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EXISTENCE OF SOLUTIONS TO DIFFERENTIAL INCLUSIONS AND TO TIME OPTIMAL CONTROL PROBLEMS IN THE AUTONOMOUS CASE*

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Abstract. We prove existence of solutions to upper semicontinuous differential inclusions and to time optimal control problems under conditions that are strictly weaker than the usual assumption of convexity.

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1. Introduction.

The condition of convexity with respect to the variable gradient has been of universal use in the calculus of variations, in optimal control, and in differential inclusions to prove the existence of solutions. In fact, convexity is the property required in order to pass to a weak limit along a sequence, be it a minimizing sequence or a sequence of successive approximations, preserving the properties that are needed. This approach, however, because of its generality, need not always provide the best results, since it does not take into account possible additional information such as, for instance, the presence of symmetries in the problem. One is led to think that, by suitably exploiting these symmetries, the convexity condition could be substantially reduced. The purpose of the present paper is to show that, for the simplest of such symmetries, the time invariance in the problem of the existence of solutions to upper semicontinuous differential inclusions, convexity can be replaced by a strictly weaker condition, our almost convexity, below. Moreover, we show that, in the case of autonomous control systems of the form

$$x'(t) = f(x(t), u(t)), \quad u(t) \in U(x(t))$$

for the existence of a time optimal solution, Filippov's classical assumption of convexity of the images of the map $F(x) = f(x, U(x))$ can be replaced by the weaker assumption of almost convexity of the same images. As will be shown, our assumption does not imply that the set of solutions to the differential inclusion is closed in the space of continuous functions with uniform convergence, as happens in the case of the assumption of convexity, but only that the sections of this set of solutions are closed. This property is sufficient to establish the existence of time optimal solutions.

2. Main results.

The following is our assumption of almost convexity.

DEFINITION 1. *Let X be a vector space. A set $K \subset X$ is called* almost convex *if for every $\xi \in coK$ there exist λ_1 and λ_2, $0 \le \lambda_1 \le 1 \le \lambda_2$, such that $\lambda_1\xi \in K, \lambda_2\xi \in K$.*

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Every convex set is almost convex. If a set K is almost convex and $0 \in co(K)$, then $0 \in K$. Typical cases of almost convex sets are $K = \partial C$, with C a convex set not containing the origin, or $K = \{0\} \cup \partial C$, C a convex set containing the origin.

It is our purpose to prove the following theorem.

THEOREM 1. *Let $\Omega \subset R^N$ be open, and let F, from Ω to the nonempty subsets of R^N, be upper semicontinuous with bounded, closed, and almost convex values. Then the Cauchy problem*

$$y'(s) \in F(y(s)), \quad y(0) = x_0 \in \Omega$$

admits a solution defined on some interval $[-\delta, \delta]$, $\delta > 0$. Moreover, for every $\tau \in [-\delta, \delta]$, the attainable set at τ, $A_{x_0}(\tau)$, is closed and coincides with $A^{co}_{x_0}(\tau)$, the attainable set at τ of the convexified problem

$$y'(s) \in coF(y(s)), \quad y(0) = x_0.$$

Remarks. (1) The upper semicontinuous map F, from R to the closed subsets of R, defined by $F(x) = -\text{sign}(x)$ for $x \neq 0$, $F(0) = \{-1, +1\}$, is not almost convex at $x = 0$, and the corresponding Cauchy problem with the condition $y(0) = 0$ admits no local solution.

(2) Under the condition of almost convexity, the attainable sets are closed for every (small) τ, but the set of solutions need not be closed in $C(I)$, unlike in the convex case.

The following corollary to Theorem 1, to be compared with Theorem 1 of Filippov [3], shows that, in the case of autonomous control systems, for the existence of a time optimal solution, Filippov's assumption that the set $f(x, U(x))$ is convex can be replaced by the weaker assumption that the same set is almost convex.

COROLLARY 1. *Let $f(x, u)$ be continuous for $x \in \Omega$ and $u \in U(x)$, and let the set valued map $U(x)$, from Ω to the nonempty compact subsets of R^N, be upper semicontinuous. Moreover, assume that the set*

$$F(x) = f(x, U(x))$$

is almost convex for every $x \in \Omega$. Let x_0 and x_1 be given in Ω, and assume that for some $\tilde{t} \geq 0$, $x_1 \in A_{x_0}(\tilde{t})$. Then the problem of reaching x_1 from x_0 in minimum time admits a solution.

For the proof of Theorem 1 we shall need the following preliminary result.

THEOREM 2. *Let F be upper semicontinuous. Let $x : [a, b] \to R^n$ be a solution to*

$$y'(t) \in co(F(y(t))), \quad y(a) = x_a.$$

Assume that there are two integrable functions $\lambda_1(.), \lambda_2(.)$, from $[a, b]$ to R, satisfying $0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t)$ and such that, for almost every $t \in [a, b]$, we have $\lambda_1(t)x'(t) \in F(x(t))$ and $\lambda_2(t)x'(t) \in F(x(t))$. Then there exists $t = t(s)$, a nondecreasing absolutely continuous map of the interval $[a, b]$ onto itself, such that the map $\tilde{x}(s) = x(t(s))$ is a solution to

$$y'(s) \in F(y(s)), \quad y(a) = x_a.$$

Moreover, $\tilde{x}(a) = x(a)$ and $\tilde{x}(b) = x(b)$.

Proof. (a) When $x'(t) = 0$, we shall assume, without loss of generality, that $\lambda_2(t) = 1$. Consider the set

$$C = \{t \in I : 0 \in F(x(t))\}.$$

From the continuity of x and the upper semicontinuity of F, we obtain that C is closed. Without loss of generality we shall assume that, for t in C, $\lambda_1(t)x'(t) = 0$.

(b) Let $[\alpha, \beta]$ be an interval, and assume that, on this interval, there exist two functions $\lambda_1(.), \lambda_2(.)$ with the properties stated above. In addition, assume that $\lambda_1(t) > 0$ a.e. We claim that there exist two measurable subsets of $[\alpha, \beta]$, having characteristic functions χ_1 and χ_2 such that $\sum \chi_i = \chi_{[\alpha,\beta]}$, and an absolutely continuous function $s = s(t)$ on $[\alpha, \beta]$, $s(\alpha) - s(\beta) = \alpha - \beta$, such that

$$s'(t) = \chi_1(t)\lambda_1(t) + \chi_2(t)\lambda_2(t).$$

This conclues the proof of this claim. □

Redefine $\lambda_1(t)$ and $\lambda_2(t)$ on a set of measure zero to have both functions positive for every $t \in [\alpha, \beta]$. Set $p(.)$ to be $\frac{1}{2}$ when $\lambda_1(t) = \lambda_2(t) = 1$, to be $\frac{\lambda_2 - 1}{\lambda_2 - \lambda_1}$ otherwise. With this definition we have that $0 \leq p(t) \leq 1$ and that both equalities

$$1 = p(t) + (1 - p(t))$$

and

$$1 = p(t)\lambda_1(t) + (1 - p(t))\lambda_2(t)$$

hold true. In particular, we have

$$\int_\alpha^\beta 1dt = \int_\alpha^\beta [p(t) + (1 - p(t))]dt = \int_\alpha^\beta \left[\frac{p(t)\lambda_1(t)}{\lambda_1(t)} + \frac{(1 - p(t))\lambda_2(t)}{\lambda_2(t)}\right] dt.$$

We wish to apply Liapunov's theorem on the range of measures, to infer the existence of two measurable subsets having characteristic functions $\chi_1(.), \chi_2(.)$ such that $\sum \chi_i = \chi_{[\alpha,\beta]}$ and with the property that

$$\int_\alpha^\beta 1dt = \int_\alpha^\beta \left[\chi_1(t)\frac{1}{\lambda_1(t)} + \chi_2(t)\frac{1}{\lambda_2(t)}\right] dt.$$

However, it is not obvious that the function $\frac{1}{\lambda_1(t)}$ is integrable, and thus the results of [2] need not be applicable. For this purpose we shall use a device already used in [1]. Consider the sequence of disjoint sets

$$E^n = \left\{t \in [\alpha, \beta] : n < \frac{1}{\lambda_1(t)} \leq n + 1\right\}.$$

We have that $\cup E^n = [\alpha, \beta]$. Applying Liapunov's theorem to each E^n, we infer the existence of two sequences of measurable subsets E_1^n, E_2^n, having characteristic functions χ_1^n, χ_2^n, such that for every n,

$$\int_{E^n} 1dt = \int_{E^n} \left[\chi_1^n(t)\frac{1}{\lambda_1(t)} + \chi_2^n(t)\frac{1}{\lambda_2(t)}\right] dt.$$

Set $\cup E_1^n = E_1$, $\cup E_2^n = E_2$ and $\chi_1 = \sum \chi_1^n$, $\chi_2 = \sum \chi_2^n$. For each m, the function

$$\sigma^m(t) = \sum_{n=0}^m \left[\chi_1^n(t)\frac{1}{\lambda_1(t)} + \chi_2^n(t)\frac{1}{\lambda_2(t)}\right]$$

is positive, and the sequence converges pointwise monotonically to

$$\sigma(t) = \chi_1(t)\frac{1}{\lambda_1(t)} + \chi_2(t)\frac{1}{\lambda_2(t)}.$$

Moreover, the sequence of sets $V^m = (\cup_{n=0}^m E^n)_m$ is monotonically increasing to $[a, b]$, so that $\int_\alpha^\beta 1dt = \lim_m \int_{V^m} 1dt$. Hence

$$\int_\alpha^\beta 1dt = \lim_m \int s^m(t)dt = \int \lim_m s^m(t)dt,$$

so that we obtain

$$\int s(t)dt = \int \left[\chi_1(t)\frac{1}{\lambda_1(t)} + \chi_2(t)\frac{1}{\lambda_2(t)}\right] dt = \int_\alpha^\beta 1dt.$$

Define $s'(t) = \sigma(t)$. Then $\int_\alpha^\beta s'(t)dt = \beta - \alpha$. This proves the claim.

(c) Consider the case in which C is empty. In this case, it cannot be that $\lambda_1(t) = 0$ on a set of positive measure, and the previous point (b) can be applied to the interval $[a, b]$. Set $s(t) = a + \int_a^t s'(\tau)d\tau$. By the previous point (b), s is strictly monotonic increasing and maps $[a, b]$ onto itself. Let $t = t(s)$ be its inverse, so that, in particular, $t(a) = a$: we have that $1 = s'(t(s))t'(s)$. Consider the map $\tilde{x}(s) = x(t(s))$. We have

$$\begin{aligned}
\frac{d}{ds}\tilde{x}(s) &= x'(t(s))t'(s) = x'(t(s))\frac{1}{s'(t(s))} \\
&= x'(t(s))\frac{1}{s'(t(s))}\chi_1(t(s)) + x'(t(s))\frac{1}{s'(t(s))}\chi_2(t(s)) \\
&= x'(t(s))\lambda_1(t(s))\chi_1(t(s)) + x'(t(s))\lambda_2(t(s))\chi_2(t(s)) \\
&\in F(x(t(s)) = F(\tilde{x}(s)).
\end{aligned}$$

Hence the theorem is proved in this case.

(d) From now on we shall assume that C is nonempty. Set $c = \sup C$, so that $c \in C$. The complement of C is open relative to $[a, b]$; it consists of at most countably many nonoverlapping open intervals (a_i, b_i), with the possible exception of one of the form $[a_{i_i}, b_{i_i})$ with $a_{i_i} = a$, and one $(a_{i_f}, b_{i_f}]$ with $a_{i_f} = c$. For each i apply point (b) to the interval (a_i, b_i) to infer the existence of K_1^i and K_2^i, two subsets of (a_i, b_i) with characteristic functions $\chi_1^i(t)$ and $\chi_2^i(t)$ such that $\chi_1^i(t) + \chi_2^i(t) = \chi_{(a_i,b_i)}$ and such that, setting

$$s'(t) = \chi_1^i(t)\frac{1}{\lambda_1(t)} + \chi_2^i(t)\frac{1}{\lambda_2(t)},$$

we obtain

$$\int_{a_i}^{b_i} s'(\tau)d\tau = b_i - a_i.$$

(e) On $[a, c]$ set

$$s'(t) = \frac{1}{\lambda_2(t)}\chi_C(t) + \sum \left(\chi_1^i(t)\frac{1}{\lambda_1(t)} + \chi_2^i(t)\frac{1}{\lambda_2(t)}\right),$$

where the sum is over all intervals contained in $[a,c]$, i.e., with the exception of $(c,b]$. We have that

$$\int_a^c s'(\tau)d\tau = \kappa \le c-a$$

since $\lambda_2(t) \ge 1$ and $\int_{a_i}^{b_i} s'(\tau)d\tau = b_i - a_i$. Setting $s(t) = a + \int_a^t s'(\tau)d\tau$, we obtain that s is an invertible map from $[a,c]$ to $[a,\kappa]$.

(f) Define $t = t(s)$ from $[a,\kappa]$ to $[a,c]$ to be the inverse of $s(.)$. Extend $t(.)$ as an absolutely continuous map $\tilde{t}(.)$ on $[a,c]$, setting

$$\tilde{t}'(s) = 0$$

for $s \in [\kappa, c]$. We claim that the function $\tilde{x}(s) = x(\tilde{t}(s))$ is a solution to

$$y'(s) \in F(y(s)), \quad y(a) = x_a$$

on the interval $[a,c]$. Moreover, we claim that it satisfies $\tilde{x}(c) = x(c)$.

To prove the claim, notice that, as in (c), we have that for s in $[a,\kappa]$, $\tilde{t}(s) = t(s)$ is invertible and

$$\begin{aligned}\frac{d}{ds}\tilde{x}(s) &= x'(\tilde{t}(s))\frac{1}{s'(\tilde{t}(s))} = x'(t(s))\frac{1}{s'(t(s))} \\ &= x'(\tilde{t}(s))\left[\lambda_2(t(s))\chi_C(t(s)) + \sum\left(\chi_1^i(t(s))\lambda_1(t(s)) + \chi_2^i(t(s))\lambda_2(t(s))\right)\right] \\ &\in F(x(t(s)) = F(\tilde{x}(s)).\end{aligned}$$

In particular, from $t(\kappa) = c$ we obtain $\tilde{x}(\kappa) = x(c)$. On $[\kappa, c]$, $\tilde{x}$ is constant and we have

$$\frac{d}{ds}\tilde{x}(s) = 0 \in F(x(c)) = F(\tilde{x}(\kappa)) = F(\tilde{x}(s)).$$

This proves the claim.

(g) It is left to define the solution on $[c,b]$. On it, $\lambda_1(t) > 0$ and the construction of points (b) and (c) can be repeated to find a solution to

$$y'(s) \in F(y(s)), \quad y(c) = x(c)$$

on $[c,b]$. This completes the proof. □

Proof of Theorem 1. By the upper semicontinuity of F, there exist a ball $B = B[x_a,\rho]$ and a positive real M such that F is bounded by M on $B = B[x_a,\rho]$; set $\delta = \frac{\rho}{M}$ and consider the interval $I = [a-\delta, a+\delta]$. On I a solution x to the Cauchy problem

$$y'(s) \in coF(y(s)), \quad y(a) = x_a$$

exists.

Fix any $\tau \in I$. Since the attainable set at τ, $A_{x_0}(\tau)$, is contained in the attainable set for the solutions to the convexified problem, $A_{x_0}^{co}(\tau)$, it is enough to show that $A_{x_0}^{co}(\tau) \subset A_{x_0}(\tau)$.

By assumption, for every x, $F(x)$ is almost convex; hence for almost every $t \in I$ there exist two nonempty sets $\Lambda_1(t)$ and $\Lambda_2(t)$ such that $\lambda_1 x'(t)) \in F(x(t))$ for

$\lambda_1 \in \Lambda_1(t)$, $\lambda_2 x'(t)) \in F(x(t))$ for $\lambda_2 \in \Lambda_2(t)$, and $0 \leq \lambda_1 \leq 1$, $1 \leq \lambda_2$. Set $Z = \{t : x'(t) = 0\}$: there is no loss of generality in assuming that, for $t \in Z$, $\Lambda_1(t) = \Lambda_2(t) = \{1\}$. We claim that the set valued map $t \to \Lambda_1(t)$ is measurable. Applying Lusin's theorem to x', write $I \setminus Z$ as $(\cup K_i) \cup N$, where the measure of N is 0, each K_i is compact, and the restriction to K_i of x' is continuous on K_i. Then, by the continuity on K_i of x' and of x and the upper semicontinuity of F, it follows that the map Λ_1 has a closed graph on $K_i \times R^N$; since, in addition, its values are closed subsets of $[0,1]$, it is upper semicontinuous. It follows then that Λ_1 is measurable on I. The proof that $t \to \Lambda_2(t)$ is measurable is similar, with the difference that the values of Λ_2 need not be bounded. In this case, write $I \setminus Z$ as the countable union of the sets $M_n = \{t : \|x'(t)\| \geq 1/n\}$. On each M_n, Λ_2 has an upper bound, since F is bounded, and the same reasoning as in the previous point can be applied.

Hence, by standard arguments, there exist measurable selections $\lambda_1(.)$ and $\lambda_2(.)$ from the maps $\Lambda_1(.)$ and $\Lambda_2(.)$.

Apply Theorem 2 to the interval $[a, \tau]$ or $[\tau, a]$ to prove Theorem 1. □

Proof of Corollary 1. Let $t^* = \inf\{t \in [0, \tilde{t}] : x_1 \in A_{x_0}(t)\}$. Let (t_n) be decreasing to t_* and let x_n be solutions to the differential inclusion

$$x'(t) \in F(x(t))$$

such that $x_n(0) = x_0$ and $x_n(t_n) = x_1$. A subsequence of this sequence converges uniformly to x_*, and it is known that x_* is a solution to

$$x'(t) \in coF(x(t)).$$

Then, $x_*(t_*) \in A^{co}_{x_0}(t_*)$, and by Theorem 1 this set coincides with $A_{x_0}(t_*)$. Hence, x_* is the solution to the minimum time problem, and t_* is the minimum time required. □

REFERENCES

[1] A. Cellina and G. Colombo, *On a classical problem of the calculus of variations without convexity assumptions*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, 7 (1990), pp. 97–106.

[2] L. Cesari, *Optimization—Theory and Applications*, Springer-Verlag, New York, 1983.

[3] A. F. Filippov, *On certain questions in the theory of optimal control*, Vestnik Moskov. Univ., Ser. Mat. Mech. Astr., 2 (1959), pp. 25–32; translated in SIAM J. Control, 1 (1962), pp. 76–84.