-to-one correspondence. This can be done by the following steps. Take a circle $\mathcal{S}_0$ and think of it as a retract of a standard torus via a continuous map. Consider a Killing field acting on this circle and look for solution curves of the Kida class as traveling waves on a tubular neighborhood of $\mathcal{S}_0$ via perturbation of the circular solution. Conditions for closure and periodicity as well as prescriptions for the winding number $W$ of the solution curve are uniquely determined by the constants of integration. Solution curves are thus given in the form of torus knots in their standard form. Torus knots $\mathcal{T}_{pq}$ can then be isotoped to any other topologically equivalent configuration $\mathcal{K}_{pq}$ by a second map $\phi_2: \mathcal{T}_{pq} \rightarrow \mathcal{K}_{pq}$ [Fig. 5(b)].
III. TORUS KNOTS IN CYLINDRICAL POLAR COORDINATES

In this section we want to show how to generate torus knot solutions by standard perturbation techniques from the circle under the action of a Killing field. For illustration we give an example of this procedure by taking the first flow $V_1$. Torus knots generated by $V_1$ are not new and originally have been found by Kid

31 in the context of vortex filament motion. Successively Keener

32 working on similar problems, found "higher-order" torus knot solutions like cable knot solutions. The solutions found by Kid

33 are expressed in terms of elliptic integrals and therefore are not so easy to be manipulated analytically. The technique employed by Keener is also difficult to be used and is quite laborious, since there one must solve the set of intrinsic equations together with the set of the Frenet-Serret equations together with the closure condition for the solution curve. In this paper we propose an alternative technique that avoids all these difficulties relying on the use of cylindrical polar coordinates $(r,\alpha,z)$, natural to the problem at hand. The closure condition for the solution curve can thus be automatically fulfilled and torus knot solutions can be easily expressed in explicit analytic form.

A. $V_1$ in cylindrical polar coordinates

The action of $V_1$ on the circle $S_0$ can be studied by rewriting $V_1$ in cylindrical polar coordinates. To do this let us write $V_1$ as

$$V_1 = \hat{c}b = X' \wedge X'', \quad (10)$$

where $X = (x,y,z)$ is the position vector in an external (fixed) reference frame $O,x,y,z$. Equation (10) is equivalent to the system

$$\begin{align*}
\dot{x} &= y'y'' - z'y'', \\
\dot{y} &= z'x'' - x'z'', \\
\dot{z} &= x'y'' - y'x''
\end{align*} \quad (11)$$

In cylindrical polar coordinates we have $X(s,t) = [r(s,t),\alpha(s,t),z(s,t)]$, where

$$r = r \cos \alpha, \quad y = r \sin \alpha. \quad (12)$$

Differentiating Eqs. (12) with respect to $t$ and rewriting Eqs. (11) in terms of cylindrical polar coordinates, we have

$$\begin{align*}
\dot{r} &= (y'z'' - z'y'') \cos \alpha + (z'x'' - x'z'') \sin \alpha, \\
\dot{\alpha} &= \dot{y}' - (y'z'' - z'y'') \sin \alpha + (z'x'' - x'z'') \cos \alpha, \\
\dot{z} &= x'y'' - y'x''.
\end{align*} \quad (13)$$

Differentiating Eqs. (12) with respect to $s$ and substituting in Eqs. (13), we obtain the required set of relations. After some straightforward algebraic manipulation, $V_1$ can be rewritten in the following form:

$$\begin{align*}
\dot{r} &= r'z' - 2r' \alpha' z' - r' \alpha^2 z', \\
\dot{\alpha} &= -r'z'' + r'z - r' \alpha z', \\
\dot{z} &= 2r'^2 \alpha' + r' \alpha' z'' - r' \alpha' z' - r' \alpha^2 z''
\end{align*} \quad (14)$$

B. Torus knot solutions

In cylindrical polar coordinates the circle $S_0$ is simply given by

$$r = r_0, \quad \alpha = s/r_0, \quad (15)$$

so that in this coordinate system $V_1$ is reduced to the form

$$\dot{z} = -\frac{r_0}{r_0} \sin \alpha, \quad z = -\frac{t}{r_0}, \quad (16)$$

where $t$ is a scaled "time" parameter. Linear perturbation analysis from the circular solution has been carried out in the context of vortex filament motion under $V_1$ in Ref. 33. Following the same procedure let

$$\begin{align*}
\dot{r} &= r_0 + \epsilon \dot{r}_1, \quad \alpha = \frac{s}{r_0} - \epsilon \alpha_1, \\
\dot{z} &= \frac{t}{r_0} + \epsilon \alpha_1, \quad \dot{z} = -\frac{t}{r_0} \epsilon \alpha_1, \quad (17)
\end{align*}$$

where $\epsilon = o(1)$. Let us linearize Eqs. (14) by Eqs. (17). Taking the terms up to the order of $\epsilon$, the first of Eqs. (14) becomes

$$\epsilon \dot{r}_1 = \left( \frac{1}{r_0} + \epsilon \alpha_1 \right) \left( \frac{1}{r_0} \epsilon \alpha_1 \right) \epsilon z'' + \cdots + O(\epsilon^2)$$

which reduces to

$$\dot{r}_1 = \frac{1}{r_0} \epsilon \alpha_1. \quad (18)$$

Similarly for the second and the third of Eqs. (14), we have

$$\dot{\alpha}_1 = -\frac{1}{r_0} \epsilon \alpha_1. \quad (19)$$

and

$$\dot{z}_1 = \frac{t}{r_0} - \frac{1}{r_0} \epsilon \alpha_1. \quad (20)$$

Combining together Eq. (18) and Eq. (20), we have

$$\dot{r}_1 = \frac{r_0}{r_0} \epsilon \alpha_1. \quad (21)$$

We look for traveling wave solutions, for which

$$r_1 = r_1(\xi), \quad \alpha_1 = \alpha_1(\xi), \quad z_1 = z_1(\xi), \quad (22)$$

where $\xi = s - at$ and $a$ is a real constant (the propagation velocity of the wave on the curve). Since $\partial / \partial t = -a \partial / \partial \xi$, we can substitute the first of Eqs. (22) into Eq. (21) to obtain

$$r_1' = \left( a^2 + \frac{4}{r_0} \right) r_1 = A \xi + B. \quad (23)$$

The condition for having a closed space curve is automatically satisfied by taking $r_1(s) = r_1(s + L)$ and periodicity of the solution is guaranteed by putting $A = B = 0$. Hence

$$r_1 = k \sin \left( \frac{2\pi}{L}, \xi + \beta \right). \quad (24)$$

where $L$ denotes the length of the perturbed curve, $q$ the number of wraps around the torus in the meridian plane.
(or, equivalently, the number of intersections of the knotted curve with the longitude), and $k$, and $\beta_k$ two positive constants. This yields

$$a^2 = \frac{1}{r_0} + \left(\frac{2q\pi}{L}\right)^2.$$

Equation (20), by Eqs. (22) and (24), becomes

$$z_1 = \frac{akL}{2q\pi} \cos \left(\frac{2q\pi}{L} \xi + \beta_k\right),$$

and Eq. (19), by the second of Eqs. (22), can be written as

$$\alpha_1 = \frac{1}{\alpha_0} \xi = \frac{kL}{2q\pi} \cos \left(\frac{2q\pi}{L} \xi + \beta_k\right).$$

The set of equations is now complete. To have a solution in the form of torus knot $T_{p,q}$ ($p$, $q$ coprime) we must have

$$\alpha(s+L) = \alpha(s) + 2\pi n,$$

where $p$ is the number of wraps around the torus in the longitudinal plane (or, equivalently the number of intersections of the curve with the meridian); since $\alpha_1$ is also periodic in $L$, the condition (28) yields

$$L/r_0 = 2\pi n,$$

so that relation (25) becomes

$$a^2 = \frac{1}{r_0} + \left(\frac{2q\pi}{2\pi n r_0}\right)^2 = \frac{1}{\alpha_0} \left(\frac{g}{p} - 1\right).$$

In summary, by the above relations and by Eqs. (24), (26), and (27), torus knot solutions $T_{p,q}$ are given in the form:

$$r = r_0 + e_k \sin \left(\left(\frac{g}{p} \xi \right) \frac{k}{r_0} + \beta_k\right),$$

$$\alpha = \frac{1}{r_0} + e_k \cos \left(\left(\frac{g}{p} \xi \right) \frac{k}{r_0} + \beta_k\right),$$

$$z = \frac{L}{r_0} + e_k \left[1 - \left(\frac{g}{p} \xi \right) \frac{k}{r_0} + \beta_k\right],$$

where $(s/r_0) \in [0,2\pi n]$. As an example, the knot $T_{3,1}$ generated by $V_1$ and determined by Eqs. (31) is shown in Fig. 6.

IV. SOLITON INVARIANTS AS POLYNOMIAL INVARIANTS OF TORUS KNOTS

The relation between intrinsic equations of nonlinear strings in the $X$ space and soliton solutions in the $\psi$ space via the Hasimoto map has been discussed above in Sec. II. As already pointed out, the corresponding set of intrinsic equations associated with the first Killing field $V_1$ is given by the so-called Da Rios–Betchoch equations, and these can be reduced by the Hasimoto map to the nonlinear Schrödinger equation (NLS). The latter is well-known to be an example of an infinite-dimensional, completely integrable Hamiltonian system5 having soliton solutions and an infinite sequence of conserved quantities in involution satisfying conservation laws. Similar considerations are true for the class of soliton equations associated with other Killing fields listed in Fig. 2.35,36

In particular, for all solutions $\psi(s,t)$ to NLS, we have a conservation law defined by

$$\frac{\partial}{\partial t} D[s,\psi(s,t)] = \frac{\partial}{\partial s} F[s,\psi(s,t)],$$

where $D(\cdot)$ is the conserved density and $F(\cdot)$ is the conserved flow.37 The functional

$$\mathcal{F}[s] = \int_s \left[ D[\psi(u,t)] \right] du$$

is an invariant provided that the integral exists and the integrand satisfies the appropriate boundary conditions. As Zakharov and Shabat1 showed, these invariants can be derived by considering NLS as an isospectral flow for an associated linear problem. NLS has a countable family of so-called polynomial conservation laws.34,35 Since the Hasimoto map $h:X(s,t) \rightarrow \psi(s,t)$ is a well-defined one-to-one onto transformation (and therefore is invertible), we can employ the recurrence formula given by Zakharov and Shabat to calculate this family of invariants in terms of geometric quantities such as curvature and torsion of the space curve.

The recurrence formula for the densities of NLS takes the form

$$D_{n+1} = e \frac{\partial}{\partial s} \left( \frac{D_n}{c} \right) - i\tau D_n + \sum_{j+k=n} D_j D_k, \quad n=1,2,\ldots,$$

where $D_1 = c^2/4$ and $i = \sqrt{-1}$. The associated invariants for torus knots (scaled by $i^{(2n+1)}/2^{(n+2)}$) are given by integral expressions of the kind

$$I_n = \int_{\mathcal{F}_{p,q}} D_n(s,t) ds = \text{const}, \quad n=1,2,\ldots,\quad (34)$$

This procedure has been used to calculate the first conserved quantities,39,42 with densities

$$D_1 = c^2, \quad D_2 = e^2 \tau, \quad D_3 = c^2 r^2 + e^2 - (c^2/4),$$

$$D_4 = (c^2 r^2 + e^2 - 2c e^2 - \alpha^2) \tau, \quad \ldots.$$
It is interesting to note that the integrals associated with the first two densities are the invariants discovered by Betzchov in his study of vortex filament motion, and are the same conserved quantities arising in the study of the Bernoulli elastic string model. In fact, as a consequence of Noether's theorem, when \( V_1 \) and \( V_2 \) are acting on elastic rods, solitonlike elastic waves are produced together with the conservation of energy and momentum, identified with the above quantities; a physical interpretation of some of the invariants listed above has been given in the context of vortex filament motion, where it has also been proved that other global quantities, not captured by the recurrence formula (33), remain conserved under \( V_1 \).

It is immediately evident that two fundamental quantities are not captured: the total torsion \( I_0 \) (as it can be directly checked by the second of the da Rios–Betchov Eqs. (5)) and of course the total length \( I_1 \)

\[
I_0 = \oint_{\mathcal{F}_{pq}} \tau \, ds, \quad I_1 = \oint_{\mathcal{F}_{pq}} P \, ds. \tag{35}
\]

Two other global quantities (vector quantities)

\[
I_2 = \oint_{\mathcal{F}_{pq}} X \times X' \, ds \tag{36}
\]

and

\[
I_3 = \oint_{\mathcal{F}_{pq}} X \times (X \times X') \, ds \tag{37}
\]

are also conserved and not captured by the recurrence relation (33).

Since both \( c \) and \( \tau \) on dimensional grounds are \([L^{-1}]\) (inverse of a length), it is interesting to note that on dimensional grounds all these invariants can be related (as in a cascade process) as \( I_{-n} \rightarrow [L^{-n}] \).

Since torus knots generated by Killing fields are like frozen solutions in the ambient space, it is understandable that they must have an infinite number of global constraints conserved. Indeed, these are the integral invariants \( I_{-n} \). With torus knot solutions \( \mathcal{F}_{pq} \) in the form given by Eqs. (31), it is quite straightforward to calculate the geometry of the knot by its intrinsic parameters, and then to evaluate the constants \( I_{-n} \). This can be done simply by recalling the definition of tangent as \( t = X' \) and using Frenet–Serret equations to calculate curvature and torsion of the curve. By Eqs. (12) and (31) and taking the terms up to the order of \( \epsilon \), \( \mathcal{F}_{pq} \) have curvature given by

\[
c(s,t) = \frac{1}{r_0} \left[ 1 - \frac{ek_r}{r_0} \left( \frac{q}{p} \right)^2 - 1 \right] \sin \beta \cos \alpha \tag{38}
\]

and torsion given by

\[
\tau(s,t) = \frac{ek_r}{r_0} \left( \frac{q}{p} \right)^2 - 1 \left( \frac{q}{p} \right)^2 - 1 \sin \gamma \cos \alpha \]

where \( \alpha \) and \( \gamma \) are simply given by

\[
\alpha = -\frac{s}{r_0} + \epsilon \left( \frac{p}{q} \right) \frac{K_r}{r_0} \cos \gamma, \quad \gamma = \left( \frac{q}{p} \right) \frac{s-at}{r_0} + \beta_k. \tag{40}
\]

Invariants \( I_{-n} \) can now be easily calculated by Eqs. (39) and (40) and are polynomial invariants of the torus knots \( \mathcal{F}_{pq} \) generated by \( V_1 \).

Among the infinite family of global invariants, however, there are at least some quantities that are locally invariant on the curve. For example, if you consider traveling wave solutions simply given by \( c = c(s-at) \) and \( \tau = \tau(s-at) \), then Eqs. (5) are reduced to

\[
\frac{1}{2} \left( c^2 \right)' = -\left( \epsilon^2 \tau \right)', \tag{41}
\]

which integrated give the two constants

\[
K_1 = c^2 \left( \tau \frac{a}{2} \right), \tag{42}
\]

\[
K_2 = c^2 \left( \frac{1}{2} c^2 + \frac{1}{2} \right)
\]

that are "locally" conserved on the curve. These local invariants impose a much stronger constraint on the geometry of the curve and should be regarded as a particular subset of invariants for frozen geometries.

V. CONCLUDING REMARKS

In this paper we have shown how to interpret the nonlinear dynamics of a class of one-dimensional physical systems exhibiting soliton behavior in terms of Killing fields acting as generators of torus knots. Torus knot solutions under the action of Killing fields have been related to the complete integrability of one-space-dimensional soliton equations and to the existence of traveling wave solutions for these systems. As an example, the simplest Killing field that is associated with the nonlinear Schrödinger equation has been considered. Torus knot solutions have been obtained explicitly in cylindrical polar coordinates via perturbation techniques from the circular solution. Then, by the Hasimoto map, the soliton conserved quantities have been interpreted in terms of global geometric quantities and it has been shown how to express these quantities as polynomial invariants for torus knots.

The techniques here employed are of general interest, since any other choice of a Killing field of the list in Fig. 2 would have led to similar conclusions. This is due to the particular Poisson structure of the Hamiltonian flows (i.e.,
the Killing fields on \( S^2 \) that are generating the class of soliton equations, which the nonlinear Schrödinger equation, the modified Korteweg-de Vries equation and other "higher-order" soliton equations belong  to.\[^{16}\]

Geometric properties of soliton solutions are not new. In this work, however, we have interpreted soliton dynamics in terms of global geometry of torus knots presenting the associated conservation laws as polynomial invariants of torus knots in their standard form. Since the class of soliton equations associated with the Killing fields have also an infinity of polynomial conservation laws,\[^{35,36}\] the procedure used here can also be applied to other soliton equations for choices of Killing fields different from ours. Similarly, the associated polynomial invariants can be interpreted via the Hasimoto map as global geometric invariants for the solution curve. In our case we have pointed out that these global geometric invariants are given on dimensional grounds in (decreasing) powers of length, as in a cascade process.

This aspect seems to be very intriguing indeed. On one hand it poses the question of understanding how much of the geometric structure we analyzed here is inherited in the so-called prolongation structure of soliton theory. On the other hand dimensional power laws of these invariants seem to suggest that it would be possible to interpret these quantities as coefficients of a polynomial series for each particular knot type. Since torus knot solutions under Killing action are like frozen solutions in the ambient space—so that their topology is frozen—this approach indicates a direct way to use soliton concepts of nonlinear theory in order to classify topologically distinct structures in a unique manner. In general, this interplay between nonlinear dynamics and global geometry provides fundamental links between the physics and the topology of nonlinear one-dimensional systems.\[^{10}\] In this sense the soliton theory of one-dimensional systems and the topology of knots have a common playground.

ACKNOWLEDGMENTS

The author would like to thank Professor H. K. Moffatt for his help and his valuable suggestions, Professor K. C. Millett and Professor J. E. Marsden for fruitful discussions and Dr. J. S. Langer for his kind hospitality at the Institute for Theoretical Physics at UCSB, where part of this work was initiated. Financial support from UK Science and Engineering Research Council Grant No. GR/G46745 is also acknowledged.

Erratum: “Torus knots and polynomial invariants for a class of soliton equations” [Chaos 3, 83 (1993)]

Renzo L. Ricca
Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom

(Received 14 September 1994; accepted for publication 20 September 1994)

In the paper “Torus knots and polynomial invariants...”\(^1\) an error has been found in Eq. (20). By taking into account terms up to the order of \(\epsilon\), the corrected equation should read

\[
\dot{z}_1 = -r_0 + \frac{1}{r_0^2} - \frac{r_1}{r_0}, \tag{20}
\]

and similar corrections \((-1\) instead of \(+2\)\) are needed in (21) and (39), and \(+1\) instead of \(-2\)\) in (23). Therefore, Eq. (25) becomes

\[
a^2 = -\frac{1}{r_0} + \left(\frac{2q\pi}{L}\right)^2, \tag{25}
\]

and Eq. (30) becomes

\[
a^2 = -\frac{1}{r_0} + \left(\frac{2q\pi}{L}\right)^2 = \frac{1}{r_0^2} \left[\left(\frac{q}{p}\right)^2 - 1\right]. \tag{30}
\]

Since \(a^2 > 0\), then \((q/p) > 1\). According to Massey’s theorem, we have nontrivial torus knot solutions for \(q > 1\) and \(p > 1\) relative co-prime. Hence \((q/p) > 1\), with the third of Eqs. (31) that takes the form

\[
z = \frac{\dot{z}}{r_0} + \epsilon k_x \sqrt{1 - \left(\frac{p}{q}\right)^2} \cos \left(\frac{q}{p} \frac{\xi}{r_0} + \beta_0\right). \tag{31}
\]

The corrected Eqs. (31) are in perfect agreement with the solutions of Kids;\(^2\) moreover, by our method it is possible to show\(^3\) that these solutions are stable for \((q/p) \geq 1\) and unstable for \((q/p) < 1\) (although \(S_{p,q}\) and \(S_{q,p}\) are topologically equivalent). The torus knot solution \(S_{2,3}\) shown in Fig. 6 in the paper\(^1\) must therefore be replaced with the solution \(S_{2,3}\) shown here (Fig. 1). These connections between topological aspects and nonlinear dynamics have interesting implications that we hope to investigate further in the near future.

ACKNOWLEDGMENTS

I thank M.R.E. Proctor who pointed out the error in the paper.

