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ON THE BOUNDED SLOPE CONDITION AND THE VALIDITY OF THE EULER LAGRANGE EQUATION*

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Abstract. Under the bounded slope condition on the boundary values of a minimization problem for a functional of the gradient of u, we show that a continuous minimizer w is, in fact, Lipschitzian. An application of this result to prove the validity of the Euler Lagrange equation for w is presented.

Key words. bounded slope condition, weak maximum principle, Euler Lagrange equation

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1. Introduction.

The bounded slope condition was introduced by Hartman and Nirenberg [5] and, in a variational context, by Stampacchia [9], with the purpose of obtaining pointwise bounds a.e. on the norm of the gradient $\nabla u(x)$ of a solution u to a minimum problem of the form

$$(P) \qquad \text{minimize} \int_\Omega f(\nabla u(x))\, dx \quad \text{on} \quad u - u^0 \in W_0^{1,1}(\Omega).$$

The purpose of this paper is to extend the applicability of this condition and to use the result so obtained to prove the validity of the Euler Lagrange equation for the minimizer, without assuming growth conditions from above for the integrand f. More precisely, our Theorem 4.1 below extends Stampacchia's theorem to a wider class of integrands f, while requiring less regularity on the solutions. Stampacchia's result is based on the a priori assumption that the solution is Lipschitzian, and it yields an estimate on the value of the Lipschitz constant. Our result requires that the solution be continuous, and it derives that it is, in fact, Lipschitzian. This step demands a different proof: Stampacchia's proof was based on the fact that the minimizer satisfies the Euler Lagrange equation; however, without the a priori assumption of Lipschitzianity, proving the validity of the Euler Lagrange equation under the conditions on f required by Stampacchia's theorem is still an open and challenging problem. As a consequence of our Theorem 4.1, we provide in Theorem 4.7 a result on the validity of the Euler Lagrange equation for the minimizer that does not require, as do those commonly used in the literature, growth assumptions *from above* on the integrand f.

For the proof of Theorem 4.1 we use the method of translations. This method has been used in contexts similar to the one here by, e.g., Brezis and Stampacchia [2], Brezis and Sibony [1], and, more recently, Treu and Vornicescu [10]. In all of the above papers the functional considered is

$$\int_\Omega [f(\nabla u(x)) + g(u(x))]\, dx,$$

and the argument used depends on g being strictly monotone; i.e., the case $g = 0$ is excluded. In the proof presented here, we develop an argument that allows us to

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extend the method of translations to the case $g = 0$, the case of interest in this paper. Theorem 4.2 below, instrumental to the proof of our main result, is a weak maximum principle under rather general assumptions on f.

2. Notations and preliminary results. The closed ball of radius ρ about the origin is B_ρ. The subgradient of a convex function f is denoted by ∂f and its domain by $Dom(\partial f)$. The closure of its domain, $cl(Dom(\partial f))$, is a convex set [8]. A (possibly extended valued) convex function f is called *strictly convex* if it is strictly convex on its effective domain. A *face* of a convex set is a convex extremal subset. The collection of the relative interiors of the faces of a convex set is a partition of the convex set. We say that a set Ω has the *segment property* if, given $x^0 \in \partial\Omega$, there exist a neighborhood U^0 containing x^0 and a nonzero vector k such that $x + tk \in \Omega$ whenever $x \in \overline{\Omega} \cap U^0$ and $t \in (0, 1]$. Every convex set has the segment property. Let Ω be bounded and open. We say that $u \in W^{1,1}(\Omega)$ satisfies $u \le 0$ on $\partial\Omega$ in the sense of $W^{1,1}(\Omega)$ if $u^+ \in W_0^{1,1}(\Omega)$.

3. The bounded slope condition $(BSC)_K$.

DEFINITION 3.1. *Let K be a positive real, Ω a bounded convex set. The boundary datum u^0 satisfies $(BSC)_K$ if for every $x^0 \in \partial\Omega$ there exist vectors $k^+(x^0)$ and $k^-(x^0)$, $\|k^+(x^0)\| \le K, \|k^-(x^0)\| \le K$, such that for every $x \in \partial\Omega$ we have*

$$u^0(x) - u^0(x^0) \le \langle k^+(x^0), x - x^0\rangle$$

and

$$u^0(x) - u^0(x^0) \ge \langle k^-(x^0), x - x^0\rangle.$$

The validity of $(BSC)_K$ for some K depends on the smoothness of $\partial\Omega$ and of u^0, as can be seen from the classical results of Miranda [7] and of Hartman [6].

Following Stampacchia, let us call an integrand f *regular* if $f \in C^2(\Re^N)$ and the $N \times N$ matrix of partial derivatives is positive definite at every point. Stampacchia's theorem [9] is as follows.

THEOREM 3.2. *Let f be a regular integrand. Let $u(x)$ be a minimizing function for problem (P) among all Lipschitz functions which have the same boundary values $u^0(x)$ satisfying $(BSC)_K$. If, moreover, $u \in C^1(\Omega) \cap H^2(\Omega)$, then*

$$\max_{x \in \overline{\Omega}} |u_{x_i}| \le K.$$

4. Main results. It is our purpose to prove the following theorem, our main result.

THEOREM 4.1. *Let Ω be open, bounded, and convex; let f be a (possibly extended valued) lower semicontinuous strictly convex function. Let $u^0 : \Omega \to \Re$ be Lipschitzian and let it satisfy $(BSC)_K$. Let w in $C(\Omega) \cap W^{1,1}(\Omega)$ be a solution to problem (P):*

$$\textit{minimize} \int_\Omega f(\nabla u(x))\, dx \, : u - u^0 \in W_0^{1,1}(\Omega).$$

Then w is Lipschitzian and, for almost every x in Ω, $\|\nabla w(x)\| \le K$.

Under the conditions of Stampacchia's theorem on f and Ω and assuming that the boundary datum satisfies $(BSC)_K$, the fact that the solution is *Lipschiztian* implies that the Lipschitz constant of the solution is K. Our Theorem 4.1 says that under the conditions of Theorem 4.1 on f and Ω and assuming that the boundary datum satisfies

$(BSC)_K$ for *some* constant K, knowing that the solution is *continuous* implies that the solution is *Lipschitzian.*

The following theorem is a generalized version of the weak maximum principle, to be used in the proof of Theorem 4.1.

THEOREM 4.2. *Let Ω in $\Re^N$ be open and bounded with the segment property; let f be a (possibly extended valued) lower semicontinuous, convex function. Let $u^0(x)$ in $W^{1,1}(\Omega)$ and $\ell(x) = \langle a, x\rangle + b$ be given such that in $\partial\Omega$, $u^0(x) \leq \ell(x)$ in the sense of $W^{1,1}(\Omega)$. If the infimum in problem (P) is finite and attained by some function w, then the inequality*

$$w(x) \leq \ell(x) \quad \textit{for almost every } x \in \Omega$$

follows from either (i) *or* (ii) *below:*

(i) $a \notin Int(Dom(\partial f))$.

(ii) $a \in Int(Dom(\partial f))$ *and the face of $epi(f)$ whose relative interior contains $(a, f(a))$ has dimension less than N.*

(The latter condition is immediate when f is strictly convex.)

Remark. Let f be the indicator function of the unit disk $D \subset \Re^2$; i.e., $f(\xi) = 0$ when $\|\xi\| \leq 1, f(\xi) = +\infty$ otherwise. Let $\overline{\Omega}$ be D and u^0 be $= 0$. Finally, let $\ell(x) = 0$. Then $a = 0 \in Int(Dom(f))$ and for $\|x\| = 1$, $\ell(x) \geq u^0(x)$. However, the function $w : \Omega \to \Re$ defined by $w(x) = 1 - \|x\|$, for which $\|\nabla w\| = 1$ a.e. in Ω, is a solution to the minimization problem (P) for the given f and u^0, but it is not true that $\ell(x) \geq w(x)$ a.e. in Ω. The face of $epi(f)$ containing $(0, 0)$ in its relative interior is of dimension $N = 2$.

For the proof of Theorem 4.2 we shall need the following lemma.

LEMMA 4.3. *Let f, Ω, $u^0(x)$, and ℓ be as in Theorem 4.2. Let $w - u^0$ be in $W_0^{1,1}(\Omega)$. Let $E^+ = \{x \in \Omega : w(x) > \ell(x)\}$. Then, for every $l \in \Re^N$,*

$$\int_{E^+} \langle l, \nabla w(x) - a\rangle \, dx = 0.$$

Proof. Since $u^0(x) \leq \ell(x)$ on $\partial\Omega$ in the sense of $W^{1,1}(\Omega)$, we have

$$0 \leq (w - \ell)^+ = [(u^0 - \ell) + (w - u^0)]^+ \leq (u^0 - \ell)^+ + (w - u^0)^+,$$

so that $(w - \ell)^+ \in W_0^{1,1}(\Omega)$, i.e., $(w - \ell) \leq 0$ on $\partial\Omega$ in the sense of $W^{1,1}(\Omega)$. Hence [11, Lemma 1.59] there exists a sequence (ψ_n), $\psi_n \in C^\infty(\overline{\Omega})$ and $\psi_n(x) \geq 0$ for x in $\partial\Omega$, converging to $(\ell - w)$ in $W^{1,1}(\Omega)$. Let $w_n = \ell - \psi_n$ and assume we have selected a subsequence of the sequence (w_n) converging to w pointwise as well as in $W^{1,1}(\Omega)$. Let $E^- = \{x \in \Omega : w(x) < \ell(x)\}$, $E_0 = \{x \in \Omega : w(x) = \ell(x)\}$, $E_n = \{x \in \Omega : w_n(x) - \ell(x) > 0\}$. Then $\chi_{E_n}(x) \to 1$ for almost every x in E^+, and $\chi_{E_n}(x) \to 0$ for almost every x in E^-.

We have

$$\int_{E^+} \langle l, \nabla w(x) - a\rangle \, dx = \int_\Omega \langle l, \nabla w(x) - a\rangle \chi_{E_n} \, dx + \int_\Omega \langle l, \nabla w(x) - a\rangle (\chi_{E^+} - \chi_{E_n}) \, dx.$$

The last integral is the sum of the same integral over E^+, over E^-, and over E^0. The first two integrals tend to zero from an application of the dominated convergence theorem; the third is zero since, on E^0, $\nabla w(x) = a$ a.e. Hence

$$\int_\Omega \langle l, \nabla w(x) - a\rangle (\chi_{E^+} - \chi_{E_n}) \, dx \to 0.$$

Moreover,

$$\int_\Omega \langle l, \nabla w(x) - a\rangle \chi_{E_n}\, dx = \int_{E_n} \langle l, \nabla w_n(x) - a\rangle\, dx + \int_{E_n} \langle l, \nabla w(x) - \nabla w_n(x)\rangle\, dx.$$

The second integral tends to zero since $w_n \to w$ in $W^{1,1}(\Omega)$. To prove the lemma it suffices to show that $\int_{E_n} \langle l, \nabla w_n(x) - a\rangle\, dx = 0$.

Let $\mathbf{l} = l/\|l\|$. Let P^l be the plane through the origin orthogonal to $\mathbf{l}$, O^l the projection of Ω on P^l, and $L^l(x'), x' \in O^l$, the line $\{x' + \mathbf{l}\tau; \tau \in \Re\}$. The intersection of a line $L^l(x')$ with the open set E_n can be described as $\{x' + \mathbf{l}\tau : \tau \in \cup_i(\alpha_i(x'), \beta_i(x'))\}$, where some or all of the points of $x' + \mathbf{l}\alpha_i(x')$ and of $x' + \mathbf{l}\beta_i(x')$ can belong to $\partial\Omega$. Then

$$int_{E_n} \langle l, \nabla w_n(x) - a\rangle\, dx = \int_{O^l} \left(\int_{E_n \cap L^l(x')} \langle l, \nabla w_n(x' + \mathbf{l}\tau) - a\rangle\, d\tau \right) dx'$$

$$= \int_{O^l} \left(\Sigma_i \int_{\alpha_i(x')}^{\beta_i(x')} \langle l, \nabla w(x' + \mathbf{l}\tau) - a\rangle\, d\tau \right)\, dx'.$$

We have

$$\int_{\alpha_i(x')}^{\beta_i(x')} \langle l, \nabla w_n(x' + \mathbf{l}\tau) - a\rangle\, d\tau = \int_{\alpha_i(x')}^{\beta_i(x')} \|l\| \frac{d}{d\tau}[w_n(x' + \mathbf{l}\tau) - \ell(x' + \mathbf{l}\tau)]d\tau$$

$$= \|l\|\{[w_n(x' + \mathbf{l}\beta_i(x')) - \ell(x' + \mathbf{l}\beta_i(x'))] - [w_n(x' + \mathbf{l}\alpha_i(x')) - \ell(x' + \mathbf{l}\alpha_i(x'))]\}.$$

For each i, when $x' + \mathbf{l}\alpha_i(x')$ is in Ω, w_n and ℓ coincide, and the same is true for $x' + \mathbf{l}\beta_i(x')$. Since at $\partial\Omega$, $w_n(x) \leq \ell(x)$ for all $x' + \mathbf{l}\alpha_i(x')$ while $x' + \mathbf{l}\alpha_i(x')$ is the limit of points where $w_n(x) > \ell(x)$, and the same is true for $x' + \mathbf{l}\beta_i(x')$, we have that the last integral is zero. This ends the proof that $\int_{E^+} \langle l, \nabla w(x) - a\rangle\, dx = 0$. ∎

Proof of Theorem 4.2. We wish to prove that E^+ has measure zero. We assume that it is not so and will show that this leads to a contradiction in either case (i) or (ii).

We must have that $\nabla w(x)$ is a.e. in $Dom(f)$, hence in $cl(Dom(\partial f))$; otherwise the integral would not be finite.

(a) Assume (i), i.e., $a \notin Int(Dom(\partial f))$. Then a can be separated by a hyperplane from the convex and closed set $cl(Dom(\partial f))$, i.e., there exists $h \neq 0$ such that $\langle h, a\rangle \geq \sup_{d \in Dom(\partial f)} \langle h, d\rangle$. Hence for almost every $x \in \Omega$, in particular for almost every $x \in E^+$, we have the following inequality:

$$\langle h, \nabla w(x) - a\rangle \leq 0.$$

The proof of case (i) continues in step (e) below.

(b) Assume (ii). Fix k in $\partial f(a)$. Let $\eta^+ = (w - \ell)^+$; since

$$0 \leq (w - \ell)^+ = (w - u^0 + u^0 \ell)^+ \leq (w - u^0)^+ (u^0 - \ell)^+,$$

and both maps at the right-hand side are in $W_0^{1,1}(\Omega)$, so is $(w-\ell)^+$. We thus have

$$(w-\eta^+)(x) = \begin{cases} w(x) & \text{if } w(x) \le \ell(x), \\ \ell(x) & \text{otherwise,} \end{cases}$$

$$\nabla(w-\eta^+)(x) = \begin{cases} \nabla w(x) & \text{if } w(x) \le \ell(x), \\ a & \text{otherwise.} \end{cases}$$

Hence, by the convexity of f and applying Lemma 4.3, we obtain

$$\int_\Omega (f(\nabla w(x)) - f(\nabla(w-\eta^+)(x)))\, dx = \int_{E^+} (f(\nabla w(x)) - f(a))\, dx$$

$$\ge \int_{E^+} \langle k, \nabla w(x) - a\rangle \, dx = 0.$$

(c) Since w is a minimizer, we also have

$$0 \ge \int_\Omega (f(\nabla w(x)) - f(\nabla(w-\eta^+)(x)) = \int_{E^+} (f(\nabla w(x)) - f(a))\, dx \ge 0;$$

hence, from the conclusion of (b),

$$\int_{E^+} \{f(\nabla w(x)) - [f(a) + \langle k, \nabla w(x) - a\rangle]\} dx = 0.$$

The integrand above is nonnegative, so we obtain that, a.e. in E^+, $f(\nabla w(x)) = f(a) + \langle k, \nabla w(x) - a\rangle$, i.e., the $N+1$-dimensional vector $(f(\nabla w(x)), \nabla w(x))$ belongs, for almost every x, to H, the intersection of the epigraph of f with the hyperplane $z = f(a) + \langle k, \xi - a\rangle$. H is a face of *epi*(f): it is either of dimension less than N or its dimension is N. Let $H^N = \{\xi : f(\xi) = f(a) + \langle k, \xi - a\rangle\}$ be its projection on $\Re^N$.

(d) By assumption, the face of *epi*(f), containing $(a, f(a))$ in its relative interior, has dimension less than N, and so does F_a^N, its projection on $\Re^N$. F_a^N is a face of H^N: there is a (nonzero) N-vector τ that properly separates F_a^N from H^N, $\langle \tau, \xi - a\rangle = 0$, $\xi \in F_a^N$, $\langle \tau, \xi - a\rangle \le 0$, $\xi \in H^N$, and there is some $z \in (H^N \setminus F_a^N)$ such that $\langle \tau, z - a\rangle < 0$. Hence, for $\xi \in H^N$,

$$f(\xi) \ge f(a) + \langle k, \xi - a\rangle \ge f(a) + \langle k + \tau, \xi - a\rangle.$$

Since, for $x \in E^+$, we have $\nabla w(x) \in H^N$, in particular we have, for $x \in E^+$,

$$f(\nabla w(x)) \ge f(a) + \langle k + \tau, \nabla w(x) - a\rangle.$$

Again, since w is a minimizer, we have

$$0 \ge \int_\Omega (f(\nabla w(x)) - f((\nabla(w-\eta^+)(x))) = \int_{E^+} (f(\nabla w(x)) - f(a))\, dx$$

$$\ge \int_{E^+} \langle k + \tau, \nabla w(x) - a\rangle \, dx = 0,$$

where the last equality follows from Lemma 4.3. Hence

$$\int_{E^+} \{f(\nabla w(x)) - [f(a) + \langle k + \tau, \nabla w(x) - a\rangle]\} dx = 0,$$

and, since $f(\nabla w(x)) \geq f(a)+\langle k+\tau, \nabla w(x)-a\rangle$ a.e. in E^+, it follows that $f(\nabla w(x)) = f(a) + \langle k+\tau, \nabla w(x) - a\rangle$. From the conclusion of (c) we obtain that, a.e. in E^+,

$$\langle \tau, \nabla w(x) - a\rangle = 0.$$

(e) By the conclusion of (a) in case (i) and by the above construction in case (ii), there is a nonzero N-dimensional vector, $k^\perp$, such that $\langle k^\perp, \nabla w(x) - a\rangle \leq 0$ a.e. in E^+. Choose a line $\{x' + k^\perp t\}$ intersecting E^+ on a set of positive measure and such that $t \to w(x' + ht)$ is absolutely continuous. Let $T^+ = \{t : x' + ht \in E^+\}$ and let t^+ be in T^+, i.e., such that for $x^+ = x' + k^\perp t^+$, $\eta^+(x^+)$ is positive. Since the gradient of η^+ is

$$\nabla \eta^+(x) = \begin{cases} 0 & \text{on } \Omega \setminus E^+, \\ \nabla w(x) - a & \text{otherwise,} \end{cases}$$

we have

$$0 < (w-\ell)^+(x' + t^+ h) = \int_{-\infty}^{t^+} \left(\frac{d}{dt}(w-\ell)^+(x'+th) \right) dt$$

$$= \int_{(-\infty, t^+] \cap T^+} \langle h, \nabla w(x'+th) - a\rangle \, dt$$

$$\leq \int_{(-\infty, t^+] \cap T^+} \left(\sup_{d \in Dom(\partial f)} \{\langle h, d - a\rangle\} \right) dt \leq 0,$$

a contradiction. So E^+ has measure zero. ☐

For the proof of Theorem 4.1 we shall need the following preliminary results.

LEMMA 4.4. *Let $\Omega^i, i = 1, 2$, be open and let g^i be in $W_0^{1,1}(\Omega^i)$ and such that for almost every x in Ω^i, $g^i(x) \geq 0$. Then* $\min(g^1(x), g^2(x)) \in W_0^{1,1}(\Omega^1 \cap \Omega^2)$.

Proof. Let $g_n^i : \Omega^i \to \Re, i = 1, 2$, be two sequences of Lipschitzian maps with compact support in Ω^i, g_n^i converging to g^i in $W_0^{1,1}(\Omega^i)$ and pointwise a.e. Set $G(x) = \min(g^1(x), g^2(x))$, $E^1 = \{x \in (\Omega^1 \cap \Omega^2) : g^1(x) < g^2(x)\}$, $E^2 = \{x \in (\Omega^1 \cap \Omega^2) : g^2(x) < g^1(x)\}$, $E^0 = \{x \in (\Omega^1 \cap \Omega^2) : g^1(x) = g^2(x)\}$; set also $G_n(x) = \min((g_n^1(x), g_n^2(x))$: the maps G_n are Lipschitzian with compact support contained in $(\Omega^1 \cap \Omega^2)$. One has

$$\int_{\Omega^1 \cap \Omega^2} |G - G_n| = \int_{E^1} |G - G_n| + \int_{E^2} |G - G_n| + \int_{E^0} |G - G_n|.$$

To evaluate the first integral, set $E_n^{1,1} = E^1 \cap \{x : g_n^1 < g_n^2\}, E_n^{1,2} = E^1 \cap \{x : g_n^2 < g_n^1\}, E_n^{1,0} = E^1 \cap \{x : g_n^1 = g_n^2\}$. Then

$$\int_{E^1} |G - G_n| = \int_{E_n^{1,1}} |g^1 - g_n^1| + \int_{E_n^{1,0}} |g^1 - g_n^1| + \int_{E_n^{1,2}} |g^1 - g_n^2|,$$

and the first two integrals converge to zero since $g_n^1 \to g^1$ in $W^{1,1}(\Omega^1)$. Also, g_n^2 converges pointwise to g^2; hence $\chi_{E_n^{1,2}} \to 0$ pointwise a.e. The sequence $(|g^1 - g_n^2|)$ is equiintegrable, since g_n^2 converges in $L^1(\Omega^2)$; by Egoroff's theorem $\int_{E_n^{1,2}} |g^1 - g_n^2| \to 0$. Similarly for the other cases and for $\int_{\Omega^1 \cap \Omega^2} \|\nabla G - \nabla G_n\|$. ☐

LEMMA 4.5. *Let f be strictly convex on its effective domain. For every pair (a, b) in its effective domain, $a \neq b$, for every $\lambda, 0 < \lambda < 1$, we have*

$$f(a + \lambda(b - a)) - f(a) + f(b - \lambda(b - a)) - f(b) < 0.$$

Proof. Consider the restriction of f to the line oriented from a to b. Under the conditions of the Lemma, the map $c \to \frac{f(c+\lambda(b-a))-f(c)}{\lambda(b-a)}$ is strictly monotonic. ∎

Proof of Theorem 4.1. It is convenient to set $\Psi^+(x) = \inf_{x^0 \in \partial\Omega} \langle k^+(x^0), x - x^0 \rangle + u^0(x^0)$ and $\Psi^-(x) = \sup_{x^0 \in \partial\Omega} \langle k^-(x^0), x - x^0 \rangle + u^0(x^0)$; the maps Ψ^+ and Ψ^- are Lipschitzian with Lipschitz constant K. Applying Theorem 4.2 to each of the affine maps $\langle k^+(x^0), x - x^0 \rangle + u^0(x^0)$, we infer that the solution w satisfies $w \leq \Psi^+$. Applying the same theorem to the problem $\tilde{P}$ whose data are $\tilde{f}(\xi) = f(-\xi)$ and $\tilde{u}^0 = -u^0$, we obtain $\Psi^- \leq w$.

To prove the theorem it is enough to show that there cannot exist a unit vector $\mathbf{v}$, a scalar $M > K$, and a set $E \subset \Omega$ with $\mu(E) > 0$ such that, for x in E, $\langle \nabla w(x), \mathbf{v} \rangle > M$. Let us assume that $M, \mathbf{v}, E$ exist and derive a contradiction.

(a) There exists a representative of w that is absolutely continuous on almost every line parallel to $\mathbf{v}$. Since it coincides with w a.e. in Ω, on almost every such line $\{x = t\mathbf{v} + a : t \in \Re\}$, it coincides with w for almost every t; by continuity, they coincide for all t on every such line. Hence w is absolutely continuous on almost every line parallel to $\mathbf{v}$. On a plane orthogonal to $\mathbf{v}$ there exists a set of points of positive $(N - 1)$ measure, such that lines parallel to $\mathbf{v}$ through these points meet E in a set of positive one-dimensional measure. Let us fix one such line; let x^* be a point on it that is at once in E and such that the map $t \to w(x^* + t\mathbf{v})$ is differentiable at $t = 0$ with derivative

$$\frac{d}{dt}[w(x^* + t\mathbf{v})]|_{t=0} = \langle \nabla w(x^*), \mathbf{v} \rangle = M + \zeta, \qquad \zeta > 0.$$

Then there exists $h^* > 0$ such that for every $0 < h \leq h^*$

$$w(x^* + h\mathbf{v}) - w(x^*) - Mh > 0.$$

(b) We wish to prove the following claim. Let x^{**} be a point in Ω such that $t \to w(x^{**} + t\mathbf{v})$ is differentiable at $t = 0$ with derivative $D^{**} > M$, and let $h^{**} > 0$ be such that for every $0 < h \leq h^{**}$, $x^{**} + h\mathbf{v}$ is in Ω and

$$w(x^{**} + h\mathbf{v}) - w(x^{**}) - Mh > 0.$$

Then $t \to w(x^{**} + t\mathbf{v})$ is affine on $[0, h^{**}]$ with derivative D^{**}.

Proof of the claim. Fix any $h \in (0, h^{**}]$. On the convex set $\Omega_h = \Omega \cap (\Omega - h\mathbf{v})$ both $x \to w(x)$ and $x \to w(x + h\mathbf{v})$ are defined. By assumption, the set

$$E_h^+ = \{x \in \Omega_h; w(x + h\mathbf{v}) > w(x) + hM\}$$

is an open subset of Ω_h containing x^{**}. For $x \in E_h^+$, we have that $y = x + h\mathbf{v}$ is such that $y - h\mathbf{v}$ is in Ω and $w(y - h\mathbf{v}) < w(y) - hM$. The set

$$E_h^- = \{y \in \Omega_{-h}; w(y - h\mathbf{v}) < w(y) - hM\}$$

is a translate of E_h^+ : $E_h^- - h\mathbf{v} = E_h^+$. Let $\eta_h^+(x)$ on Ω_h be $(w(x + h\mathbf{v}) - w(x) - hM)^+$ and $\eta_h^-(x)$ on Ω_{-h} be $(w(x - h\mathbf{v}) - w(x) + hM)^-$. We wish to show that η_h^+ and η_h^-

are admissible variations, i.e., that they are in $W_0^{1,1}(\Omega)$. From the Lipschitzianity of Ψ^+ and of Ψ^- we obtain

$$\eta^+(x) \le \Psi^+(x+h\mathbf{v}) - Mh - w(x) \le \Psi^+(x) - w(x),$$

$$\eta^+(x) \le w(x+h\mathbf{v}) - Mh - \Psi^-(x+h\mathbf{v}) + Kh \le w(x+h\mathbf{v}) - \Psi^-(x+h\mathbf{v});$$

i.e., $\eta_h^+ \le \min(\Psi^+(x) - w(x), w(x+h\mathbf{v}) - \Psi^-(x+h\mathbf{v}))$.

Apply Lemma 4.4 with $\Omega^1 = \Omega, \Omega^2 = \Omega_h, g^1(x) = \Psi^+(x) - w(x), g^2(x) = w(x+h\mathbf{v}) - \Psi^-(x+h\mathbf{v})$ to infer that η_h^+ is an admissible variation, and the same is true for η_h^-. Since w is a minimum, we must have that for all λ

$$\int_\Omega f(\nabla w(x) + \lambda \nabla \eta_h^+(x))\, dx \ge \int_\Omega f(\nabla w(x))\, dx,$$

$$\int_\Omega f(\nabla w(x) + \lambda \nabla \eta_h^-(x))\, dx \ge \int_\Omega f(\nabla w(x))\, dx.$$

We have

$$\nabla \eta_h^+ = \begin{cases} \nabla w(x+h\mathbf{v}) - \nabla w(x) & \text{if } x \in E_h^+, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nabla \eta_h^- = \begin{cases} \nabla w(x-h\mathbf{v}) - \nabla w(x) & \text{if } x \in E_h^-, \\ 0 & \text{otherwise,} \end{cases}$$

so that the above inequalities yield

$$\int_{E_h^+} f(\nabla w(x) + \lambda[\nabla w(x+h\mathbf{v}) - \nabla w(x)]) - f(\nabla w(x))\, dx \ge 0,$$

$$\int_{E_h^-} f(\nabla w(x) + \lambda[\nabla w(x-h\mathbf{v}) - \nabla w(x)]) - f(\nabla w(x))\, dx \ge 0.$$

Making the change of variables $y = x + h\mathbf{v}$ and adding the two inequalities, one obtains

$$\int_{E_h^+} \{f(\nabla w(x) + \lambda[\nabla w(x+h\mathbf{v}) - \nabla w(x)]) - f(\nabla w(x)) + f(\nabla w(x+h\mathbf{v})$$

$$-\lambda[\nabla w(x+h\mathbf{v}) - \nabla w(x)]) - f(\nabla w(x+h\mathbf{v}))\}\, dx \ge 0.$$

From Lemma 4.5 we obtain that, for every x such that $\nabla w(x) \ne \nabla w(x+h\mathbf{v})$, the integrand is negative. Since E_h^+ is a nonempty open set, this is a contradiction unless, a.e. in E_h^+, $\nabla w(x) = \nabla w(x+h\mathbf{v})$.

The set E_h^+ contains a ball B_h about x^{**}; for x in this ball, $\nabla w(x) - \nabla w(x+h\mathbf{v}) = 0$ a.e. By the continuity of w, there exists a constant C such that, on $B_h, w(x) - w(x+h\mathbf{v}) = C$. Then, since the limit

$$\lim_{t\to 0} \frac{1}{t}[w(x^{**} + t\mathbf{v}) - w(x^{**})]$$

exists and equals D^{**}, so does

$$\lim_{t\to 0} \frac{1}{t}[w(x^{**} + t\mathbf{v} + h\mathbf{v}) - w(x^{**} + h\mathbf{v})].$$

In particular, the derivative at $t = 0$ of the map $t \to w(x^* + h\mathbf{v} + t\mathbf{v})$ exists and equals D^{**}. This reasoning holds for every $0 < h \le h^{**}$, thus proving the claim.

(c) The previous claim applies at x^*. Hence the map $t \to w(x^* + t\mathbf{v})$ is affine on $[0, h^*]$ with derivative $M + \zeta$. Let $[0, \Lambda]$ be the maximal interval on which this map is affine. We claim that $x^* + \Lambda\mathbf{v}$ is in $\partial\Omega$. If it is in Ω for some $\varepsilon > 0$, then so is $x^* + (\Lambda + \tau)\mathbf{v}$ for $0 \le \tau < \varepsilon$. Choose λ in $(0, \Lambda)$. The map $t \to w(x^* + t\mathbf{v})$ is differentiable at λ with derivative $\langle \nabla w(x), \mathbf{v}\rangle$. Moreover we have that $w(x^* + \Lambda\mathbf{v}) - w(x^* + \lambda\mathbf{v}) = (M + \zeta)(\Lambda - \lambda)$, i.e.,

$$w((x^* + \lambda\mathbf{v}) + (\Lambda - \lambda)\mathbf{v}) - w(x^* + \lambda\mathbf{v}) - (\Lambda - \lambda)M = \zeta M.$$

Hence, by the continuity of w, for all $\tau \le \varepsilon_1 < \varepsilon$,

$$w((x^* + \lambda\mathbf{v}) + (\Lambda - \lambda + \tau)\mathbf{v}) - w(x^* + \lambda\mathbf{v}) - (\Lambda - \lambda + \tau)M > 0.$$

The point $x^* + \lambda\mathbf{v}$ can be used as x^{**} with $h^{**} = (\Lambda - \lambda + \varepsilon_1)$. Applying the claim of part (b), we have that the map $t \to w(x^* + t\mathbf{v})$ is affine on $[0, \Lambda + \varepsilon_1]$, contradicting the maximality of Λ. Hence $x^* + \Lambda\mathbf{v}$ is in $\partial\Omega$.

(d) Let x^{***} be $x^* + \Lambda\mathbf{v}$; since u^0 is continuous, the conditions $u^0 \le \ell$ and $u^0 \ge \ell$ on $\partial\Omega$ in $W^{1,1}$ in sense and pointwise coincide. Thus, by Theorem 4.2, for every $x \in \Omega$ (in particular for x^*)

$$u^0(x^{***}) + \langle k^-(x^{***}), x - x^{***}\rangle \le w(x) \le u^0(x^{***}) + \langle k^+(x^{***}), x - x^{***}\rangle.$$

Hence, from point (c),

$$w(x^{***}) = w(x^*) + \|x^* - x^{***}\|(M + \zeta) > w(x^*) - \langle k^-(x^{***}), x^* - x^{***}\rangle \ge u^0(x^{***}),$$

while, for every $x \in \Omega$,

$$w(x) \le u^0(x^{***}) + \langle k^+(x^{***}), x - x^{***}\rangle.$$

The above inequalities are incompatible whenever $\|x - x^{***}\|$ is sufficiently small. This is a contradiction. ∎

COROLLARY 4.6. *Under the same assumptions on* Ω, f, *and* u^0 *as in Theorem* 4.2, *let solutions to problem* (P) *be continuous. Then problem* (P) *and problem* $(P)_K$,

$$\textit{minimize} \int_\Omega f(\nabla u(x))\,dx \quad \textit{on} \quad u - u^0 \in W_0^{1,1}(\Omega) \textit{ and } \|\nabla u(x)\| \le K,$$

are equivalent, in the sense that they have the same solutions.

Known results on the validity of the Euler Lagrange equation for a minimizer w hold under growth assumptions *from above* on f, i.e., under slow growth assumptions. (An exception to this statement is [4], whose results are for integrands f that tend to $+\infty$ at the boundary of $Dom(f)$, under conditions different from those presented here.)

THEOREM 4.7. *Let* $f : \Re^N \to \Re$ *be* C^1, *strictly convex, and such that for some* α *and* $\beta > 0$, $f(\xi) \ge \alpha + \beta\|\xi\|^p$, $p > N$. *Let* Ω *be bounded and convex and let* u^0

satisfy $(BSC)_K$ *for some constant* K. *Let* w *be a solution to problem* (P). *Then* w *is Lipschitzian and it satisfies the Euler Lagrange equation in the sense that*

$$\int_\Omega \langle \nabla f(\nabla w(x)), \nabla \eta(x)\rangle \, dx = 0$$

for every Lipschitzian η, $\eta|_{\partial\emptyset} = 0$.

Proof. From the growth assumptions we know that $w \in W^{1,p}(\Omega)$; hence we know that it is continuous. Theorem 4.1 applies and $\|\nabla w(x)\| \le K$ a.e. in Ω. Fix η and let λ be so small that $\lambda\|\nabla\eta\| \le 1$. Let $M = \max_{\xi\in B_{K+1}}\{\|\nabla f(\xi)\|\}$. Since w is a minimum, one has

$$0 \le \left(\frac{1}{\lambda}\right)\int (f(\nabla w + \lambda\nabla\eta) - f(\nabla w)) = \int \langle \nabla f(\nabla(w(x) + \sigma(x)\lambda\nabla\eta(x)), \nabla\eta(x)\rangle \, dx,$$

and the term under the integral sign, that converges pointwise to $\langle \nabla f(\nabla w(x)), \nabla\eta(x)\rangle$ as $\lambda \to 0$, is bounded in norm by M. Hence, applying the dominated convergence theorem, the result follows. □

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