SOLUTIONS TO VARIATIONAL PROBLEMS AND THE STRONG MAXIMUM PRINCIPLE

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1. Introduction

The purpose of this Note is to present a few known results, remarks and open problems concerning the Strong Maximum Principle. In its classical form, this principle states that a solution to

$$\Delta u(x) = 0$$

on an open connected region Ω, that is such that $u(x) \geq 0$ on Ω, if it touches O at a point $x^* \in \Omega$, has to be identically zero. This principle, strictly connected with Hopf's boundary point Lemma, has been generalized to more general equations and is considered to be a basic tool in several situations. It is, for instance, the basis for the Moving Plane Method (see, for instance, [1].)

We wish to present results and open problems on this Principle, in more general situations.

The equation

$$\Delta u(x) = 0$$

is the Euler Lagrange equation for the functional

$$\int_\Omega \frac{1}{2}\|\nabla u(x)\|^2\, dx.$$

Being the Lagrangean convex, u is not only a solution to the Euler Lagrange equation, but actually a solution to the problem of minimizing the integral functional among all u satisfying given boundary conditions. It is this point of view that we wish to follow here.

2. The minimization of $\int_\Omega f(\|\nabla u(x)\|)\, dx$

In the present section we consider properties of solutions to Problem (P)

$$(P) \qquad \text{minimize} \int_\Omega f(\|\nabla u(x)\|)\, dx \text{ on } u^0 + W_0^{1,1}(\Omega).$$

under very general conditions on f; in fact, $f : \Re \to \Re^+$ is any (possibly extended valued) lower semicontinuous, convex function, such that $f(0) = 0$. The motivation for investigating this problem was the desire of understanding what makes the Strong Maximum Principle true; for instance, when f is convex, but there is no interval, containing the origin, such that f is differentiable there (it is very easy to build such an example), is it still true? The following Theorem , as presented in [4], provides a necessary and sufficient condition on the convex function f in order

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that continuous solutions to (P) satisfy a Strong Maximum Principle on any open connected Ω.

We shall say that the integrand f has the *Strong Maximum Principle Property* if for any open connected Ω, for every *continuous* non-negative solution to Problem (P), $u(x^0) = 0$ for some $x^0 \in \Omega$ implies $u(x) \equiv 0$ on Ω. The following is the result.

Theorem 1. *Let $f : \Re \to \Re^+$ be a (possibly extended valued) lower semicontinuous, convex function such that $f(0) = 0$. Then: f has the Strong Maximum Principle Property if and only if both conditions: i) $\partial f(0) = \{0\}$ and ii) $(\partial f)^{-1}(0) = \{0\}$ hold true.*

The conditions of the previous Theorem can be restated as follows.

Proposition 1. *Let f be as in Theorem 2; then conditions i) and ii) are equivalent to*

j) $\lim_{t\to 0^+} f(t)/t = 0$ and jj) $f(t) > 0$ for $t > 0$

Proposition 2. *A further equivalent condition is k) $\partial f(0) = \{0\}$ and kk) $\partial f^*(0) = \{0\}$*

This last condition is, to this author's opinion, particularly elegant: it expresses the fact that

When a convex function f satisfies this criterion, so does its polar f^*.

Example 1. Consider

$$f(\xi) = \begin{cases} e^{-\frac{1}{\xi^2}} & \xi \neq 0 \\ \\ 0 & \xi = 0 \end{cases}$$

Then f is in C^∞ and f and all of its derivatives are 0 at 0. Still, $f(\xi) > 0$ for $\xi > 0$ and conditions $j)$ and $jj)$ are satisfied. Hence conditions $k)$ and $kk)$ are satisfied. Hence the Strong Maximum Principle Property holds for f. Then the Strong Maximum Principle Property holds for f^* (whose explicit expression is not easy to compute)

Example 2. Consider

$$f(\xi) = \begin{cases} 0 & \|\xi\| \leq 1 \\ \\ = \infty & \text{elsewhere} \end{cases}$$

whose polar is

$$f^*(p) = \|p\|.$$

Then, neither f nor f^* satisfy the criterion.

Moreover, one should notice that, for the validity Strong Maximum Principle, f need not be differentiable other than at 0.

The situation we have is as follows: there are two solution to the minimum problem, u and 0; one of the two lies on the same side of the other: then the two solutions cannot touch, otherwise they coincide. As such it is a strong uniqueness result among solutions.

Should we expect this uniqueness result to hold *no matter* what are the solutions? For a *linear* problem, i.e. when the Euler Lagrange equation is linear, the answer is yes, but, in general, what can be said? The function $u \equiv 0$ is a very special solution to a any minimization problem for a Lagrangean f with the properties required in this section. However, a larger class of solutions should possibly be considered.

3. A class of solutions for the minimization of $\int_\Omega f(\nabla u(x))\,dx$

In this section f is any extended valued, lower semicontinuous, convex function. We wish to present here a result that, although implicitely presented in [3], is not clearly stated in this form. By a *solution* u^* to the problem of minimizing $\int_\Omega f(\nabla u(x)\,dx$ we mean a function such that the value of the integral is *minimum* among all functions in the same space, having the same boundary data as u^*: in general, among all functions u such that $u - u^*$ belongs to $W_0^{1,1}(\Omega)$. By *effective domain* $(Dom(f))$ of a convex extended valued function $f : R^N \to R$, we mean the set of points where f is finite.

Theorem 2. *Let Ω be an open, bounded set, and $f : \mathbb{R}^N \to \mathbb{R}$ an extended valued, lower semicontinuous, convex function. Then, for every $a \in Dom(f)$ and every $r \in \mathbb{R}$, the affine map*

$$\ell(x) = \langle a, x \rangle + r$$

is a solution to the problem of minimizing the functional

$$\mathcal{J}(u) = \int_\Omega f\left(\nabla u(x)\right) dx,$$

in the class of functions assuming the same boundary values

$$\mathcal{S} = \left\{ u \in W^{1,1}\left(\Omega\right), u(x) - \ell(x) \in W_0^{1,1}\left(\Omega\right) \right\}.$$

Proof. a) Assume first that $a \in Dom(\partial f)$. Then, for every $p \in \partial f(a)$ and every ξ,

$$f(\xi) \geq \langle \xi - a, p \rangle + f(a),$$

so that, for every $u \in \mathcal{S}$,

$$\int_\Omega f\left(\nabla u(x)\right) dx \geq \int_\Omega \langle \nabla u(x) - a, p \rangle\, dx + \int_\Omega f(a) dx.$$

Since, by the divergence Theorem,

$$\int_\Omega \langle \nabla u(x) - a, p \rangle\, dx = 0,$$

we infer that

$$\int_\Omega f\left(\nabla u(x)\right) dx \geq \int_\Omega f(a) dx = \int_\Omega f\left(\nabla\left(\langle a, x \rangle + r\right)\right) dx,$$

thus proving that $\ell(x)$ is a solution.

b) Suppose that $a \notin Dom(\partial f)$. In this case a belongs to the boundary of the convex set $Dom(f)$, and there exists an hyperplane separating a and $Dom(f)$, i.e. a non-zero vector v such that, for every $\xi \in Dom(f)$,

$$\langle v, \xi \rangle \leq \langle v, a \rangle.$$

From Lemma of [3], we infer that, for every function $u \in \mathcal{S}$, we have that

$$\langle v, \nabla u(x) \rangle = \langle v, a \rangle,$$

i.e. $\nabla u(x)$ belongs, for almost every x, to the $N-1$ dimensional plane

$$\langle v, \xi \rangle = \langle v, a \rangle.$$

Equivalently, we can compute the value of $\mathcal{J}$ on u as

$$\int_\Omega f\left(\nabla u(x)\right)dx = \int_\Omega f_1\left(\nabla u(x)\right)dx,$$

where f_1 is the extended valued, lower semicontinuous, convex function defined as

$$f_1(\xi) = \begin{cases} f(\xi) & \text{on the plane } \langle v, \xi\rangle = \langle v, a\rangle \\ \\ +\infty & \text{elsewhere.}\end{cases}$$

The domain of this new lagrangean f_1 is contained in an affine space having at most dimension $N-1$.

c) We apply the reasoning above to the function f_1; either the subdifferential of f_1 is defined at a, and we obtain that $\ell(x) = \langle a, x\rangle + b$ is a solution, or it is not, so that there is a new convex function f_2, whose domain is contained in a $N-2$ dimensional affine space, such that $f\left(\nabla u(x)\right) = f_1\left(\nabla u(x)\right) = f_2\left(\nabla u(x)\right)$ for almost every x.

d) In N steps we either reach the conclusion of point a), that $\ell(x)$ is a solution, or there exists a vector b such that $\nabla u(x) = b$, almost everywhere in Ω. The boundary condition can be satisfied only if $a = b$, hence the only function u satisfying the boundary conditions is $\ell(x) = \langle a, x\rangle + r$, that therefore is a solution.

□

For instance, for

$$f(\xi) = \begin{cases} 1 - \sqrt{1 - \|\xi\|^2} & \|\xi\| \le 1 \\ +\infty & \text{elsewhere;}\end{cases}$$

for every vector a of norm 1, $\ell(x) = t\langle a, x\rangle$ is a solution, even though $\partial f(a)$ is empty.

4. Comments

Are results similar to those of Theorem valid when, instead of comparing u to the special solution 0, we compare it with any affine function $\ell(x) = \langle a, x\rangle + r$, with $a \in Dom(f)$, as presented in Section 3?

5. The Strong Maximum Principle for functionals that are not rotationally symmetric

In this section we consider the functional

$$J(u) = \int_\Omega f(\nabla u(x))dx = \int_\Omega \frac{1}{2}\left(\sum_{i=1}^N f_i(u_{x_i}^2)u_{x_i}^2\right)dx,$$

whose Euler-Lagrange equation will be written

$$\sum_{i=1}^N g_i(u_{x_i}^2)u_{x_i x_i} = 0. \tag{1}$$

We assume that f is strictly convex, that $f(0) = 0$ and that $g_i : [0, +\infty) \to [0, +\infty)$ are continuous functions. In particular, 0 is still a solution (and so are the affine functions of section 3).

In the case in the above equation we have $g_i \equiv 1$, for every i, then equation 1 is $\Delta u = 0$, and we find the classical problem of Hopf. On the other hand, when there exists $i \in \{1, \ldots, N\}$ such that $g_i \equiv 0$ on an interval $I = [0, T] \subset \mathbb{R}$, the Strong Maximum Principle does not hold. Indeed, in this case, it is always possible to define a function u assuming minimum in Ω and such that $\sum_{i=1}^{N} g_i(u_{x_i}^2) u_{x_i x_i} = 0$. For instance, let $g_N(t) = 0$, for every $t \in [0, 2]$. The function

$$u(x_1, \ldots, x_N) = \begin{cases} -(x_N^2 - 1)^4 & \text{if } -1 \leq x_N \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

satisfies (1) in $\mathbb{R}^N$. The convexity of the Lagrangean is enough for proving that 0 and the affine functions are solutions, but not for the validity of the Strong Maximum Principle. One would like to understand what are the additional properties needed for its validity.

We are interested in the case when $0 \leq g_i(t) \leq 1$ and it does not exist i such that $g_i \equiv 0$ on an interval. Since g_i could assume value zero, the equation (1) is degenerate elliptic.

A sufficient condition for the validity of the Strong Maximum principle is provided by the following Theorem. In it, some technical conditions are needed (we refer to the original paper [2]), but the main assumption concerns the behaviour of the improper Riemann integral

$$\int_0^{\xi} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = \lim_{\hat{\xi} \to 0} \int_{\hat{\xi}}^{\xi} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta$$

seen as an extended valued function G,

$$G(\xi) = \int_0^{\xi} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta,$$

where we mean that $G(\xi) \equiv +\infty$ whenever the integral diverges.

Theorem 3 (Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set. Let $u \in W^{1,2}(\Omega) \cap C\left(\overline{\Omega}\right)$ be a weak supersolution to*

$$\sum_{i=1}^{N} g_i(u_{x_i}^2) u_{x_i x_i} = 0.$$

In addition to some regularity and growth assumptions on g_i,as presented in [2]*, assume that $G(\xi) \equiv +\infty$. Then, if u attains its minimum in Ω, it is a constant.*

Unfortunately, this result and similar (rather precise) results presented in [2] do not seem to have a simple geometrical interpretation and do not shed light on what is really essential for the validity of the Strong Maximum Principle.

6. The minimization of $\int_\Omega (f(\|\nabla u(x)\|) + g(u(x)))\, dx$

We see the Strong Maximum Principle as a strong uniqueness result for solutions to variational problems. So far, a generic solution u and the special solution 0 have been compared. Since the Euler Lagrange equation for a solution (assuming that a solution to the minimum problem satisfies the Euler Lagrange equation, something that, under general conditions, still has to be proved!) is

$$\text{div} \nabla f(\nabla u(x)) = g'(u(x)), \tag{2}$$

if we wish 0 to be a solution, we should have $f(0) = 0$ and $g'(0) = 0$. It was Vazquez [6] the first to consider the validity of the Strong Maximum Principle for the equation

$$\Delta(u(x)) = \beta(u(x))$$

with β a nondecreasing function $R \to R$, such that $\beta(0) = 0$. (Vazquez's equation is more general, see the comments in the following section) and to provide a necessary and sufficient condition on β in order that the Strong Maximum Principle be true (since here $f(\xi) = \frac{1}{2}\|\xi\|^2$ is fixed, these conditions are on β, i.e. on g'.) The condition he found is: the Strong Maximum Principle holds if and only if the integral $\int (\beta(s)s)^{-1/2}\, ds$ diverges at $s = 0+$.

This result of Vazquez was extended to functions f more general than $\frac{1}{2}\|\xi\|^2$: in [5], Pucci, Serrin and Zou, consider the case of general elliptic equations involving nonlinear operators of the form $\mathrm{div}(A(|\nabla u|)\nabla u) = \beta(u)$, where A is a continuous function such that $t \mapsto tA(t) =: \Omega(t)$ is continuously differentiable and strictly increasing and $\Omega(t) \to 0$ as $t \to 0$. Define $H(t) = t\Omega(t) - \int_0^t \Omega(s)ds$. Setting $B(s) = \int_0^s \beta(t)dt$, the main result of the paper says that the strong maximum principle holds if $\liminf_{t\to 0} H(t)/(t\Omega(t)) > 0$ and either $\beta(s) \equiv 0$ on $[0, \tau)$, or $\int_0^\delta ds/H^{-1}(B(s)) = +\infty$.

To compare the language of this paper to the language we use here, notice that

$$\nabla_\xi f(\|\xi\|) = f'(\|\xi\|)\frac{\xi}{\|\xi\|}$$

for $\|\xi\| \neq 0$, so that

$$A(\|\xi\|) = \frac{f'(\|\xi\|)}{\|\xi\|}$$

or

$$f'(\|\xi\|) = \|\xi\|A(|\xi|)$$

and $\Omega(\|\xi\|)$ is $f'(\|\xi\|)$. Moreover, when f is differentiable and convex,

$$H(t) = t\Omega(t) - \int_0^t \Omega(s)ds = tf'(t) - f(t) = f^*(f'(t)),$$

where f^* is the *polar* of f. Hence the condition

$$\liminf_{t\to 0} \frac{H(t)}{t\Omega(t)} > 0$$

can be stated either

$$\limsup_{t\to 0} \frac{\frac{f(t)}{t}}{f'(t)} < 1 \tag{3}$$

or

$$\liminf_{t\to 0} \frac{f^*(f'(t))}{tf'(t)} > 0.$$

With the change of variables $q = f'(t)$, this last condition can be expressed as

$$\liminf_{q\to 0} \frac{\frac{f^*(q)}{q}}{(f^*)'(q)} > 0. \tag{4}$$

Hence, comparing conditions 3 and 4, one can see that this condition is *not* symmetric in f and f^*.

In the case $f(\|\xi\|) = \frac{1}{2}\|\xi\|^2$, we have that $f^* = f$, and

$$0 < \lim_{t \to 0} \frac{\frac{\frac{1}{2}t^2}{t}}{t} = \frac{1}{2} < 1$$

Hence, this condition is satisfied (obviously, in both the equivalent forms 3 and 4).

Let f be as in Example 2: we have

$$\frac{f(t)}{t} = \frac{e^{-\frac{1}{t^2}}}{t}; \qquad f'(t) = e^{-}\frac{1}{t^2}\frac{2}{t^3}$$

so that

$$\lim_{t \to 0} \frac{\frac{f(t)}{t}}{f'(t)} = \lim_{t \to 0} \frac{t^2}{2} = 0$$

and 3 is satisfied. Now, consider the polar of f, so that f is the polar to this polar: condition 4 is *not* satisfied, since

$$\lim_{q \to 0} \frac{\frac{f(q)}{q}}{f'(q)} = 0.$$

Still, as we have remarked, this function "polar of f" satisfies the condition of Theorem 1 of section 2.

7. Comments

Is there a result, for the minimizers of $\int_\Omega (f(\|\nabla u(x)\|) + g(u(x)))\, dx$, that, in the case of $g = 0$, yields back the results of Theorem 1?

8. Again on the minimization of $\int_\Omega (f(\|\nabla u(x)\|) + g(u(x)))\, dx$: solutions or supersolutions?

For the problems considered in Sections 2 and 6, the function identically 0 is a solution and so is u. In this case, it is totally immaterial whether we show that $u(x) \geq 0$ and $u(x) = 0$ at one point imply $u(x) \equiv 0$, or that $u(x) \leq 0$ and $u(x) = 0$ at one point imply $u(x) \equiv 0$, in other words whether we prove a Strong Maximum Principle or a Strong Minimum Principle.

In most classical papers, the Strong Maximum Principle is stated for solutions to

$$\Delta u(x) = f(x)$$

and, in Vazquez's paper, for solutions to

$$\Delta(u(x)) - \beta(u(x)) = f(x).$$

If we assume (as in [6]) that $f \leq 0$, a function u satisfying $\Delta u(x) = f(x)$ can be seen

1) as a supersolution to the problem

$$\Delta u(x) = 0,$$

i.e., a function that stays *above* the minimum to

$$\int_\Omega \frac{1}{2}\|\nabla u(x)\|^2\, dx;$$

in this case, 0 is a minumum, at it makes sense to compare u to 0; however, u is only a supersolution, hence it cannot be that both a Maximum Principle and a

Minimum Principle hold: a solution to $x'' = -1$ is a supersolution to $x'' = 0$ and it is not true that if u attains a maximum on Ω, it is a constant.

2) as a solution to

$$\Delta u(x) = f(x)$$

i.e., a minimum to

$$\int_\Omega \left(\frac{1}{2} \|\nabla u(x)\|^2 + f(x)u(x) \right) dx.$$

This case is different from those of section 6, since we do not assume, in the language of section 6, that $g'(0) = 0$, so that 0 is not any more a solution, at it makes little sense to compare u with 0. However, there should be *another* solution to which u could be compared: for instance, for the problem of minimizing

$$\int_\Omega (f(\|\nabla u(x)\|) + u(x))\, dx,$$

is there a solution, to be used instead of 0, such that a corresponding Strong Maximum Principle (and a Strong Minimum Principle) holds?

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