UNIQUENESS AND COMPARISON RESULTS FOR FUNCTIONALS DEPENDING ON ∇u AND ON u.

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ABSTRACT. We consider a variational problem, where the Lagrangean is of the form $f(\nabla u) + \alpha u$. We introduce an explicit family of solutions to this problem and we prove uniqueness and comparison results. We show that as α tends to zero, the solutions converge to solutions of the limiting variational problem where $\alpha = 0$.

1. Introduction

We consider the problem of minimizing the functional

$$\mathcal{J}(u) = \int_\Omega [f(\nabla u(x)) + \alpha u(x)]\; dx, \tag{1}$$

where $\alpha \neq 0$ is a constant, on a suitable class of functions. We shall say that a function $w \in W^{1,1}(\Omega)$ is a *solution* if the functional attains its minimum among all the functions in $W^{1,1}(\Omega)$ that satisfy the same boundary conditions as w. Our purpose is to provide a comparison result and a result on uniqueness of solutions. A comparison result is a statement of the kind: "for w and v solutions (satisfying different boundary conditions), $w \leq v$ on $\partial\Omega$ implies $w \leq v$ on Ω". A uniqueness result instead concerns solutions with the same boundary datum.

The assumptions we make on the convex function f are very general, and apply to convex Lagrangeans that can be extended valued and not necessarily differentiable; moreover, f can be such that the domain of the subdifferential is strictly smaller than the domain of the function. Apart from this, the main point is that we do not assume that f is strictly convex: in this sense, this paper is a sequel to [1]. Notice, however, that the uniqueness result we present does not hold no matter what the boundary data are (a result of this kind would be rather unlikely, without any assumption of strict convexity), but only for a restricted class of boundary conditions.

These boundary conditions are those satisfied by a family of solutions, that we are going to describe. We emphasize the fact that this family is explicit, i.e. that it can be computed directly from the Lagrangean f.

2. A family of solutions

In what follows, f^* is the *polar* of f; for its main properties, we refer to [3] and [2].

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Theorem 1. *Let Ω be an open bounded set, enough regular so that the Divergence Theorem holds, and let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be an extended valued, convex, lower semicontinuous function. For x_0 and θ in $\mathbb{R}^N$ and $c \in \mathbb{R}$, consider the function*

$$\omega_\alpha(x) = \frac{N}{\alpha} f^* \left(\theta + \frac{x - x_0}{N}\alpha\right) + c.$$

If ω_α is defined on Ω and belongs to $W^{1,1}(\Omega)$, then it is the only minimum of the functional

$$\mathcal{J}(u) = \int_\Omega [f(\nabla u(x)) + \alpha u(x)]\, dx,$$

in the class of functions

$$\mathcal{S} = \left\{u \in W^{1,1}(\Omega), u - \omega_\alpha \in W_0^{1,1}(\Omega)\right\}.$$

As an Example, the polar to the convex function

$$f(\xi) = \begin{cases} \|\xi\| - \sqrt{\|\xi\|} & \text{if } \|\xi\| \geq \sqrt{1} \\ 0 & \text{if } \|\xi\| \leq \sqrt{1} \end{cases}$$

is given by

$$f^*(p) = \begin{cases} \frac{1}{4(1-\|p\|)} & \text{if } \frac{1}{2} \leq \|p\| < 1 \\ \|p\| & \text{if } \|p\| \leq \frac{1}{2} \end{cases}.$$

For $N = 1$, $\alpha = 1$, $\Omega = (-1, 1)$, $\theta = 0$, $x_0 = 0$, the function

$$\omega(x) = f^*(x)$$

is defined on Ω but does not belong to $W^{1,1}(\Omega)$. However, one has the following result.

Remark 1. *If, in addition, f has superlinear growth, then, for Ω bounded, for every x_0 and θ, the function $\omega_\alpha(x)$ is defined on $\overline{\Omega}$ and belongs to $W^{1,\infty}(\Omega)$.*

Indeed, by the superlinear growth of f, we obtain that $f^*(p) < +\infty$ for every $p \in \mathbb{R}^N$. In particular, for every θ and x_0 in $\mathbb{R}^N$, the convex function $\omega_\alpha(x)$ is defined and Lipschitzian on $\overline{\Omega}$.

The proof of Theorem 1 is direct and does not depend on the validity of the Euler Lagrange equation, since, to this author's knowledge, the validity of this equation, for the class of problems considered in this paper, is yet to be established. We shall need the following lemma.

Lemma 1. *Let f be convex, let $\xi_1 \neq \xi_2$ and let p be both in $\partial f(\xi_1)$ and in $\partial f(\xi_2)$. Then, f^* is not differentiable at p.*

Proof. Since $p \in \partial f(\xi_2)$, for $\lambda > 0$ we have

$$\frac{1}{\lambda}(f^*(p + \lambda(\xi_2 - \xi_1)) - f^*(p)) =$$

$$\frac{1}{\lambda}\left(\sup_\xi\{\langle p + \lambda(\xi_2 - \xi_1), \xi\rangle - f(\xi)\} - (\langle p, \xi_2\rangle - f(\xi_2))\right) \geq \langle(\xi_2 - \xi_1), \xi_2\rangle.$$

Analogously, since $p \in \partial f(\xi_1)$, we have

$$\frac{1}{\lambda}(f^*(p + \lambda(\xi_1 - \xi_2)) - f^*(p)) \geq \langle(\xi_1 - \xi_2), \xi_1\rangle$$

so that, if f^* is differentiable at p, we obtain

$$\langle(\xi_2-\xi_1),\xi_1\rangle \geq \langle \nabla f^*(p), \xi_2-\xi_1\rangle \geq \langle(\xi_2-\xi_1),\xi_2\rangle,$$

i.e.,$\|\xi_2-\xi_1\|^2 \leq 0$. □

Proof of Theorem 1. Consider the case $\alpha > 0$; in this case, ω_α is a convex function (it would be concave for $\alpha < 0$.)

a) We shall prove that for every $u \in \mathcal{S}$, we have $\mathcal{J}(\omega_\alpha) \leq \mathcal{J}(u)$. By assumption, the effective domain of the convex function ω_α contains $\overline{\Omega}$; hence, ω_α is locally Lipschitzian, hence differentiable almost everywhere, on Ω. Let $x \in \Omega$ be such that $\nabla\omega_\alpha(x)$ exists, and set $p(x) = \theta + \frac{x-x_0}{N}\alpha$. Then $f^*(p)$ is differentiable in p at $p = p(x)$, with gradient $z = \nabla f^*(p) = \nabla f^*(p(x))$. Since $z \in \nabla f^*(p)$ implies $p \in \partial f(z)$, we obtain that

$$p(x) \in \partial f\left(\nabla f^*\left(p(x)\right)\right) = \partial f(z),$$

so that, for every $q \in \mathbb{R}^N$,

$$f(q) - f(z) \geq \langle p(x), q - z\rangle .$$

We have that

$$\nabla\omega_\alpha(x) = \nabla f^*(\theta + \frac{x-x_0}{N}\alpha) = \nabla f^*(p(x)) = z.$$

Hence, in particular, for $u(x) \in \mathcal{S}$ and for almost every $x \in \Omega$, we obtain

$$f\left(\nabla u(x)\right) - f\left(\nabla\omega_\alpha(x)\right) \geq \langle p(x), \nabla u(x) - \nabla\omega_\alpha(x)\rangle , \tag{2}$$

so that

$$\int_\Omega \left[f\left(\nabla u(x)\right) - f\left(\nabla\omega_\alpha(x)\right)\right] dx \geq \int_\Omega \left\langle \theta + \frac{\alpha}{N}(x-x_0), \nabla u(x) - \nabla\omega_\alpha(x)\right\rangle dx.$$

We have that $u - \omega \in W_0^{1,1}(\Omega)$; recalling the divergence theorem, we obtain

$$\int_\Omega \langle\theta, \nabla u(x) - \nabla\omega_\alpha(x)\rangle\, dx = \int_\Omega \mathrm{div}((u(x)-\omega_\alpha(x))\theta)\, dx = 0$$

and

$$\int_\Omega \left\langle \frac{\alpha}{N}(x-x_0), \nabla u(x) - \nabla\omega_\alpha(x)\right\rangle dx =$$

$$\int_\Omega \left[\mathrm{div}((u(x)-\omega_\alpha(x))\frac{\alpha}{N}(x-x_0)) - \frac{\alpha}{N}(u(x)-\omega_\alpha(x))\mathrm{div}(x-x_0)\right] dx =$$

$$-\int_\Omega \alpha(u(x)-\omega_\alpha(x))\, dx,$$

so that

$$\int_\Omega \left[f\left(\nabla u(x)\right) - f\left(\nabla\omega_\alpha(x)\right)\right] dx \geq -\int_\Omega \alpha\left(u(x)-\omega_\alpha(x)\right) dx,$$

i.e.

$$\int_\Omega \left[f\left(\nabla u(x)\right) + \alpha u(x)\right] dx \geq \int_\Omega \left[f\left(\nabla\omega_\alpha(x)\right) + \alpha\omega_\alpha(x)\right] dx.$$

Since u is arbitrary in $\mathcal{S}$, the above inequality shows that ω_α is a solution.

b) Let w be another solution. Because it is a solution, we have

$$\int_\Omega \left(\left[f\left(\nabla w(x)\right) - f\left(\nabla\omega_\alpha(x)\right)\right] + \alpha[w(x)-\omega_\alpha(x)]\right) dx = 0. \tag{3}$$

Since

$$-\int_\Omega \langle p(x), \nabla w(x) - \nabla\omega_\alpha(x)\rangle\, dx = -\int_\Omega \langle \theta + \frac{x-x_0}{N}\alpha, \nabla w(x) - \nabla\omega_\alpha(x)\rangle\, dx =$$

$$= \alpha \int_\Omega [w(x) - \omega_\alpha(x)]\, dx,$$

from 3) we obtain

$$\int_\Omega \left([f(\nabla w(x)) - f(\nabla\omega_\alpha(x))] - \langle p(x), \nabla w(x) - \nabla\omega_\alpha(x)\rangle\right)\, dx = 0.$$

From 2) applied to the solution w, we have that

$$[f(\nabla w(x)) - f(\nabla\omega_\alpha(x))] - \langle p(x), \nabla w(x) - \nabla\omega_\alpha(x)\rangle \geq 0,$$

so that, for a.e. $x \in \Omega$, we obtain

$$f(\nabla w(x)) - f(\nabla\omega_\alpha(x)) - \langle p(x), \nabla w(x) - \nabla\omega_\alpha(x)\rangle = 0.$$

Next, we claim that $p(x) \in \partial f(\nabla w(x))$ as well. In fact, for every ξ, from the above equality we obtain

$$f(\xi) - f(\nabla w(x)) = f(\xi) - [f(\nabla\omega_\alpha(x)) + \langle p(x), \nabla w(x) - \nabla\omega_\alpha(x)\rangle] =$$

$$f(\xi) - [f(\nabla\omega_\alpha(x)) + \langle p(x), \xi - \nabla\omega_\alpha(x)\rangle + \langle p(x), \nabla w(x) - \xi\rangle] \geq \langle p(x), \nabla w(x) - \xi\rangle$$

so that, by definition, $p(x) \in \partial f(\nabla w(x))$.

Apply Lemma 1: by assumption, we have that f^* is differentiable at $p = p(x)$, hence we infer that $\nabla w(x) = \nabla\omega_\alpha(x)$. The point x was arbitrary and the two functions satisfy the same boundary condition, so that $w = \omega_\alpha$. □

The following is our comparison result; here, by saying that at $\partial\Omega$ we have $v \geq u$ we mean, as usual, that $(u - v)^+ \in W_0^{1,1}(\Omega)$.

Corollary 1 (Comparison Theorem)**.** *Let w be a solution to the minimization of (1), in the class of those functions satisfying the same boundary conditions as w; assume that, on $\partial\Omega$, we have $w \leq \omega_\alpha$. Then, $w \leq \omega_\alpha$ a.e. in Ω.*

Proof. Set $\eta = (w - \omega_\alpha)^+$, so that $\eta \in W_0^{1,1}(\Omega)$. We have that $\tilde{\omega}_\alpha = \omega_\alpha + \eta$ is such that $\tilde{\omega}_\alpha - \omega_\alpha \in W_0^{1,1}(\Omega)$, while, defining $\tilde{w} = w - \eta$, we have that $\tilde{w}$ satisfies $\tilde{w} - w \in W_0^{1,1}(\Omega)$. Set $E^+ = \{x \in \Omega : \eta(x) > 0\}$: on E^+, $\nabla\tilde{\omega}_\alpha = \nabla w$ and $\tilde{\omega}_\alpha = w$, while $\nabla\tilde{w} = \nabla\omega_\alpha$ and $\tilde{w} = \omega_\alpha$.

Since ω_α is a solution on the set $\omega - \omega_\alpha \in W_0^{1,1}(\Omega)$, we have

$$0 \leq \int_\Omega [f(\nabla\tilde{\omega}_\alpha(x)) + \alpha\tilde{\omega}_\alpha(x))]\, dx - \int_\Omega [f(\nabla\omega_\alpha(x)) + \alpha\omega_\alpha(x))]\, dx$$

$$= \int_{E^+} [f(\nabla w(x)) + \alpha w(x))]\, dx - \int_{E^+} [f(\nabla\omega_\alpha(x)) + \alpha\omega_\alpha(x))]\, dx.$$

In the same way, since w is a solution on the set $v - w \in W_0^{1,1}(\Omega)$, we obtain

$$0 \leq \int_{E^+} [f(\nabla\omega_\alpha(x)) + \alpha\omega_\alpha(x))]\, dx - \int_{E^+} [f(\nabla w(x)) + \alpha w(x))]\, dx$$

so that

$$\int_{E^+} [f(\nabla\omega_\alpha(x)) + \alpha\omega_\alpha(x))]\, dx = \int_{E^+} [f(\nabla w(x)) + \alpha w(x))]\, dx$$

$$= \int_{E^+} [f(\nabla\tilde{\omega}_\alpha(x)) + \alpha\tilde{\omega}_\alpha(x))]\, dx$$

and $\tilde{\omega}_\alpha$ is a further solution to the minimization of (1) on $\{u : u - \omega_\alpha \in W_0^{1,1}(\Omega)\}$. This solution differs from ω_α when $m(E^+) > 0$, a contradiction to Theorem 1. Hence, $m(E^+) = 0$. □

3. The limit as $\alpha \to 0$

The functions ω_α are undefined for $\alpha = 0$. We are interested in the question of whether the functions ω_α converge to a limit as $\alpha \to 0$, and, if this is the case, what is the relation of this limit with the solutions to the minimum problem with $\alpha = 0$, i.e., to the problem

$$\text{minimize} \int_\Omega f(\nabla u(x))\, dx, \qquad u - u_0 \in W_0^{1,1}(\Omega). \tag{4}$$

In particular, write the arbitrary constant c as $c = -\frac{N}{\alpha} f^*(\theta) + \beta$ (β arbitrary) and consider the family of solutions to problem 1) given by

$$\omega_{(\alpha,\theta,\beta)}(x) = \frac{N}{\alpha} f^*\left(\theta + \frac{x - x_0}{N}\alpha\right) - \frac{N}{\alpha} f^*(\theta) + \beta$$

A remarkable feature of this class of solutions to problem 1 is provided by the following result:

Theorem 2. *Let f be an extended valued, convex, lower semicontinuous function with superlinear growth. Then:*

a) when f^* is differentiable at θ, as α tends to 0, the function $\omega_{(\alpha,\theta,\beta)}$ converges to the affine map $\langle \nabla f^*(\theta), x - x_0\rangle + \beta$, a solution to Problem 4).

b) in general, as α tends to 0^+, the function $\omega_{(\alpha,\theta,\beta)}$ converges to $h^+_{\theta,x_0,\beta}$, the solution to Problem 4, presented in Theorem 1 of [1]; as α tends to 0^-, $\omega_{(\alpha,\theta,\beta)}$ converges to $h^-_{\theta,x_0,\beta}$.

Proof. a) From the assumption of differentiability we have that

$$\lim_{\alpha\to 0} \frac{N}{\alpha}\left[f^*\left(\theta + \frac{x - x_0}{N}\alpha\right) - f^*(\theta)\right]$$
$$= \langle \nabla f^*(\theta), x - x_0\rangle .$$

b) By assumption, $\partial f^*(\theta)$ is non-empty, so by Theorem 23.4 of [3], we have

$$\lim_{\alpha\to 0^+} \frac{N}{\alpha} f^*\left(\theta + \frac{x - x_0}{N}\alpha\right) - \frac{N}{\alpha} f^*(\theta)) = \sup_{k\in\partial f^*(\theta)} \langle k, x - x_0\rangle .$$

By Theorem 1 of [1], the map $\sup_{k\in\partial f^*(\theta)} \langle k, x - x_0\rangle$ is a solution to 4) and the claim follows. Analogously for the second Claim, taking into account that the function $\omega_{(\alpha,\theta,\beta)}$ becomes concave in this case.

□

Example. Consider the problem

$$\text{minimize} \int_\Omega [G(u'(x)) + \alpha u(x)]\, dx$$

where $\alpha > 0$ and G is

$$G(\xi) = \begin{cases} \sqrt{2}\|\xi\| & \text{if } \|\xi\| \le \sqrt{2} \\ 1 + \frac{1}{2}\|\xi\|^2 & \text{if } \|\xi\| \ge \sqrt{2} \end{cases} . \tag{5}$$

We have

$$G^*(p) = \begin{cases} 0 & \text{if } \|p\| \leq \sqrt{2} \\ \frac{1}{2}\|p\|^2 - 1 & \text{if } \|p\| \geq \sqrt{2} \end{cases}.$$

In particular, for $\|\theta\| = \sqrt{2}$, so that $G^*(\theta) = 0$, and $x_0 = 0$, we obtain

$$\omega_{(\alpha,\theta,0)} = \begin{cases} 0 & \text{if } \|\theta + \frac{\alpha}{N}x\| \leq \sqrt{2} \\ \frac{N}{\alpha}\left(\frac{1}{2}\|\theta + \frac{\alpha}{N}x\|^2 - 1\right) & \text{if } \|\theta + \frac{\alpha}{N}x\| \geq \sqrt{2} \end{cases}.$$

As $\alpha \downarrow 0$, the set of points $\{x : \|\theta + \frac{\alpha}{N}x\| \leq \sqrt{2}\}$ grows to the half space $\{x : \langle x, \theta\rangle \leq 2\}$ and $\omega_{(\alpha,\theta,0)}$ converges pointwise to

$$\begin{cases} 0 & \text{if } \langle x, \theta\rangle \leq 2 \\ \langle \theta, x\rangle & \text{if } \langle x, \theta\rangle \geq 2 \end{cases}.$$

This last function is

$$\sup_{k \in \partial G^*(\theta)} \{\langle k, x\rangle\} = \left(I_{\partial G^*(\theta)}\right)^*(x),$$

a solution to 4).

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