A VIEW ON DIFFERENTIAL INCLUSIONS

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1. Ordinary Differential Inclusions

A differential incusion is a relation of the kind

$$\frac{d}{dt}x(t) \in F(t, x(t)),$$

that we call an ordinary differential inclusion, or

$$\nabla u(x) \in F(x, u(x)),$$

that we call a gradient inclusion. In the first case, one seeks a function $x(\cdot)$, in general in the class of absolutely continuous functions, such that, for almost every t in some interval, the derivative $\frac{d}{dt}x(t)$ exists and is contained in the set $F(t, x(t))$. In addition, initial or boundary conditions may be prescribed. In the second case, an open region Ω is given and one seeks a function u in some Sobolev space $W^{1,p}(\Omega)$ whose Sobolev gradient, for almost every $x \in \Omega$, is contained in $F(x, u(x))$. In addition, auxiliary conditions, in general in the form of Dirichlet boundary conditions, are prescribed.

To this author's knowledge, this kind of problems appears in papers by Zaremba (1936) and Marchaud (1938), in a form different from the one presented here: at the left hand side a "multivalued derivative" appears [26]. It was Wazewski (1961) [23] who showed that these inclusions could be equivalently presented as inclusions with a regular single valued derivative at the left. The reason that led to their introduction into the mathematical world is not clear; it seems that there was no real motivation from the needs of some model; rather, it was their interest as mathematical objects that arose the interest of a few mathematicians. At the beginning, it was the problem of the existence of a solution to the Cauchy Problem that was considered: show the existence of a function satisfying the inclusion over some interval (a, b) and such that $x(t^0) = x^0$, with t^0 given in (a, b).

What happens is that, for the case of an inclusion, showing the existence of a solution is (considerably) more difficult than showing the existence of a solution for the case of an ordinary differential equation

$$\frac{d}{dt}x(t) = f(t, x(t)).$$

This fact is contrary to intuition: in fact, shouldn't it be that, proving that the point x' equals the point f, is *more difficult* than merely proving that it is in a set F? Let us look more closely, then, to what is actually happening in the proof.

Let x_n be a sequence of approximate solutions, such that

$$\frac{d}{dt}x_n(t) - f(t, x_n(t)) = \varepsilon_n,$$

and let us assume that (x_n) converges to a limit x_*.

Since f is continuous in the variable x, we have that $f(\cdot, x_n(\cdot))$ converges as well to $f(\cdot, x_*(\cdot))$, so that $\frac{d}{dt}x_n(t) = f(t, x_n(t)) + \varepsilon_n \to f(t, x(t))$. Hence, from the fact that the approximate solutions x_n converge, one obtains that their derivatives converge and, passing to the limit, that $\frac{d}{dt}x(t) = f(t, x(t))$. In the case of inclusions, knowing that x_n converge, even when the set-valued map F is very regular (for instance, constant!) there is no reason to think that their derivatives should converge (consider the case $F \equiv R^N$.) This lack of convergence of the derivatives is the main source of difficulties for the case of inclusions.

At this point, the reader who approches differential inclusions for the first time may very well ask: shouldn't one simply assume (or prove) the existence of a single valued continuous function $f(t, x)$ "contained" in the set $F(t, x)$, so that a solution to the differential equation

$$\frac{d}{dt}x(t) = f(t, x(t))$$

would authomatically be a solution to the inclusion

$$\frac{d}{dt}x(t) \in F(t, x(t))?$$

(such a function f, when it exists, with the regularity prescribed, is called a "selection" from the set-valued map F). The fact is that the class of mappings F such that a continuous selection f exists is very limited and is seldom considered in papers on differential inclusions, for two reasons: the first is that for this class, there are no mathematical difficulties, hence no fun. The second is that, when differential inclusions appear in some model, they never satisfy the (stringent) requirements for the existence of continuous selections. It is again not very intuitive the fact that a set valued map can be rather regular *without* admitting continuous selections.

The theorems one finds in the literature about differential inclusions are rather technical, with assumptions that do not appear very understandable to those new to the field. The reason is that, while to describe the behaviour of a point valued function is easy (a point can only displace itself), a set, besides displacing can get larger or smaller (possibly in a discontinuous way); a set can be closed or not, can be convex or not. All these different facts are relevant to the problem of the existence of solutions and to their properties. On the other hand, if we were to consider only very regular inclusions, cutting the links with *discontinuous* phenomena, differential inclusions would lose much of their interest.

The class of differential inclusion that were first considered consists in those whose right hand side is upper semicontinuous (in the sense of inclusion) taking values that are (non empty) compact convex sets; the properties of solutions to these inclusions are very similar to the properties of solutions to ordinary differential equations. Many different tools can be used to prove existence of solutions (for instance, a fixed point argument; a result on the approximation of these set valued maps by means of continuous single valued maps.) The standard proof is essentially the same proof as that of Tonelli's weak lower semicontinuity theorem, in the Calculus of Variations, where one wants to minimize

$$\int_a^b L(t, x(t), x'(t))\, dt.$$

One considers $(x_n)_n$, a minimizing sequence, in the case of the minimum problem, or a sequence of approximate solutions, for the case of a differential inclusion. From some *a priori* estimates for the solutions to the differential inclusion, or from the coercivity assumptions for the functional one is trying to minimize, one infers that the derivatives x'_n are weakly pre-compact in L^1 (or in L^p); an application of Mazur's Lemma yields the strong convergence of a sequence of convex combinations of x'_n. At this point, exploiting either the convexity of the values of F (for the case of the differential inclusion), or the convexity of L with respect to the variable x', (for the variational prolem), one obtains the result.

One might notice that the essential identity of the two proofs, the one of weak lower semicontinuity in the calculus of Variations and the other of the existence of solutions for upper semicontinuous convex-valued differential inclusions, has been recognized rather recently; still, Tonelli's work dates from about 1920 and the proof for differential inclusions from about 1960: mathematics proceeds in strange ways.

From the sketch of the Proof above, one can see the importance of the assumption that images be convex: in fact, the convergence of the sequence of derivatives one can hope to obtain is the *weak* convergence. In order that a set be closed for the weak convergence it is needed that the set be convex. However, at this point, it might be worth remarking that the assumption of convexity is essential for this *method of proof* of the existence of solutions, *not* for the existence of solutions: a different method might yield the proof of the existence of solutions under different assumptions. In fact, mathematicians interested in this class of problems, were greatly puzzled by the problem of proving existence of solutions without assuming convexity of the images.

In 1971 an article by A.F. Filippov [15] appeared, presenting the proof of existence of solutions to the Cauchy problem for a differential inclusion whose right hand side was continuous (in the Hausdorff metric) with compact (but not necessarily convex) values. Thirty years have passed since then, and the author still remembers the emotion he proved in reading (and - slowly !- understanding) that proof. Through a very clever and simple construction, Filippov was able to build a sequence of approximate solutions whose derivatives were piecewise constant maps. To the opposite of what happens in the single-valued case, however, in the case of a differential inclusion it was *not* possible to arbitrarily decrease the jumps of the derivatives at the the points of the partition of the time interval, by decreasing the mesh of the partition. To overcome this difficulty, Filippov was able to build a sequence of approximate solutions whose derivatives yield a (strongly) (pre) compact sequence in L^∞! This author knows of no other instance of a such a construction; compactness in L^p is relatively easy to obtain, since it is possible to approximate in L^p integrable functions by smooth (continuous) maps, and apply the Ascoli-Arzelà construction to the approximations; however, this process cannot be applied in L^∞!

Alternative methods of proof were later provided to prove results in the spirit of this theorem of Filippov. To simplify the construction, L^1 was used, as in the selection approach of Antosiewicz-Cellina (1975)[2]; here the continuous selections were not taken for the set valued map F itself (since, in general, continuous selections do not exist), but, rather, for the map $y(\cdot) \to \int_{t0}^{t} F(s, y(s))\, ds$, seen as a map from the space of absolutely continuous maps (with the norm the L^1 norm of the derivative) to (the subsets of) itself, the, so called, Nemytskij operator. Another interesting method, based on Baire's theorem, was introduced by De Blasi and Pianigiani [13]. This method can be applied only to those problems where, in case there is a solution, there are many, consists in using Baire's theorem as a non empty intersection theorem, and in applying it to suitably defined approximate solutions. This method has led to results that seem strange at first sight. Before mentioning one such result, let us notice that in general, in an infinite dimensional space, a Cauchy Problem for a differential equation with a continuous right hand side need not have a solution: in fact, there is a lack of compactness due to the infinite dimensionality. By Baire's theorem, in an infinite dimensional Hilbert space H, De Blasi and Pianigiani build a surprising example of a differential inclusion, that in some sense is the union of countably many ordinary equations, more precisely they build a continuous closed-valued map,F, from $I \times H$ to H, and a sequence of continuous single valued functions f_n that are all the possible continuous selections of F, and such that the Cauchy Problem for F,

$$x'(t) \in F(t,x), \quad x(0) = 0$$

admits solutions, while none of the problems

$$x'(t) = f_n(t,x), \quad x(0) = 0$$

admits any solution.

Another interesting approach that has been devised to overcome the difficulties posed by the lack of convexity of the values of the set valued map F, is the method of directionally continuos selections; the starting point was an observation in a little known paper by A. Pucci (1971)[21], itself an extension of earlier work by Cambini and Querci (1969) [8], that, in order to obtain existence theorems for an ordinary differential equation it was not really necessary to have a right hand side continuous with respect to the space variable, but that a weaker form of continuity, essentially only in the direction of the flow, was sufficient for this purpose. A. Bressan (1988) [6] was able to show that set valued maps F, even without the assumption of convexity, hence, in general, not admitting continuous selections, under reasonable assumptions, do instead admit selections that are only directionally continuous, hence, by the previuos results, enough regular so that the corresponding Cauchy Problems do admit solutions.

The difficulties posed by the lack of convexity arise not only in order to prove existence of solutions, but also to study their properties. Assume we have proved that the Cauchy problems (under some conditions) do admit solutions, and consider the map that to each initial point at time zero associates the set of solutions through this point. This is a new set valued map, certainly non convex-valued (since it is not convex valued even for ordinary differential equations, whenever the problem lacks uniqueness). What are the properties of this map? How does one define a "well posed problem"? In the single valued case, a problem is called well posed if the corresponding map "initial point to the solution through it" is (single valued and) continuous. And in the set valued case? Can one prove the existence of continuous selections for this map even without convexity of the images?

In a review to a book on Differential Inclusions, (that appeared on Bull. A.M.S. 1995) the present author expressed the opinion that "differential inclusions have been a formidable gymnasium for the creation of ideas" to treat non convex problems. However, the positive comment on the "formidable gymnasium" actually concealed a darker side of the author's opinion on differential inclusions: that is, that this instrument, mathematically challenging and stimulating as it had been, was essentially of no use in describing the real word.

Since then, this author's opinion has changed a bit.

Consider the following example. Circa 1660, Fermat stated the well known principle that light, to travel from a point to a second point in space, through a medium where the speed of the light is (possibly) variable, among all the possible paths that join the two points, follows the path that minimizes the time. Fermat's principle predates of thirty years the celebrated Brachystocrone Problem of Johan Bernouilli, and, remarkably, its aim is to explain a physical phenomenon that is *discontinuous* : Fermat's aim was to characterize the path followed by light in passing though two media with different speed coefficients, as water and air, and in particular to explain why a stick, partially sunk under water, seems to be broken to an observer.

We can model the problem in the following way: the speed of light at the point x, $\rho(x)$, is given, and so are the two points P_1 e P_2. Among all solutions to he differential inclusion

$$x'(t) \in \rho(x(t))\, \partial B \quad \text{ or, equivalentlty, } \quad \|x'(t)\| = \rho(x(t))$$

joining P_1 and P_2, prove that there is a solution that travels from P_1 to P_2 in minimum time. It is clear that we would like to prove this theorem under conditions on the function ρ that allow it to be *discontinuous* , otherwise we do not prove what Fermat had in mind to prove. Today we can prove that if ρ is upper semicontinuous and positive, Fermat's minimum time problem admits a solution (Cellina, Ferriero e Marchini, 2005). Why did it take so long to prove such a simple and natural result? The mathematics involved in it are rather elementary, and the tools needed (some measure theory) have been known for about a century. The fact is that the problem presents itself as a non standard problem: the set valued map $x \to \rho(x)\, \partial B$ not only is not convex valued,but is not upper semicontinuous (in the sense of inclusion), nor is lower semicontinuous but enjoys the weaker property that the map $x \to co(\rho(x)\, \partial B)$ is upper semicontinuous, so we cannot reason in the standard, traditional way of passing to weak limits and so on. What does the trick is the fact that the map itself is of very special form.

A remark on the mathematical assumption that ρ be upper semicontinuous. Assume that we have two media, A and B: medium A in the region $y > 0$ and B in the region $y < 0$, with speed $\rho_A = 1$ and $\rho_B = 2$. Assigning the line $y = 0$ to either one of the two media is probably a physically meaningless operation. Mathematically, assigning it to A, makes the speed ρ lower semicontinuous, while, assigning it to B, makes ρ upper semicontinuous. Our result assures the existence of a minimum time solution , no matter what P_1 and P_2 are, when we make the second choice. To the opposite, one can show that the first choice would make the result false.

An analogous result holds for the (similar) problem of the brachystocrone; again the model present a lack of convexity, so that it is not lower semicontinuous, to use the standard language of the Calculus of Variations. Still, the existence of a solution for the problem follows along the same lines as for Fermat's Principle.

A comment. In modelling the diffusion of light it would have been natural to use differential inclusions; actually, one could even say that it would have been natural to introduce this instrument specially for this purpose. A little thinking on the problem would have suggested rather easily a proof of existence of solutions to the minimum time problem. Instead, the Brachystocrone problem, that could be modeled as a minimum time problem for a simple differential inclusion (here the gravitational field is constant), is presented, often in the introduction of books on the Calculus of Variations, in a very innatural way, as a problem of minimizing an integral functional, of the kind that, in later chapters, will be faced with the tools of the Direct Method of the Calculus of Variations; it is only too bad that this formulation of the Brachystocrone Problem does not satisfy the requirements of the Direct Method, since the corresponding integral functional is not coercive. A simpler model, based on a differential inclusion, is disregarded. It was H.J. Sussmann e J.C. Willems (2000) that noticed this fact: problems that could have been easily formulated and possibly solved have been either disregarded or made difficult, only to present them in a way that was considered "more standard".

The author cannot see a reason to explain why the mathematical community has been so much negletting the instrument of differential inclusions.

2. Gradient inclusions.

In the case of gradient inclusions, η is a scalar, x a vector. Given an open set Ω and a function "boundary datum" η_0, we look for solutions to the inclusion

$$\nabla\eta(x) \in K(x), \text{for a.e. } x \in \Omega$$

such that $\eta|_{\partial\Omega} = \eta_0|_{\partial\Omega}$. The choice of the letter η to denote a solution, is made on purpose, to point out an important interpretation of this problem. In the case the boundary condition is $\eta|_{\partial\Omega} = 0$, the map η is usually called a "variation", and in the Calculus of Variations it is costumary to use η. To gain insight in the problem, assume that we are given an open region Ω in R^2 and the boundary condition $\eta|_{\partial\Omega} = 0$. Assume that the set K is constant, and consists in the three vertices of an equilateral triangle. Can we solve the problem? Can we find a lipschitzian function η that is zero at the boundary of Ω (no restrictions on Ω, it can be any (bounded) open set) whose gradient (that exists almost everywhere) takes only the three prescribed values (and, obviously, not zero)? The difficulty, again, comes from the fact that the set K is not convex: otherwise it would contain zero, and the solution would be trivial. The answer to this problem comes from a more general result of a few years ago, (Cellina,1992) [9]. The problem

$$\nabla\eta(x) \in K, \quad \eta|_{\partial\Omega} = \langle a, x\rangle$$

has a solution if and only if

$$a \in \text{int}(co(K)).$$

The solutions obtained are such that their level sets are union of sets that are polar to the set K (polar sets are always convex, since they are intersections of halfspaces, even when the sets are not). Itself, the above result is a special case of a more complex problem, that of deciding whether a minimum problem of the form

$$\text{minimize} \qquad \int_\Omega f(\nabla u(x))\, dx$$

under affine boundary conditions, admits solutions; again, f need not be convex. To give an idea of the construction, begin by considering the problem of buiding a function that is zero at the boundary of *some* region, whose gradient takes the prescribed values, without considering at all the given region Ω. This problem can be solved by considering a "pyramid" whose base (where the function is zero) is the polar to the given sets of gradients, and the gradients of the pyramid, the slopes of the sides, are the points of K. This construction can be made with the base as small as we please (and proportionally reducing the height, so as not to change the gradients) and "centered" at any point. So, by a covering argument, we can cover Ω with countably many such sets "base of the pyramids" and take care that the size of the basis, hence the heights, tend to zero when the center of the pyramid approaches the boundary of Ω. In this way we obtain a function that is Lipschitzian, has gradients a.e. in the set K, and satisfies the condition of being zero at the boundary of Ω.

The above results make evident the difference in theorems where convexity is assumed from those where it is not. In case convexity is assumed, we can use the powerful tool of weak convergence; hence the effort in these theorems is to show (and this can be very difficult) that there are weakly converging subsequences, from the minimizing sequence or from a sequence of approximate solutions. In the case convexity is not assumed, instead, the emphasis is on a construction for the solution.

Exploiting the technique of Baire's Theorem (since, even for this problem, if a solution exists, then there are many solutions), several results have been obtained for the case where the right hand side is not constant (always under the assumption that $0 \in int(co(K(x)))$): the first paper was by Bressan e Flores Bazan [7]; many interesting results followed, by Marcellini and Dacorogna, that wrote an important book on the subject [12]; by Zagatti [25][24]; De Blasi and Pianigiani [14].

The author likes to think that these theorems are a result of the "formidable gymnasium for the creation of ideas". It is true that the methods one can use when the independent variable is a space vector are very different from those one employs when the variable is the time t: essentially, this difference boils down to the fact that in R^N there is no indefinite integral. in fact, when the variable is t, one defines a function by defining its derivative first, and then the function itself by integration. While doing so, one can use all those "tricks" connected to this integration. A trick, or a theorem, that has been widely used to avoid the assumption of convexity, has been Liapunov's Theorem, on the convexity of images of non-atomic vector valued measures. Roughly speaking, this Theorem asserts that integration by itself is a convexification, so that, after integration, some convexity always exists. In some cases, this convexity might be sufficient for our purposes. To the opposite, when the independent variable is the vector x, this process cannot be followed: one cannot define pointwise a vector function, and hope that it be the gradient of some other function. Still, although the methods in the two cases are different, the underlying ideas remain the same.

Let us go back to the idea of a variation. Assume we are minimizing a functional as

$$\int_\Omega [L(\nabla u(x)) + g(u(x))]\, dx$$

under some boundary conditions. In order to find necessary conditions, satisfied by a solutions, one has to explore a a neighborhoud of the solution u, i.e., one has to consider functions $u + \eta$, whose gradient is $\nabla u + \nabla \eta$. But is it obvious that non-trivial variations η do always exist? The convex function L could have an effective domain (the set where it takes finite values) smaller than R^N. As an example, let us consider

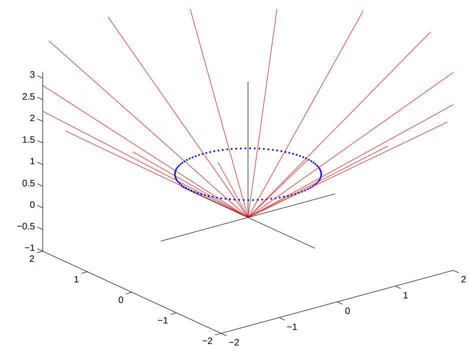
$$L(\xi) = \begin{cases} 1 - \sqrt{1 - \|\xi\|^2} & \text{, when } \|\xi\| \leq 1 \\ +\infty & \text{, elsewhere} \end{cases}$$

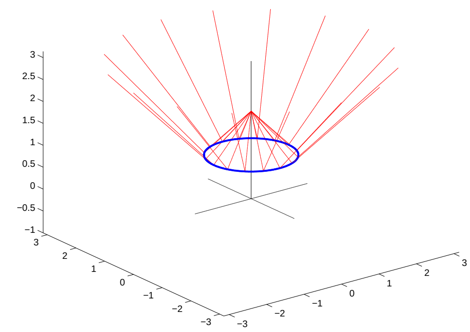
For this integrand L, the value of the integral functional is finite if and only if the gradient of u takes values in the closed unit ball of R^N. Assume we have a solution

u (a minimum for this functional, under suitable boundary conditions, exists, since the functional is convex and coercive), and, to find necessary conditions, we want to compare the values of the functional at u and at $u+\eta$. However it might happen that for every non trivial (not identically zero) η the functional assumes value $+\infty$. This fact wil happen only under very special conditions, when the boundary data force the gradient of the solution u to take values on the boundary of B. The situation is not clear, since one can have non trivial variations η even in some cases where $\|\nabla u(x)\| = 1$, η. The problem reduces to ask whether, given u, there are solutions η to the problem

$$\nabla \eta(x) \in -\nabla u(x) + B, \quad \eta|_{\partial\Omega} = 0$$

For this problem, it may very well happen that $0 \in \partial(-\nabla u(x) + B)$! In fact, consider the function $u(x, y) = \sqrt{x^2 + y^2}$. We have that the norm of its gradient is one (with the exception of the origin). For this function, the following is true: when Ω is the disk $x^2 + y^2 < R^2$, there exist non trivial variations η ; when Ω is the annulus $r^2 < x^2 + y^2 < R^2$, there are non non trivial variations! hence, the existence or non-existence of variations depends on the geometrical properties of Ω and cannot be described by *local* conditions.





Bertone and Cellina (2005), have recently proposed a conjecture, yielding a necessary and sufficient condition for the existence of non-trivial solutions. As is custumary, by saying that a vector function $p \in L^1_{loc}(\Omega)$ is such that div $(p) = 0$ we mean that, for every $\eta \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \langle p(x), \nabla\eta(x)\rangle dx = 0.$$

Conjecture. *Let* $\mathbf{D} \subset \mathbb{R}^N$. *Let* u *be a solution to*

$$\nabla u(x) \in co(\mathbf{D}).$$

Then, the following a) and b) are in alternative:

a) there exists a nontrivial $\eta \in W_0^{1,\infty}(\Omega)$, *solution to*

$$\nabla\eta(x) \in -\nabla u(x) + \mathbf{D} \tag{1}$$

b) there exists a vector function $p \in L^1_{loc}(\Omega)$, $p(x) \neq 0$ *a.e., such that* $\operatorname{div}(p) = 0$, *and*

$$\langle p(x), \nabla u(x)\rangle = \sup_{k\in\mathbf{D}} \langle p(x), k\rangle \tag{2}$$

for almost every $x \in \Omega$.

The above is a conjecture in the sense that it is not proved in this full generality (although the authors believe that it is true): the above conjecture is proved under some additional regularity assumption. In particular, the results apply to the function $u(x, y) = \sqrt{x^2 + y^2}$ on Ω.

3. The case where u is a vector.

When u is a vector, hence the derivatives form a matrix Du, one is interested in minimizing an integral functional of the form

$$\int_\Omega W(Du(x))\, dx$$

The theory is essentially the same as for the scalar case, with the difference that, in order to obtain the weak lower semicontinuity of the functional, the assumption of convexity is replaced by a more abstract condition, called quasi-convexity. Since the years ’80, several articles (see for instance: Allaire [1]; Avellaneda [3]; Ball [4]; James [5] [17]; Kinderleherer [18];Kohn [19] [20]; Sverak), considered the case where the energy function W has the form of a ”double well”. The reason for the interest in the problem came from applications, to explain the fact that in some material two different crystal structures are present either together or in ways connected to the history of the phase transformations.

The ”two well problem”, as proposed by Sverak in 1993 [22], was: given an open region $\Omega \subset R^2$, two square matrices A and B, such that $\det A$ and $\det B$ are positive, find a solution to the differential inclusion

$$Du(x) \in SO(2)A \cap SO(2)B$$

satisfying $u|_{\partial\Omega} = 0$. Essentially the idea is that the energy function W equals zero only on the set $SO(2)A\cap SO(2)B$ and it is positive elsewhere: assuming that where it is positive actually takes value $= \infty$, we have the identity of the two problems. The energy function is not convex (more precisely, it is not quasi-convex) and one cannot apply the direct method to prove the existence of solutions to the minimum problem. Actually in these papers, the problem of the existence of a ”true” solution was not even considered. The results seeked were on a relaxed model for the problem, and consideration was on the infimum of the problem, independently on whether this infimum was actually attained or not. Rather, at that time, mathematicians were essentially interested in the properties of minimizing sequences. About solutions to (non convex, non quasi-convex) differential inclusions, the following result, obtained Reshetniak in 1967 (see the citation in [16]), was known: the boundary value problem for the gradient inclusion

$$Du(x) \in SO(3)I, \quad u|_{\partial\Omega} = 0$$

has no solution; in other words, it is impossible to find a vector function u, $u = 0$ at the boundary of Ω, whose gradient takes values in $SO(3)$; notice that in this case, $0 \in int(coSO(3)I)$, hence the result obtained for gradient inclusions, when u is a scalar, is not true in the vector case.It was natural to try with *two* rotations. Somewhat surprisingly, Cellina Perrotta (1995) [10] prove that the two well problem

$$Du(x) \in SO(3)I \cap SO(3)I^{-}, \quad u|_{\partial\Omega} = 0$$

where I^{-} is the matrix expressing the other orientation of R^3, admits a solution. The construction of the solution is (to this author's opinion) rather interesting, since the basic construction of "pyramids", that will lead to a solution of the problem on the assigned region Ω through a covering theorem, is itself a result of an infinite process, trough an induction argument.

This result was *not* the solution to the microstructure problem since one of the two matrices has a negative determinant; however, for the first time, it was shown that such a problem could indeed have a true solution.Il problema delle microstrutture fu in seguito risolto da Muller e Sverak (1998). A class of problems of optimal design leads to differential inclusions in the vector valued case: the different components of the vector are the different criteria that ought to be optimized at once. For an interesting problem raised by Kohn and Strang (non quasi-convex), Marcellini and Dacorogna [11] provided a constructive proof of the existence of a solution. The idea that has been behind all the work done on problems that lack convexity, i.e. that of proving the existence of solutions not by passing to the limit, but by an actual construction, has been used more and more.

4. Conclusions

The purpose of this talk has been to express some thoughts on how mathematics interacts with the real world, particularly on how physical phenomena are modeled, considering the (unpretentious) example of the development of the theory of differential inclusions. Two conclusions can be drawn, both rather obvious.

The first is that this interaction can be rather oblique and crooked.

Differential inclusions, born essentially as a mathematical curiosity, have been developed within the mathematical community, that was attracted by their technical difficulty and pushed to develop new ideas to overcome these difficulties (mainly the lack of convexity). On the other hand the subject in itself has had few points of contact with either mathematical modelling or different areas of mathematical research. In turn, the new ideas and methods that have been thus obtained did show that problems that seemed to be out of reach, were actually solvable. By this increased confidence in the possibilities of the tool, today differential inclusions are used from the beginning to mathematical modelling.

The second is the obvious remark that everybody emploies, to describe the world, the language and the tools he already owns in his intellectual baggage. Had Fermat seen a differential inclusion before considering geometrical optics, possibly the development of mathematical analysis would have been different.

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