ON HOPF'S LEMMA AND THE STRONG MAXIMUM PRINCIPLE

S. BERTONE, A. CELLINA, AND E. M. MARCHINI

Abstract. In this paper we consider Hopf's Lemma and the Strong Maximum Principle for supersolutions to

$$\sum_{i=1}^{N} g_i(u_{x_i}^2)u_{x_i x_i} = 0$$

under suitable hypotheses that allow g_i to assume value zero at zero.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set and, on Ω, consider the operator

$$F(u) = \sum_{i=1}^{N} g_i(u_{x_i}^2)u_{x_i x_i}, \tag{1}$$

where $g_i : [0, +\infty) \to [0, +\infty)$ are continuous functions.

Ω is called regular if, for every $z \in \partial\Omega$, there exists a tangent plane, continuously depending on z, and Ω satisfies the interior ball condition at z if there exists an open ball $B \subset \Omega$ with $z \in \partial B$. When F is elliptic, two classical results hold.

Hopf's Lemma:
Let Ω be regular, let u be such that $F(u) \leq 0$ on Ω. Suppose that there exists $z \in \partial\Omega$ such that

$$u(z) < u(x), \quad \text{for all } x \text{ in } \Omega.$$

If, in addition, Ω satisfies the interior ball condition at z, we have

$$\frac{\partial u}{\partial \nu}(z) < 0,$$

where ν is the outer unit normal to B at z.

The Strong Maximum Principle:
Let u be such that $F(u) \leq 0$ on Ω, then if u it attains minimum in Ω, it is a constant.

In 1927 Hopf proved the Strong Maximum Principle in the case of second order elliptic partial differential equations, by applying a comparison technique, see [11]. For the class of quasilinear elliptic problems, many contributions have been given,

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to extend the validity of the previous results, as in [1, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 18].

In the case in equation (1) we have $g_i \equiv 1$, for every i, then $F(u) = \Delta u$, and we find the classical problem of the Laplacian, see [7, 9].

On the other hand, when there exists $i \in \{1, \dots, N\}$ such that $g_i \equiv 0$ on an interval $I = [0, T] \subset \mathbb{R}$, the Strong Maximum Principle does not hold. Indeed, in this case, it is always possible to define a function u assuming minimum in Ω and such that $\sum_{i=1}^N g_i(u_{x_i}^2) u_{x_i x_i} = 0$. For instance, let $g_N(t) = 0$, for every $t \in [0, 2]$. The function

$$u(x_1, \dots, x_N) = \begin{cases} -(x_N^2 - 1)^4 & \text{if } -1 \leq x_N \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

satisfies (1) in $\mathbb{R}^N$.

We are interested in the case when $0 \leq g_i(t) \leq 1$ and it does not exist i such that $g_i \equiv 0$ on an interval. Since g_i could assume value zero, the equation (1) is degenerate elliptic.

Our interest in the problem comes, partially, from the calculus of variations. We can consider the equation

$$\sum_{i=1}^N g_i(u_{x_i}^2) u_{x_i x_i} = 0$$

as the Euler-Lagrange equation associated to the functional

$$J(u) = \int_\Omega L(\nabla u(x)) dx = \int_\Omega \frac{1}{2} \left(\sum_{i=1}^N f_i(u_{x_i}^2) u_{x_i}^2 \right) dx,$$

where $L(\nabla u)$ is strictly convex in $\{(u_{x_1}, \dots, u_{x_N}) : u_{x_i}^2 \leq \bar{t}, \text{ for every } i = 1, \dots, N\}$. Indeed, fix i. Let f_i be a solution to the differential equation

$$g_i(t) = f_i(t) + 5t f_i'(t) + 2t^2 f_i''(t), \tag{2}$$

for $t \in [0, \bar{t}\,]$. Since

$$\frac{\partial^2 L}{\partial u_{x_i}^2}(u_{x_i}^2) = f_i(u_{x_i}^2) + 5u_{x_i}^2 f_i'(u_{x_i}^2) + 2u_{x_i}^4 f_i''(u_{x_i}^2) = g_i(u_{x_i}^2),$$

we have that

$$\operatorname{div} \nabla_{\nabla u} L(\nabla u) = \sum_{i=1}^N \frac{\partial^2 L}{\partial u_{x_i}^2} u_{x_i x_i} = \sum_{i=1}^N g_i(u_{x_i}^2) u_{x_i x_i}.$$

Since J is convex, solutions to the Euler-Lagrange equation are minima (among those functions satisfying the same boundary conditions.) From our assumptions, it follows that $v \equiv 0$ is a solution. The Strong Maximum Principle can be seen as a strong uniqueness result among solutions to a variational problem: two solutions u and v are given, not crossing each other: if the two solutions meet at one point, they are identically equal. For a functional J possessing rotational symmetry, simple necessary and sufficient condition for the validity of the Strong Maximum Principle are presented in [4]. The purpose of the present paper is to investigate the same problem in the class of functional that are convex, but non possessing rotational symmetry.

The results of [4] for the validity of the Strong Maximum Principle, for equations possibly non elliptic but arising from a functional having rotational symmetry, show that this validity depends only on the behaviour of the functions g_i near zero.

In this paper, we prove, in section 3, a sufficient condition for the validity of the Hopf's Lemma and of the Strong Maximum Principle; a remarkable feature of this condition is that it concerns only the behaviour of the function $g_i(t)$ that goes fastest to zero, as t goes to zero. Hopf's lemma and the Strong Maximum Principle are essentially the same result as long as we can build subsolutions whose level lines can have arbitrarily large curvature. This need not be always possible for problems not possessing rotational symmetry. This difficulty will be evident in sections 4 and 5. In these sections, a more restricted class of equations is considered, namely when all the functions g_i, for $i = 1, \dots, N-1$, are 1 and only g_N is allowed to go to zero. In this simpler class of equations we are able to show that the condition

$$\lim_{t\to 0^+} \frac{(g_N(t))^{3/2}}{t g_N'(t)} > 0$$

is at once necessary for the validity of Hopf's Lemma and sufficient for the validity of the Strong Maximum Principle.

2. Preliminary results

We impose the following local assumptions.

Assumptions (L):

There exists $\bar{t} > 0$ such that:

i) on $[0, \bar{t}\,]$, for every $i = 1, \dots, N-1$,

$$0 \le g_N(t) \le g_i(t) \le 1;$$

ii) g_N is continuous on $[0, \bar{t}\,]$; positive and differentiable on $(0, \bar{t}\,]$;

iii) on $(0, \bar{t}\,]$, the function $t \to g_N(t) + g_N'(t)t$ is non decreasing.

Notice that, in case *ii)* above is violated, the Strong Maximum Principle does not hold; and that condition *iii)* above includes the case of the Laplacian, $g_i(t) \equiv 1$; and, finally, that under these assumptions, g_i could assume value zero at most for $t = 0$.

Moreover, the strict convexity of $L(\nabla u)$ in $\{(u_{x_1}, \dots, u_{x_N}) : u_{x_i}^2 \le \bar{t},$ for every $i = 1, \dots, N\}$ follows by the fact that g_i is positive in $(0, \bar{t}\,]$.

Since we will need general comparison theorems that depend on the global properties of the solutions, i.e. on their belonging to a Sobolev space, we will need also a growth assumption on g_i (assumption *(G)*) to insure these properties of the solutions.

Assumption (G):
Each function f_i as defined in (2), is bounded away from zero and $f_i(u_{x_i}^2)u_{x_i}^2$ is strictly convex.

Any function g_i satisfying assumptions *(L)* on $[0,\bar{t}\,]$ can be modified so as to satisfy assumption *(G)* on $[0,+\infty)$. In fact, taking f_i such that $f_i(t) = g_i(\bar{t}\,)$, for $t > \bar{t}$, it is enough to modify g_i to $(\bar{t}, +\infty)$ by setting $g_i(t) = g_i(\bar{t}\,)$, for $t > \bar{t}$.

Definition 1. *Let Ω be open, and let $u \in W^{1,2}(\Omega)$. The map u is a weak solution to the equation $F(u) = 0$ if, for every $\eta \in C_0^\infty(\Omega)$,*

$$\int_\Omega \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx = 0.$$

u is a weak subsolution ($F(u) \geq 0$) if, for every $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$,

$$\int_\Omega \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx \leq 0.$$

u is a weak supersolution ($F(u) \leq 0$) if, for every $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$,

$$\int_\Omega \langle \nabla L(\nabla u(x)), \nabla \eta(x) \rangle dx \geq 0.$$

We say that a function $w \in W^{1,2}(\Omega)$ is such that $w_{|\partial\Omega} \leq 0$ if $w^+ \in W_0^{1,2}(\Omega)$.

The growth assumption *(G)* assures that, if $u \in W^{1,2}(\Omega)$, then $\nabla L(\nabla u(x)) \in L^2(\Omega)$. The strict convexity of L implies the following comparison lemma.

Lemma 1. *Let Ω be an open and bounded set, let $v \in W^{1,2}(\Omega)$ be a subsolution and let $u \in W^{1,2}(\Omega)$ be a supersolution to the equation $F(u) = 0$. If $v_{|\partial\Omega} \leq u_{|\partial\Omega}$, then $v \leq u$ a.e. in Ω.*

The following technical lemmas will be used later.

Lemma 2. *Let $n = 2, \dots, N$ and set*

$$h_n(a) = g_N\left(\frac{t(1-a)}{n-1}\right)(1-a) + g_N(ta)a.$$

For every $0 < t \leq \bar{t}$ ($\bar{t}$ defined in assumptions (L)), $h_n(a) \geq h_n(1/n)$, for every a in $[0,1]$.

Proof. Since, on $(0,\bar{t}\,]$, the function $t \to g_N(t) + g_N'(t)t$ is non decreasing, we have that

$$h_n'(a) = -g_N\left(\frac{t(1-a)}{n-1}\right) - g_N'\left(\frac{t(1-a)}{n-1}\right)\frac{t(1-a)}{n-1} + g_N(ta) + g_N'(ta)ta \geq 0$$

if and only if $a \geq 1/n$, so that $h_n(a) \geq h_n(1/n)$, for every $a \in [0,1]$.

□

Lemma 3. *For every $0 < t \leq \bar{t}$ ($\bar{t}$ defined in assumptions (L)), we have that*

$$\sum_{i=1}^N g_N\left(t\left(\frac{x_i}{\rho}\right)^2\right)\left(\frac{x_i}{\rho}\right)^2 \geq g_N\left(\frac{t}{N}\right).$$

Proof. We prove the claim by induction on N.

Let $N = 2$. Set $a = \sin^2\theta_1$. Applying Lemma 2 we obtain that

$$g_N\left(t\left(\frac{x_1}{\rho}\right)^2\right)\left(\frac{x_1}{\rho}\right)^2 + g_N\left(t\left(\frac{x_2}{\rho}\right)^2\right)\left(\frac{x_2}{\rho}\right)^2 =$$

$${}_N(t(1-a))(1-a) + g_N(ta)a \geq g_N\left(\frac{t}{2}\right).$$

Suppose that the claim is true for $N-1$, i.e.

$$\sum_{i=1}^{N-1} g_N\left(t\left(\frac{x_i}{\rho}\right)^2\right)\left(\frac{x_i}{\rho}\right)^2 \geq g_N\left(\frac{t}{N-1}\right).$$

Let us prove it for N. Set

$$\begin{cases} y_1 = \rho\cos\theta_{N-2}\dots\cos\theta_2\cos\theta_1 \\ y_2 = \rho\cos\theta_{N-2}\dots\cos\theta_2\sin\theta_1 \\ \dots \\ y_{N-1} = \rho\sin\theta_{N-2} \end{cases}$$

and set $a = \sin^2\theta_{N-1}$. Applying Lemma 2 we obtain that

$$\sum_{i=1}^{N} g_N\left(t\left(\frac{x_i}{\rho}\right)^2\right)\left(\frac{x_i}{\rho}\right)^2 =$$

$$\sum_{i=1}^{N-1} g_N\left(t\left(\frac{y_i}{\rho}\right)^2(1-a)\right)\left(\frac{y_i}{\rho}\right)^2(1-a) + g_N(ta)a \geq$$
$$g_N\left(\frac{t(1-a)}{N-1}\right)(1-a) + g_N(ta)a \geq g_N\left(\frac{t}{N}\right),$$

and the claim is proved.

□

3. A sufficient condition for the validity of Hopf's Lemma and of the Strong Maximum Principle

Consider the improper Riemann integral

$$\int_0^\xi \frac{g_N(\zeta^2/N)}{\zeta}d\zeta = \lim_{\hat{\xi}\to 0}\int_{\hat{\xi}}^{\xi}\frac{g_N(\zeta^2/N)}{\zeta}d\zeta$$

as an extended valued function G,

$$G(\xi) = \int_0^\xi \frac{g_N(\zeta^2/N)}{\zeta}d\zeta,$$

where we mean that $G(\xi) \equiv +\infty$ whenever the integral diverges.

Given a continuous function u, a direction ν and a point z, we denote

$$\frac{\partial^+ u}{\partial\nu}(z) = \limsup_{h\to 0^-}\frac{u(z+h\nu)-u(z)}{h}. \tag{3}$$

We wish to prove the following lemma.

Lemma 4 (Hopf's Lemma)**.** *Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set. Let $u \in W^{1,2}(\Omega) \cap C\left(\overline{\Omega}\right)$ be a weak solution to*

$$\sum_{i=1}^{N} g_i(u_{x_i}^2) u_{x_i x_i} \leq 0.$$

In addition to the assumptions (L) and (G) on g_i, assume that $G(\xi) \equiv +\infty$. Suppose that there exists $z \in \partial\Omega$ such that

$$u(z) < u(x), \qquad \text{for all } x \text{ in } \Omega$$

and that Ω satisfies the interior ball condition at z. Then

$$\frac{\partial^+ u}{\partial \nu}(z) < 0,$$

where ν is the outer unit normal to B at z.

Note that the interior ball condition automatically holds if ∂U is C^2.

As an example of an equation satisfying the assumptions of the theorem above, consider the Laplace equation $\Delta u = 0$. The functions $g_i \equiv 1$ satisfy the assumptions *(L)* and *(G)*, and

$$G(\xi) = \int_0^\xi \frac{1}{\zeta} d\zeta = +\infty.$$

Another example is obtained setting

$$g_N(t) = \frac{1}{|\ln(t)|}$$

for $0 \leq t \leq 1/e$. The assumptions *(L)* and *(G)* are satisfied; moreover, for $0 \leq \xi^2/N \leq 1/e$,

$$G(\xi) = \int_0^\xi \frac{d\zeta}{\zeta |\ln(\zeta^2/N)|} = +\infty.$$

Remark 1. In this paper we will consider the operator

$$F(v) = \sum_{i=1}^{N} g_i(v_{x_i}^2) v_{x_i x_i}$$

in polar coordinates. Set

$$\begin{cases} x_1 = \rho \cos\theta_{N-1} \ldots \cos\theta_2 \cos\theta_1 \\ x_2 = \rho \cos\theta_{N-1} \ldots \cos\theta_2 \sin\theta_1 \\ \ldots \\ x_N = \rho \sin\theta_{N-1} \end{cases}$$

so that

$$v_{x_i} = v_\rho \frac{x_i}{\rho} \qquad \text{and} \qquad v_{x_i x_i} = v_{\rho\rho} \left(\frac{x_i}{\rho}\right)^2 + \frac{v_\rho}{\rho}\left[1 - \left(\frac{x_i}{\rho}\right)^2\right].$$

When v is a radial function, F reduces to

$$F(v) = \sum_{i=1}^{N} g_i\left(v_\rho^2\left(\frac{x_i}{\rho}\right)^2\right)\left[v_{\rho\rho}\left(\frac{x_i}{\rho}\right)^2+\frac{v_\rho}{\rho}\left(1-\left(\frac{x_i}{\rho}\right)^2\right)\right] =$$

$$v_{\rho\rho}\sum_{i=1}^{N} g_i\left(v_\rho^2\left(\frac{x_i}{\rho}\right)^2\right)\left(\frac{x_i}{\rho}\right)^2+\frac{v_\rho}{\rho}\sum_{i=1}^{N} g_i\left(v_\rho^2\left(\frac{x_i}{\rho}\right)^2\right)\left(1-\left(\frac{x_i}{\rho}\right)^2\right).$$

In general, we do not expect that the equation $F(v) = 0$ admits radial solutions. However we will use the expression of F valid for radial functions in order to reach our results.

Proof of Lemma 4. a) We assume that $u(z) = 0$ and that $B = B(O, r)$. Let

$$\epsilon = \min\left\{u(x) : x \in \overline{B(O, r/2)}\right\} > 0 \quad \text{and} \quad \omega = B(O, r) \setminus \overline{B(O, r/2)}.$$

b) We seek a radial function $v \in W^{1,2}(\omega) \cap C(\overline{\omega})$ satisfying

$$\begin{cases} v \text{ is a weak solution to } F(v) \geq 0 & \text{in } \omega \\ v > 0 & \text{in } \omega \\ v = 0 & \text{in } \partial B(O, r) \\ v \leq \epsilon & \text{in } \partial B(O, r/2) \\ v_\rho(z) < 0. \end{cases} \tag{4}$$

Consider the Cauchy problem

$$\begin{cases} \zeta' = -\dfrac{N-1}{\rho}\dfrac{\zeta}{g_N(\zeta^2/N)} \\ \\ \zeta(r/2) = -\dfrac{\epsilon}{r}. \end{cases} \tag{5}$$

There exists a unique local solution ζ of (5), such that

$$\int_{-\frac{\epsilon}{r}}^{\zeta(\rho)} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = \int_{r/2}^{\rho} -\frac{N-1}{s} ds = -(N-1)\ln\left(\frac{2\rho}{r}\right).$$

We claim that ζ is defined in $[r/2, +\infty)$. Indeed, suppose that ζ is defined in $[r/2, \tau)$, with $\tau < +\infty$. Since $\zeta' > 0$, ζ is an increasing function, so that $\tau < +\infty$ if and only if $\lim_{\rho\to\tau} \zeta(\rho) = 0$. But

$$-\infty = \lim_{\rho\to\tau} \int_{-\frac{\epsilon}{r}}^{\zeta(\rho)} \frac{g_N(\zeta^2/N)}{\zeta} d\zeta = \lim_{\rho\to\tau} -(N-1)\ln\left(\frac{2\rho}{r}\right),$$

a contradiction. Hence, the solution ζ of (5) is defined in $[r/2, +\infty)$.

Setting $v_\rho = \zeta$, since, for every $\rho \in (r/2, r)$,

$$-\frac{\epsilon}{r} < v_\rho(\rho) < 0,$$

we have that the function

$$v(\rho) = \int_r^\rho v_\rho(s) ds$$

solves the problem

$$v_{\rho\rho} g_N\left(\frac{v_\rho^2}{N}\right) + \frac{v_\rho}{\rho}(N-1) = 0,$$

in particular, for every $\rho \in (r/2, r)$, $v(\rho) > 0$ and $v_\rho(\rho) \leq -\eta < 0$; moreover $v(r) = 0$ and $v(r/2) \leq \epsilon$.

Since $v_{\rho\rho} \geq 0$ and $-\sqrt{t} \leq v_\rho \leq 0$, for every $\rho \in (r/2, r)$, by the hypotheses on g_i and by Lemma 3, we have that

$$\begin{aligned} F(v) = \quad & v_{\rho\rho} \sum_{i=1}^{N} g_i \left(v_\rho^2 \left(\frac{x_i}{\rho} \right)^2 \right) \left(\frac{x_i}{\rho} \right)^2 + \frac{v_\rho}{\rho} \sum_{i=1}^{N} g_i \left(v_\rho^2 \left(\frac{x_i}{\rho} \right)^2 \right) \left(1 - \left(\frac{x_i}{\rho} \right)^2 \right) \geq \\ & v_{\rho\rho} \sum_{i=1}^{N} g_N \left(v_\rho^2 \left(\frac{x_i}{\rho} \right)^2 \right) \left(\frac{x_i}{\rho} \right)^2 + \frac{v_\rho}{\rho} \sum_{i=1}^{N} \left(1 - \left(\frac{x_i}{\rho} \right)^2 \right) = \\ & v_{\rho\rho} \sum_{i=1}^{N} g_N \left(v_\rho^2 \left(\frac{x_i}{\rho} \right)^2 \right) \left(\frac{x_i}{\rho} \right)^2 + \frac{v_\rho}{\rho}(N-1) \geq \\ & v_{\rho\rho} g_N \left(\frac{v_\rho^2}{N} \right) + \frac{v_\rho}{\rho}(N-1). \end{aligned}$$

The function v solves (4), indeed, v is in $C^2(\overline{\omega})$ and it is such that $F(v) \geq 0$ and $v > 0$ in ω, $v(r) = 0$, $v(r/2) \leq \epsilon$ and $v_\rho(z) < 0$.

c) Since $u, v \in W^{1,2}(\omega) \cap C(\overline{\omega})$, v is a weak subsolution and u is a weak solution to $F(u) = 0$, and $v_{|\partial\omega} \leq u_{|\partial\omega}$, applying Lemma 1, we obtain that $v \leq u$ in ω.

We have

$$v(z) = u(z) \quad \text{ and } \quad v_\rho(z) = \frac{\partial v}{\partial \nu}(z) \leq -\eta.$$

If

$$\frac{\partial^+ u}{\partial \nu}(z) > -\eta,$$

where $\frac{\partial^+ u}{\partial \nu}$ has been defined in (3), then there exists $h < 0$ such that $v(z + h\nu) > u(z + h\nu)$, in contradiction with the fact that $v \leq u$ in ω.

□

From Hopf's Lemma we derive:

Theorem 1 (Strong Maximum Principle). *Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set. Let $u \in W^{1,2}(\Omega) \cap C\left(\overline{\Omega}\right)$ be a weak supersolution to*

$$\sum_{i=1}^{N} g_i(u_{x_i}^2) u_{x_i x_i} = 0.$$

In addition to the assumptions (L) and (G) on g_i, assume that $G(\xi) \equiv +\infty$. Then, if u attains its minimum in Ω, it is a constant.

Proof. a) Assume $\min_\Omega u = 0$ and set $\mathcal{C} = \{x \in \Omega : u(x) = 0\}$. By contradiction, suppose that the open set $\Omega \setminus \mathcal{C} \neq \emptyset$.

b) Since Ω is a connected set, there exist $s \in \mathcal{C}$ and $R > 0$ such that $B(s, R) \subset \Omega$ and $B(s, R) \cap (\Omega \setminus \mathcal{C}) \neq \emptyset$. Let $p \in B(s, R) \cap (\Omega \setminus \mathcal{C})$. Consider the line $\overline{ps}$. Moving p along this line, we can assume that $B(p, d(p, \mathcal{C})) \subset (\Omega \setminus \mathcal{C})$ and that there exists a point $z \in \mathcal{C}$ such that $d(p, \mathcal{C}) = d(p, z)$. Set $r = d(p, \mathcal{C})$. W.l.o.g. suppose that

$p = O$.

c) The set $\Omega \setminus \mathcal{C}$ satisfies the interior ball condition at z, hence Hopf's Lemma implies

$$\frac{\partial^+ u}{\partial \nu}(z) < 0.$$

But this is a contradiction: since u attains minimum at $z \in \Omega$, we have that $Du(z) = 0$.

□

4. A necessary condition for the validity of Hopf's Lemma

In this and the following section we consider the operator

$$F(u) = \sum_{i=1}^{N-1} u_{x_i x_i} + g(u_{x_N}^2) u_{x_N x_N}, \tag{6}$$

We wish to provide a necessary condition for the validity of Hopf's Lemma in a class of degenerate elliptic equations.

Consider the case

$$G(\xi) = \int_0^\xi \frac{g(\zeta^2/N)}{\zeta} d\zeta < +\infty.$$

Theorem 2. *Let g, satisfying assumptions (L) and (G), be such that, on $(0, \overline{t}\,]$, ($\overline{t}$ defined in assumptions (L)),*

$$g'(t) > 0, \quad g(t) + g'(t)t \leq 1,$$

and

$$\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} = 0.$$

Then there exist: an open regular region $\Omega \subset \mathbb{R}^N$; a radial function $u \in C^2(\Omega)$, such that $F(u) \leq 0$ in Ω, and a point $z \in \partial\Omega$ such that $u(z) = 0$, $u(z) < u(x)$ for every $x \in \Omega$ and

$$\frac{\partial u}{\partial \nu}(z) = 0$$

where ν is the outer unit normal to Ω at z.

If, in addition, we assume that

$$\frac{g(t)}{tg'(t)} \quad \text{is bounded in} \quad (0, \overline{t}\,] \tag{7}$$

then Ω satisfies the interior ball condition at z.

Remark 2. *When $\lim_{t \to 0^+} g'(t)t$ exists, it follows that*

$$\lim_{t \to 0^+} \frac{(g(t))^{3/2}}{tg'(t)}$$

exists, and that

$$g(t) + g'(t)t \leq 1\,, \;\; \text{on } (0, \overline{t}\,].$$

Indeed, we have that

$$\lim_{t\to 0^+} (g(t)+g'(t)t) = 0.$$

From

$$\int_0^{\xi^2} \frac{g(t)}{t}dt = 2\int_0^{\xi} \frac{g(\zeta^2/N)}{\zeta}d\zeta < +\infty,$$

it follows that $\lim_{t\to 0^+} g(t) = 0$. If $\lim_{t\to 0^+} g'(t)t > 0$, there exists $K > 0$ such that, when $0 < t \le \bar{t}$, $g'(t)t \ge K$, so that

$$g(t)t = \int_0^t (g(s)+g'(s)s)\, ds \ge Kt,$$

and $g(t) \ge K$, a contradiction.

The map

$$g(t) = \frac{1}{|\ln(t)|^k},$$

with $k > 2$, for $0 \le t \le 1/e$, satisfies the assumption

$$\lim_{t\to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} = 0.$$

The following lemma is instrumental to the proofs of the main results.

Lemma 5. *Let g satisfies assumptions (L) and (G). Suppose that for every $0 < t \le \bar{t}$,*

$$g'(t) \ge 0 \quad \textit{and} \quad g(t)+g'(t)t \le 1.$$

Set

$$k_1(a) = (1-a)+ag(ta) \quad \textit{and} \quad k_2(a) = -a-(1-a)g(ta)$$

For every $0 < t \le \bar{t}$ ($\bar{t}$ defined in assumptions (L)), k_1 and k_2 are non increasing in $[0,1]$.

Proof. Since, for every $0 < t \le \bar{t}$,

$$g'(t) \ge 0 \quad \text{and} \quad g(t)+g'(t)t \le 1,$$

we have that, for every $0 \le a \le 1$,

$$k_1'(a) = -1+g(ta)+g'(ta)ta \le 0$$

and

$$k_2'(a) = -1+g(ta)-(1-a)g'(ta)t = -1+g(ta)+g'(ta)ta - g'(ta)t \le 0.$$

□

Proof of Theorem 2. a) Let v be a radial function. Setting $a = \sin^2\theta_{N-1}$, (6) reduces to

$$F(v) = v_{\rho\rho}\left(1-a+ag(v_\rho^2 a)\right) + \frac{v_\rho}{\rho}\left(N-2+a+(1-a)g(v_\rho^2 a)\right).$$

Let $a = 1$, we seek a solution to

$$v_{\rho\rho}g(v_\rho^2) + (N-1)\frac{v_\rho}{\rho} = 0 \tag{8}$$

and a radius $R(1) > 0$ such that $v_\rho(R(1)+1) = 0$ and $v_\rho(\rho) < 0$, for every $\rho \in [2, R(1)+1)$. Consider the Cauchy problem

$$\begin{cases} \zeta' = -\dfrac{N-1}{\rho}\dfrac{\zeta}{g(\zeta^2)} \\ \\ \zeta(2) = -1. \end{cases} \tag{9}$$

We are interested in a negative solution ζ. Define $R(1)$ to be the unique positive real solution to

$$G(-1) - (N-1)\ln\left(\frac{R(1)+1}{2}\right) = 0,$$

i.e.

$$R(1) = 2e^{\frac{G(-1)}{N-1}} - 1.$$

The solution ζ of (9), satisfies

$$G(\zeta(\rho)) - G(-1) = \int_{\zeta(2)}^{\zeta(\rho)} \frac{g(t^2)}{t}dt = \int_2^\rho -\frac{N-1}{s}ds = -(N-1)\ln\left(\frac{\rho}{2}\right).$$

Then, for every $\rho \in (2, R(1)+1)$, $G(\zeta(\rho)) > 0$ and $\zeta(\rho) < 0$, while $\zeta(R(1)+1) = 0$. Setting $v_\rho(\rho) = \zeta(\rho)$ and

$$v(\rho) = \int_{R(1)+1}^{\rho} v_\rho(s)ds,$$

we obtain that v solves (8) and, for every $\rho \in (2, R(1)+1)$,

$$v_\rho(\rho) < v_\rho(R(1)+1) = 0 \quad \text{and} \quad v(\rho) > v(R(1)+1) = 0.$$

b) Set, for $\rho \in (1, R(1)]$, $u(\rho) = v(\rho+1)$. Since, for the function v, we have

$$v_{\rho\rho}(\rho)g(v_\rho^2(\rho)) + (N-1)\frac{v_\rho(\rho)}{\rho} = 0,$$

at $\rho+1$ we obtain

$$u_{\rho\rho}(\rho)g(u_\rho^2(\rho)) + (N-1)\frac{u_\rho(\rho)}{\rho+1} = 0. \tag{10}$$

This equality yields, for $\rho \in (1, R(1))$,

$$u_{\rho\rho}g(u_\rho^2) + (N-1)\frac{u_\rho}{\rho} = -(N-1)u_\rho\left(\frac{1}{\rho+1} - \frac{1}{\rho}\right) < 0.$$

c) Let $1/2 < a < 1$. We wish to find $R(a) \leq R(1)$ such that u is a solution to

$$F(u) = u_{\rho\rho}\left(1 - a + ag(u_\rho^2 a)\right) + \frac{u_\rho}{\rho}\left(N - 2 + a + (1-a)g(u_\rho^2 a)\right) \leq 0,$$

for $\rho \in (1, R(a))$. Since

$$\begin{aligned} F(u) = \quad & \frac{-u_\rho}{\rho(\rho+1)g((u_\rho)^2)}\left[\rho(N-1)\left(1 - a + ag((u_\rho)^2 a)\right) - \right. \\ & \left.(\rho+1)g((u_\rho)^2)\left(N - 2 + a + (1-a)g((u_\rho)^2 a)\right)\right], \end{aligned}$$

setting

$$k(\rho) = \rho(N-1)\left(1 - a + ag((u_\rho)^2 a)\right) - (\rho+1)g((u_\rho)^2)\left(N - 2 + a + (1-a)g((u_\rho)^2 a)\right),$$

we obtain that $F(u) \leq 0$ if and only if $k(\rho) \leq 0$. We have

$$
\begin{aligned}
k'(\rho) = \quad & (N-1)\left[1-a+ag((u_\rho)^2 a)\right] - g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] + \\
& \rho(N-1)a\frac{d}{d\rho}g((u_\rho)^2 a) - (\rho+1)\frac{d}{d\rho}g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] - \\
& (\rho+1)g((u_\rho)^2)(1-a)\frac{d}{d\rho}g((u_\rho)^2 a) = \\
& (N-1)\left[1-a+ag((u_\rho)^2 a)\right] - g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] - \\
& \frac{d}{d\rho}g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] - g((u_\rho)^2)(1-a)\frac{d}{d\rho}g((u_\rho)^2 a) + \\
& \rho\Bigg[(N-1)a\frac{d}{d\rho}g((u_\rho)^2 a) - \frac{d}{d\rho}g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] - \\
& g((u_\rho)^2)(1-a)\frac{d}{d\rho}g((u_\rho)^2 a)\Bigg].
\end{aligned}
$$

Assume $N \geq 2$. Applying Lemma 5, we obtain

$$
\begin{aligned}
& (N-1)\left[1-a+ag((u_\rho)^2 a)\right] - g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] \geq \\
& (N-1)g((u_\rho)^2) - g((u_\rho)^2)(N-1) = 0;
\end{aligned}
$$

from

$$
\frac{d}{d\rho}g((u_\rho)^2) \leq \frac{d}{d\rho}g((u_\rho)^2 a) \leq 0, \tag{11}
$$

we have

$$
-\frac{d}{d\rho}g((u_\rho)^2)\left[N-2+a+(1-a)g((u_\rho)^2 a)\right] - g((u_\rho)^2)(1-a)\frac{d}{d\rho}g((u_\rho)^2 a) \geq 0;
$$

since $N-2+a \geq (N-1)a$, from (11), we deduce

$$
\begin{aligned}
& \rho\Bigg[(N-1)a\frac{d}{d\rho}g((u_\rho)^2 a) - \frac{d}{d\rho}g((u_\rho)^2)(N-2+a+(1-a)g((u_\rho)^2 a)) - \\
& g((u_\rho)^2)(1-a)\frac{d}{d\rho}g((u_\rho)^2 a)\Bigg] \geq \\
& \rho\left[(N-1)a\frac{d}{d\rho}g((u_\rho)^2 a) - \frac{d}{d\rho}g((u_\rho)^2)(N-2+a)\right] \geq \\
& \rho(N-1)a\left(\frac{d}{d\rho}g((u_\rho)^2 a) - \frac{d}{d\rho}g((u_\rho)^2)\right) \geq 0.
\end{aligned}
$$

The previous inequalities imply that $k'(\rho) \geq 0$. It follows that $F(u) \leq 0$, for every $\rho \in (1, R(a))$, if and only if

$$
k(R(a)) \leq 0.
$$

We have that

$$\begin{aligned} k(R(a)) = \quad & R(a)(N-1)\left(1-a+ag((u_\rho(R(a)))^2a)\right) - \\ & (R(a)+1)g((u_\rho(R(a)))^2)\left(N-2+a+(1-a)g((u_\rho(R(a)))^2a)\right) \le \\ & (N-1)\left[R(a)(1-a)-g((u_\rho(R(a)))^2)a\right] \le \\ & c(N-1)\left[R(1)(1-a)-g((u_\rho(R(a)))^2)a\right] \le \\ & (N-1)\left[R(1)(1-a)-\frac{g((u_\rho(R(a)))^2)}{2}\right]. \end{aligned}$$

We define $R(a)$ to be a solution to

$$R(1)(1-a)-\frac{g((u_\rho(R(a)))^2)}{2}=0. \tag{12}$$

d) In order to solve (12) for the unknown $R(a)$, recalling that $1-a=\cos^2\theta_{N-1}=c^2$, let

$$h(r)=\sqrt{\frac{g((u_\rho(r))^2)}{2R(1)}}.$$

The function h is decreasing, differentiable and with inverse differentiable. We have that $|c|=h(R(1-c^2))$, so that $R(1-c^2)=h^{-1}(|c|)$, $R(1-c^2)$ is increasing in $|c|$ and

$$\lim_{c\to 0} R(1-c^2)=R(1).$$

Let $0<|\bar c|<1/2$ be such that $R(1-\bar c^2)\ge 1$, so that, for $c^2\le \bar c^2$, we have $R(1-c^2)\ge R(1-\bar c^2)\ge 1$. We have obtained that, for every $1\ge a\ge 1-|c|^2$, there exists $R(a)$ such that (12) holds. It follows that

$$k(R(a))\le\left[R(1)(1-a)-\frac{g((u_\rho(R(a)))^2)}{2}\right]=0,$$

so that the function u solves $F(u)\le 0$ for every $\rho\in(1,R(a))$.

e) Set $\Omega=\{(x_1,\dots,x_N)\in\mathbb{R}^N : \rho\in(1,R(1-c^2))$ and $|c|<|\bar c|\}$. $\Omega\subset\mathbb{R}^N$ is a connected, open and bounded set and $u\in W^{1,2}(\Omega)\cap C\left(\overline{\Omega}\right)$ is a weak solution to $F(u)\le 0$. The point $z=(R,0,\dots,0)\in\partial\Omega$ is such that $u(z)<u(x)$, for all $x\in\Omega$. We wish to show that Ω is regular in a neighborhood of $z=(R,0,\dots,0)$. Since $\frac{d}{dc}R(1-c^2)$ exists, in $(0,|\bar c|)$, to prove our claim it is sufficient to show that

$$\lim_{c\to 0}\frac{d}{dc}R(1-c^2)=0.$$

Recalling (10), we have that

$$\begin{aligned} & \frac{d}{dc}R(1-c^2)=\left(h^{-1}(c)\right)'=\frac{1}{h'(R(1-c^2))}= \\ & -\frac{\sqrt{2}R(1)}{N-1}\left(R(1-c^2)+1\right)\frac{\left(g((u_\rho(R(1-c^2)))^2)\right)^{3/2}}{(u_\rho(R(1-c^2)))^2 g'((u_\rho(R(1-c^2)))^2)}. \end{aligned} \tag{13}$$

Since

$$\lim_{c\to 0} u_\rho(R(1-c^2))=0$$

and

$$\lim_{t\to 0^+} \frac{(g(t))^{3/2}}{tg'(t)} = 0,$$

it follows that

$$\lim_{c\to 0} \frac{d}{dc} R(1-c^2) = 0.$$

Since $\frac{d}{d\theta}\cos\theta_{N-1}{}_{|\theta_{N-1}=\frac{\pi}{2}} = 1$, this shows that $\frac{d}{d\theta}R(1-\cos^2\theta_{N-1}) = 0$, and Ω is regular.

f) To prove the validity of the interior ball condition at $z = (R, 0, \ldots, 0)$, it is enough to show that the second derivative of $R(1-c^2)$ is bounded at $c = 0$, i.e. that

$$\left|\frac{1}{c}\frac{d}{dc}R(1-c^2)\right|$$

is bounded. Set

$$t(c) = (u_\rho(R(1-c^2)))^2,$$

from (13) we obtain

$$\frac{d}{dc}R(1-c^2) = -\frac{\sqrt{2}R(1)}{N-1}\left(R(1-c^2)+1\right)\frac{(g(t(c)))^{3/2}}{t(c)g'(t(c))},$$

and from (10)

$$\begin{aligned}\frac{dt(c)}{dc} = \quad & 2u_\rho(R(1-c^2))u_{\rho\rho}(R(1-c^2))\frac{d}{dc}(R(1-c^2)) = \\ & 2\sqrt{2}R(1)\frac{(g(t(c)))^{1/2}}{g'(t(c))}\end{aligned}$$

and

$$\frac{d}{dc}(g(t(c)))^{1/2} = \frac{g'(t(c))}{2(g(t(c)))^{1/2}}\frac{dt(c)}{dc} = \sqrt{2}R(1).$$

From $g(t(0)) = 0$, we obtain that

$$(g(t(c)))^{1/2} = \sqrt{2}R(1)c$$

and

$$\frac{(g(t(c)))^{3/2}}{ct(c)g'(t(c))} = \sqrt{2}R(1)\frac{g(t(c))}{t(c)g'(t(c))}.$$

From condition (7) we obtain

$$\left|\frac{1}{c}\frac{d}{dc}R(1-c^2)\right| \le M.$$

□

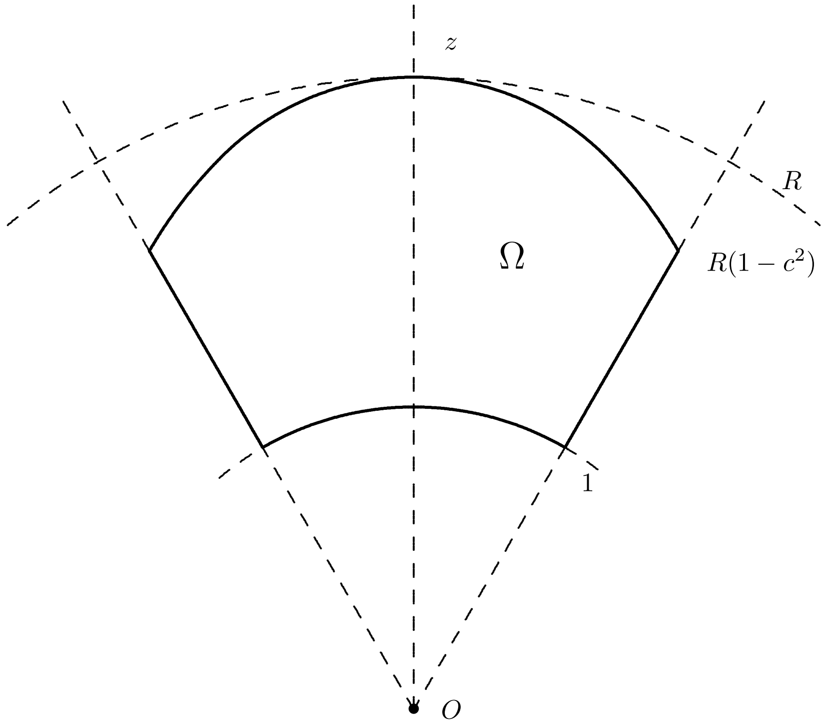

FIGURE 1. Ω in the case $N = 2$.

5. A sufficient condition for the validity of the Strong Maximum Principle

Consider the case

$$G(\xi) = \int_0^\xi \frac{g(\zeta^2/N)}{\zeta} d\zeta < +\infty.$$

We wish to prove the following theorem.

Theorem 3 (The Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be a connected, open and bounded set. Let $u \in W^{1,2}(\Omega) \cap C\left(\overline{\Omega}\right)$ be a weak supersolution to*

$$\sum_{i=1}^{N-1} u_{x_i x_i} + g(u_{x_N}^2) u_{x_N x_N} = 0,$$

where g satisfies assumptions (L) and (G) and, on $(0, \bar{t}\,]$ ($\bar{t}$ defined in assumptions (L)),

$$g'(t) \geq 0 \quad \text{and} \quad g(t) + g'(t)t \leq 1.$$

Moreover, suppose that there exists $K > 0$ such that, for every $0 < \xi^2/N \leq \bar{t}$, we have

$$\sqrt{g(\xi^2/N)} \leq K\left(e^{G(\xi)} - 1\right). \tag{14}$$

Then, if u attains its minimum in Ω, it is a constant.

Remark 3. *When the function g satisfies the condition*

$$g'(t)t \leq 2K\,(g(t))^{3/2}\,,$$

for every $0 < t \leq \bar{t}$, then it satisfies

$$g(t) + g'(t)t \leq 1$$

and

$$\sqrt{g(\xi^2/N)} \leq K\left(e^{G(\xi)} - 1\right),$$

for every $0 < \xi^2/N \leq \bar{t}$.

Indeed, since $G(\xi) < +\infty$, we have that $\lim_{t\to 0} g(t) = 0$. Hence, we can suppose that $g(t) \leq 1/(2K+1)$, for $0 < t \leq \bar{t}$, so that

$$g(t) + g'(t)t \leq g(t) + 2K\,(g(t))^{3/2} \leq 1.$$

Moreover, since $g(0) = 0$, $G(0) = 0$ and

$$\left(\sqrt{g(\xi^2/N)}\right)' \leq 2K\frac{g(\xi^2/N)}{\xi} \leq K\left(e^{G(\xi)} - 1\right)',$$

we obtain that

$$\sqrt{g(\xi^2/N)} \leq K\left(e^{G(\xi)} - 1\right).$$

Remark 4. *Among the functions g such that*

$$\lim_{t\to 0^+}\frac{(g(t))^{3/2}}{tg'(t)}$$

exists, there exists $K > 0$ such that, for every $0 < \xi^2/N \leq \bar{t}$, we have

$$\sqrt{g(\xi^2/N)} \leq K\left(e^{G(\xi)} - 1\right)$$

if and only if

$$\lim_{t\to 0^+}\frac{(g(t))^{3/2}}{tg'(t)} > 0.$$

An example of a map satisfying the assumptions of the theorem above, is given by

$$g(t) = \frac{1}{(\ln(t))^2},$$

for $0 \leq t \leq 1/e^4$. For $0 \leq \xi^2/N \leq 1/e^4$, we have

$$G(\xi) = \int_0^{\xi}\frac{1}{\zeta(\ln(\zeta^2/N))^2}d\zeta = -\frac{1}{2\ln(\xi^2/N)},$$

and

$$\sqrt{g(\xi^2/N)} = \frac{1}{|\ln(\xi^2/N)|} \leq 2\left(e^{\frac{1}{2|\ln(\xi^2/N)|}} - 1\right) = e^{G(\xi)} - 1.$$

Set

$$\mathcal{R}(\lambda, \lambda_N) = \{(x_1, \dots, x_N) : |x_i| \leq \lambda,\ \text{for } i = 1, \dots, N-1, |x_N| \leq \lambda_N\}.$$

Differently from the proof of Lemma 4 and Theorem 1, we will build a subsolution that *is not* radially symmetric. This construction is provided by next theorem.

Theorem 4. *Under the same assumptions on g as on Theorem 3, for every $r > 0$ and every ϵ, there exist: l, l_N; an open convex region $\mathcal{A} \subset \mathcal{R}(l, l_N)$; a function $v \in W^{1,2}(\omega) \cap C^1(\omega) \cap C(\overline{\omega})$, where $\omega = B(\mathcal{A}, r) \setminus \overline{\mathcal{A}}$, such that*
i) $0 \leq l \leq 2Kr$, and $0 \leq l_N \leq r/4$;
ii)

$$\begin{cases} v \text{ is a weak solution to } F(v) \geq 0 & \text{in } \omega \\ v > 0 & \text{in } \omega \\ v = 0 & \text{in } \partial B(\mathcal{A}, r) \\ v \leq \epsilon & \text{in } \partial \mathcal{A}. \end{cases} \tag{15}$$

To give a sketch of the proof we consider the construction in $\mathbb{R}^2$. Using polar coordinates (ρ, θ) and setting $a = \sin^2\theta$, the operator F, when computed on a radial function w, reduces to

$$F(w) = w_{\rho\rho}\left(1 - a + ag(w_\rho^2 a)\right) + \frac{w_\rho}{\rho}\left(a + (1-a)g(w_\rho^2 a)\right).$$

We will use this expression of F, valid only for radial functions, to build a non-radial solution of $F(u) \geq 0$.

We set $a = 1$ and w a solution to $F(w) = 0$ on $[R(1), R(1) + r)$, where $R(1)$ is such that $w(R(1)) \leq \epsilon$ and $w(R(1) + r) = 0$. The radial function w would solve $F(w) \geq 0$, for $0 \leq a \leq 1$, but the domain we would obtain is too large for our purposes.

Fix $a_1 < 1$ and find the smallest $R(a_1) > 0$ such that, setting

$$w^{a_1}(\rho) = w(\rho - R(a_1) + R(1)),$$

the function w^{a_1} is a solution to $F(w^{a_1}) \geq 0$, for every $a < a_1$. We also set

$$\mathcal{D}_0 = \{(x_1, x_2) : R(1) < \rho < R(1) + r, \sqrt{a_1} \leq \sin\theta \leq 1\}$$

and on $\mathcal{D}_0$ we define $v_0(x_1, x_2) = w(\rho(x_1, x_2))$. This function v_0 is a restriction of the function v we seek.

Fix $a_2 < a_1$ and set

$$\mathcal{D}_1 = \{(x_1, x_2) : R(a_1) < \rho_1 < R(a_1) + r, \sqrt{a_2} \leq \sin\theta \leq \sqrt{a_2}\},$$

where $\rho_1 = d((x_1, x_2), O^1)$ and $O^1 = (R(1) - R(a_1))\left(\sqrt{1-a}, \sqrt{a}\right)$, see figure 2, and on $\mathcal{D}_1$, set $v_1(x_1, x_2) = w^{a_1}(\rho_1(x_1, x_2))$. The formula

$$\bar{v}(x_1, x_2) = \begin{cases} v_0(x_1, x_2) & \text{on } \mathcal{D}_0 \\ v_1(x_1, x_2) & \text{on } \mathcal{D}_1 \end{cases}$$

defines a function $\bar{v}$ that it is a weak solution to $F(\bar{v}) \geq 0$ on $int(\mathcal{D}_0 \cup \mathcal{D}_1)$.

Iterating this procedure, we construct a function v on a domain ω, as in figure 2, solving (15).

Condition (14) implies that the domain ω is sufficiently small.

Proof of Theorem 4. Fix r; we can assume that ϵ is such that $0 < \epsilon^2/r^2 \leq \bar{t}$ and that

$$2\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)} + \frac{1}{2}\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}\left|\ln g\left(\frac{\epsilon^2}{r^2}\right)\right| \leq \frac{1}{4K}.$$

Fix the origin $O^0 = (0, \dots, 0)$, and set polar coordinates as

$$\begin{cases} x_1 = \rho \cos\theta_{N-1} \dots \cos\theta_2 \cos\theta_1 \\ x_2 = \rho \cos\theta_{N-1} \dots \cos\theta_2 \sin\theta_1 \\ \dots \\ x_N = \rho \sin\theta_{N-1}. \end{cases}$$

1) When w is a radial function, setting $a = \sin^2\theta_1$, F reduces to

$$F(w) = w_{\rho\rho}\left(1 - a + ag(w_\rho^2 a)\right) + \frac{w_\rho}{\rho}\left(N - 2 + a + (1-a)g(w_\rho^2 a)\right).$$

For $a = 1$, we seek a solution to

$$(N-1)\frac{w_\rho}{\rho} + w_{\rho\rho}g(w_\rho^2) = 0. \tag{16}$$

such that $w_\rho(R(1)) = -\epsilon/r$ and $w_\rho(\rho) < 0$, for every $\rho \in [R(1), R(1)+r)$. Consider the Cauchy problem

$$\begin{cases} \zeta' = -\dfrac{N-1}{\rho}\dfrac{\zeta}{g(\zeta^2)} \\ \\ \zeta(R(1)) = -\dfrac{\epsilon}{r}. \end{cases} \tag{17}$$

We are interested in a negative solution ζ. Define $R(1)$ to be the unique positive real solution to

$$G(-\epsilon/r) - (N-1)\ln\left(\frac{R(1)+r}{R(1)}\right) = 0,$$

i.e.

$$R(1) = \frac{r}{e^{\frac{G(-\epsilon/r)}{N-1}} - 1}.$$

Consider the unique solution ζ of (17), such that $\zeta(R(1)) = -\epsilon/r$, i.e., such that

$$G(\zeta(\rho)) - G(-\epsilon/r) = \int_{\zeta(R(1))}^{\zeta(\rho)} \frac{g(t^2)}{t}dt = \int_{R(1)}^{\rho} -\frac{N-1}{s}ds = -(N-1)\ln\left(\frac{\rho}{R(1)}\right).$$

Then, for every $\rho \in [R(1), R(1)+r)$, $G(\zeta(\rho)) > 0$ and $\zeta(\rho) < 0$, while $\zeta(R(1)+r) = 0$. Setting $w_\rho(\rho) = \zeta(\rho)$ and

$$w(\rho) = \int_{R(1)+r}^{\rho} w_\rho(s)ds,$$

we obtain that w solves (16) and, for every $\rho \in (R(1), R(1)+r)$,

$$-\epsilon/r = w_\rho(R(1)) < w_\rho(\rho) < w_\rho(R(1)+r) = 0$$

and

$$0 = w(R(1)+r) < w(\rho) < w(R(1)) \leq \epsilon.$$

2) Applying Lemma 5, we infer that the function w defined in 1) is actually a solution to

$$F(w) = w_{\rho\rho}(1-a) + \frac{w_\rho}{\rho}(N-2+a) + g(w_\rho^2 a)\left(w_{\rho\rho}a + \frac{w_\rho}{\rho}(1-a)\right) \geq 0,$$

for every $0 \leq a \leq 1$ and every $\rho \in (R(1), R(1)+r)$.

3) Let $\bar{a} < 1$. We wish to find the smallest $R(\bar{a}) > 0$ such that, setting

$$w^{\bar{a}}(\rho) = w(\rho - R(\bar{a}) + R(1)),$$

the function $w^{\bar{a}}$ is a solution to

$$F(w^{\bar{a}}) = w^{\bar{a}}_{\rho\rho}(1-\bar{a}) + \frac{w^{\bar{a}}_\rho}{\rho}(N-2+\bar{a}) + g((w^{\bar{a}}_\rho)^2\bar{a})\left(w^{\bar{a}}_{\rho\rho}\bar{a} + \frac{w^{\bar{a}}_\rho}{\rho}(1-\bar{a})\right) \geq 0, \tag{18}$$

for every $\rho \in (R(\bar{a}), R(\bar{a}) + r)$.

Since, for the function w, we have

$$(N-1)\frac{w_\rho(\rho)}{\rho} + g(w^2_\rho(\rho))w_{\rho\rho}(\rho) = 0,$$

at $\rho - R(\bar{a}) + R(1)$ we obtain

$$(N-1)\frac{w^{\bar{a}}_\rho(\rho)}{\rho - R(\bar{a}) + R(1)} + g((w^{\bar{a}}_\rho(\rho))^2)w^{\bar{a}}_{\rho\rho}(\rho) = 0.$$

This equality yields

$$(N-2+\bar{a})\frac{w^{\bar{a}}_\rho}{\rho} + \bar{a}g((w^{\bar{a}}_\rho)^2\bar{a})w^{\bar{a}}_{\rho\rho} =$$

$$\frac{w^{\bar{a}}_\rho}{\rho(\rho - R(\bar{a}) + R(1))}\left((N+2-\bar{a})(\rho - R(\bar{a}) + R(1)) - \bar{a}\rho(N-1)\frac{g((w^{\bar{a}}_\rho)^2\bar{a})}{g((w^{\bar{a}}_\rho)^2)}\right)$$

and

$$(1-\bar{a})\left(w^{\bar{a}}_{\rho\rho} + \frac{w^{\bar{a}}_\rho}{\rho}g((w^{\bar{a}}_\rho)^2\bar{a})\right) =$$

$$\frac{(1-\bar{a})w^{\bar{a}}_\rho}{\rho(\rho - R(\bar{a}) + R(1))}\left((\rho - R(\bar{a}) + R(1))g((w^{\bar{a}}_\rho)^2\bar{a}) - \frac{(N-1)\rho}{g((w^{\bar{a}}_\rho)^2)}\right).$$

Since

$$F(w^{\bar{a}}) = (N-2+\bar{a})\frac{w^{\bar{a}}_\rho}{\rho} + \bar{a}g((w^{\bar{a}}_\rho)^2\bar{a})w^{\bar{a}}_{\rho\rho} + (1-\bar{a})\left(w^{\bar{a}}_{\rho\rho} + \frac{w^{\bar{a}}_\rho}{\rho}g((w^{\bar{a}}_\rho)^2)\right),$$

we obtain that $F(w^{\bar{a}}) \geq 0$ if and only if

$$(\rho - R(\bar{a}) + R(1))\left(N-2+\bar{a} + (1-\bar{a})g((w^{\bar{a}}_\rho)^2\bar{a})\right) -$$

$$\frac{(N-1)\rho}{g((w^{\bar{a}}_\rho)^2)}\left(1-\bar{a} + \bar{a}g((w^{\bar{a}}_\rho)^2\bar{a})\right) \leq 0.$$

Set

$$k(\rho) = \quad (\rho - R(\bar{a}) + R(1))\left(N-2+\bar{a} + (1-\bar{a})g((w^{\bar{a}}_\rho)^2\bar{a})\right) -$$

$$\frac{(N-1)\rho}{g((w^{\bar{a}}_\rho)^2)}\left(1-\bar{a} + \bar{a}g((w^{\bar{a}}_\rho)^2\bar{a})\right)$$

Since

$$\frac{d}{d\rho}g((w^{\bar{a}}_\rho)^2) \leq \frac{d}{d\rho}g((w^{\bar{a}}_\rho)^2\bar{a}) \leq 0,$$

applying Lemma 5, we have that

$$k'(\rho) = \big(N-2+\bar a + (1-\bar a) g((w_\rho^{\bar a})^2 \bar a)\big) + (\rho - R(\bar a) + R(1))(1-\bar a)\frac{d}{d\rho} g((w_\rho^{\bar a})^2\bar a) -$$

$$(N-1)\left(\frac{1}{g((w_\rho^{\bar a})^2)} - \frac{\rho}{(g((w_\rho^{\bar a})^2))^2}\frac{d}{d\rho} g((w_\rho^{\bar a})^2)\right)\big(1-\bar a+\bar a g((w_\rho^{\bar a})^2\bar a)\big) -$$

$$(N-1)\frac{\rho}{g((w_\rho^{\bar a})^2)}\frac{d}{d\rho} g((w_\rho^{\bar a}(\rho))^2\bar a)\bar a \le$$

$$\frac{(N-1)\rho}{(g((w_\rho^{\bar a})^2))^2}\frac{d}{d\rho} g((w_\rho^{\bar a})^2)\big(1-\bar a+\bar a g((w_\rho^{\bar a})^2\bar a)\big) - \frac{(N-1)\rho}{g((w_\rho^{\bar a})^2)}\frac{d}{d\rho} g((w_\rho^{\bar a}(\rho))^2\bar a)\bar a \le$$

$$\frac{(N-1)\rho}{(g((w_\rho^{\bar a})^2))^2}\frac{d}{d\rho} g((w_\rho^{\bar a})^2)\big(1-\bar a+\bar a g((w_\rho^{\bar a})^2\bar a) - \bar a g((w_\rho^{\bar a})^2)\big) \le$$

$$\frac{(N-1)\rho}{(g((w_\rho^{\bar a})^2))^2}(1-\bar a) g((w_\rho^{\bar a})^2)\frac{d}{d\rho} g((w_\rho^{\bar a})^2) \le 0.$$

Since the function $k(\rho)$ is non increasing, it follows that $F(w^{\bar a}) \ge 0$, for every $\rho \in (R(\bar a), R(\bar a)+r)$, if and only if

$$R(1)\big(N-2+\bar a+(1-\bar a)g((w_\rho^{\bar a}(R(\bar a)))^2\bar a)\big) -$$

$$\frac{(N-1)R(\bar a)}{g((w_\rho^{\bar a}(R(\bar a))^2)}\big(1-\bar a+\bar a g((w_\rho^{\bar a}(R(\bar a)))^2\bar a)\big) =$$

$$R(1)\left(N-2+\bar a+(1-\bar a)g\left(\frac{\epsilon^2}{r^2}\bar a\right)\right) - \frac{(N-1)R(\bar a)}{g\left(\frac{\epsilon^2}{r^2}\right)}\left(1-\bar a+\bar a g\left(\frac{\epsilon^2}{r^2}\bar a\right)\right) \le 0,$$

i.e. if and only if

$$R(\bar a) \ge g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N-1}\frac{N-2+\bar a+g\left(\frac{\epsilon^2}{r^2}\bar a\right)(1-\bar a)}{1-\bar a+g\left(\frac{\epsilon^2}{r^2}\bar a\right)\bar a}.$$

Hence, we define

$$R(\bar a) = g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N-1}\frac{N-2+\bar a+g\left(\frac{\epsilon^2}{r^2}\bar a\right)(1-\bar a)}{1-\bar a+g\left(\frac{\epsilon^2}{r^2}\bar a\right)\bar a}.$$

4) The function $w^{\bar a}$ defined in point 3) is a solution to

$$F(w^{\bar a}) = w_{\rho\rho}^{\bar a}(1-a) + \frac{w_\rho^{\bar a}}{\rho}(N-2+a) + g((w_\rho^{\bar a})^2 a)\left(w_{\rho\rho}^{\bar a} a + \frac{w_\rho^{\bar a}}{\rho}(1-a)\right) \ge 0,$$

for every $a < \bar{a}$. Indeed, applying Lemma 5 we obtain that, for every $\rho \in (R(\bar{a}), R(\bar{a}) + r)$,

$$
\begin{aligned}
&(\rho - R(\bar{a}) + R(1)) \left(N - 2 + a + (1-a) g((w_\rho^{\bar{a}})^2 a)\right) - \\
&\frac{(N-1)\rho}{g((w_\rho^{\bar{a}})^2)} \left(1 - a + a g((w_\rho^{\bar{a}})^2 a)\right) \leq \\
&(\rho - R(\bar{a}) + R(1)) \left(N - 2 + \bar{a} + (1-\bar{a}) g((w_\rho^{\bar{a}})^2 \bar{a})\right) - \\
&\frac{(N-1)\rho}{g((w_\rho^{\bar{a}})^2)} \left(1 - \bar{a} + \bar{a} g((w_\rho^{\bar{a}})^2 \bar{a})\right) \leq 0.
\end{aligned}
$$

5) Assume we have a partition α of $[0, \pi/2]$, $\alpha = \{0 = \alpha_n < \cdots < \alpha_1 < \alpha_0 = \pi/2\}$. This partition defines two partitions of $[0,1]$, given by $c_i = \cos\alpha_i$ and $s_i = \sin\alpha_i$.

Consider the sums

$$
S_1(\alpha) = \sum_{i=1}^{n} \left(R(1-c_{i-1}^2) - R(1-c_i^2)\right) c_i = R(1)c_1 + \sum_{i=1}^{n-1} R(1-c_i^2)\,(c_{i+1} - c_i)
$$

and

$$
S_2(\alpha) = \sum_{i=1}^{n} R(s_{i-1}^2)(s_{i-1} - s_i) = R(1)(1 - s_1) + \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1}),
$$

where, in the previous equalities, we have taken into account that $R(1-c_n^2) = R(0) = 0$. Our purpose is to provide a partition α and corresponding estimates for $S_1(\alpha)$ and $S_2(\alpha)$ that are independent of ϵ.

The sums

$$
\sum_{i=1}^{n-1} R(1-c_i^2)(c_{i+1} - c_i) \quad \text{and} \quad \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1})
$$

are Riemann sums for the integrals

$$
\int_{c_1}^{1} R(1-c^2)dc \quad \text{and} \quad \int_0^{s_1} R(s^2)ds.
$$

Consider the first integral. From

$$
\frac{N - 1 - c^2 + g\left(\frac{\epsilon^2}{r^2}(1-c^2)\right)c^2}{c^2 + g\left(\frac{\epsilon^2}{r^2}(1-c^2)\right)(1-c^2)} \leq \frac{N-1}{c^2}
$$

we obtain that

$$
\begin{aligned}
\int_{c_1}^{1} R(1-c^2)dc = \quad & g\left(\frac{\epsilon^2}{r^2}\right) \frac{R(1)}{N-1} \int_{c_1}^{1} \frac{N - 1 - c^2 + g\left(\frac{\epsilon^2}{r^2}(1-c^2)\right)c^2}{c^2 + g\left(\frac{\epsilon^2}{r^2}(1-c^2)\right)(1-c^2)} dc \leq \\
& R(1) g\left(\frac{\epsilon^2}{r^2}\right) \int_{c_1}^{1} \frac{dc}{c^2}.
\end{aligned}
$$

Set

$$S_x^*(c) = \quad R(1)c + R(1)g\left(\frac{\epsilon^2}{r^2}\right)\int_c^1 \frac{db}{b^2} = R(1)\left(c + g\left(\frac{\epsilon^2}{r^2}\right)\left(\frac{1}{c} - 1\right)\right) =$$

$$R(1)g\left(\frac{\epsilon^2}{r^2}\right)\left(\frac{c}{g\left(\frac{\epsilon^2}{r^2}\right)} + \frac{1}{c} - 1\right).$$

Evaluating the last term at the minimum point $c = \sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}$, we obtain

$$S_x^*\left(\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}\right) = \quad R(1)g\left(\frac{\epsilon^2}{r^2}\right)\left(\frac{2}{\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}} - 1\right) =$$

$$\frac{2r\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}}{e^{G(\epsilon/r)} - 1} - R(1)g\left(\frac{\epsilon^2}{r^2}\right).$$

We fix $c_1 = \sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}$, so that $\alpha_1 = \arccos\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}$.

Consider the second integral

$$\int_0^{s_1} R(s^2)ds.$$

From

$$\frac{N - 2 + s^2 + g\left(\frac{\epsilon^2}{r^2}s^2\right)(1 - s^2)}{1 - s^2 + g\left(\frac{\epsilon^2}{r^2}\right)s^2} \le \frac{N - 2 + s^2 + g\left(\frac{\epsilon^2}{r^2}s^2\right)(1 - s^2)}{1 - s^2}$$

we obtain that

$$\int_0^{s_1} R(s^2)ds = \quad g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N - 1}\int_0^{s_1} \frac{N - 2 + s^2 + g\left(\frac{\epsilon^2}{r^2}s^2\right)(1 - s^2)}{1 - s^2 + g\left(\frac{\epsilon^2}{r^2}s^2\right)s^2}ds \le$$

$$g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N - 1}\int_0^{s_1} \frac{N - 2 + s^2 + g\left(\frac{\epsilon^2}{r^2}\right)(1 - s^2)}{1 - s^2}ds.$$

Set

$$S_y^*(s) = \quad R(1)(1 - s) + g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N - 1}\int_0^s \frac{N - 2 + b^2 + g\left(\frac{\epsilon^2}{r^2}\right)(1 - b^2)}{1 - b^2}db =$$

$$R(1)(1 - s) + g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N - 1}\left[\left(g\left(\frac{\epsilon^2}{r^2}\right) - 1\right)s + \frac{N - 1}{2}\ln\left(\frac{1 + s}{1 - s}\right)\right].$$

Since

$$1 - \sqrt{1 - g\left(\frac{\epsilon^2}{r^2}\right)} \le g\left(\frac{\epsilon^2}{r^2}\right),$$

evaluating the last term at the point $s_1 = \sin\alpha_1 = \sqrt{1-g\left(\frac{\epsilon^2}{r^2}\right)}$, we obtain

$$S_x^*\left(\sqrt{1-g\left(\frac{\epsilon^2}{r^2}\right)}\right) = R(1)\left(1-\sqrt{1-g\left(\frac{\epsilon^2}{r^2}\right)}\right) +$$

$$g\left(\frac{\epsilon^2}{r^2}\right)\frac{R(1)}{N-1}\left[\left(g\left(\frac{\epsilon^2}{r^2}\right)-1\right)\sqrt{1-g\left(\frac{\epsilon^2}{r^2}\right)} - \frac{1}{2}\ln\left(\frac{1-\sqrt{1-g\left(\frac{\epsilon^2}{r^2}\right)}}{1+\sqrt{1-g\left(\frac{\epsilon^2}{r^2}\right)}}\right)\right] \leq$$

$$R(1)\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}\left(\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)} + \frac{1}{2}\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}\left|\ln g\left(\frac{\epsilon^2}{r^2}\right)\right|\right).$$

To define the other points of the required partition α, consider the integrals

$$\int_{c_1}^{1} R(1-c^2)dc \quad \text{and} \quad \int_0^{s_1} R(s^2)ds.$$

Set

$$\sigma = R(1)g\left(\frac{\epsilon^2}{r^2}\right).$$

By the basic theorem of Riemann integration, taking a partition α with mesh size small enough, the value of the Riemann sums

$$\sum_{i=1}^{n-1} R(1-c_i^2)(c_{i+1}-c_i) \quad \text{and} \quad \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1})$$

differs from

$$\int_{c_1}^{1} R(1-c^2)dc \quad \text{and} \quad \int_0^{s_1} R(s^2)ds$$

by less than σ. In particular we obtain

$$\begin{aligned} S_1(\alpha) = \quad & R(1)c_1 + \sum_{i=1}^{n-1} R(1-c_i^2)\,(c_{i+1}-c_i) \leq \\ & R(1)c_1 + \int_{c_1}^{1} R(1-c^2)dc + \sigma \leq \frac{2r\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}}{e^{\frac{G(\epsilon/r)}{N-1}}-1} \leq 2Kr \end{aligned}$$

and

$$\begin{aligned} S_2(\alpha) = \quad & R(1)(1-s_1) + \sum_{i=1}^{n-1} R(s_i^2)(s_i - s_{i+1}) \leq \\ & R(1)(1-s_1) + \int_0^{s_1} R(s^2)ds + \sigma \leq \\ & \frac{r\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}}{e^{\frac{G(\epsilon/r)}{N-1}}-1}\left(2\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)} + \frac{1}{2}\sqrt{g\left(\frac{\epsilon^2}{r^2}\right)}\left|\ln g\left(\frac{\epsilon^2}{r^2}\right)\right|\right) \leq \frac{r}{4}. \end{aligned}$$

6) With respect to the coordinates fixed at the beginning of the proof, consider $x_i \geq 0$. Set

$$\mathcal{D}_0 = \{(x_1, \ldots, x_N) : R(1) < \rho < R(1) + r, \sqrt{a_1} \leq \sin\theta_{N-1} \leq \sqrt{a_0} = 1\}$$

and on $\mathcal{D}_0$ define the function

$$v_0(x_1, \ldots, x_N) = w(\rho(x_1, \ldots, x_N)).$$

By point 1), the function v_0 is of class $C^2(int(\mathcal{D}_0))$ and satisfies, pointwise, the inequality $F(v_0) \geq 0$. Having defined v_0, define v_1 as follows. Set:
for $N = 2$,

$$O_1 = (O^1_{x_1}, O^1_{x_2}) = (R(1) - R(a_1))\left(\sqrt{1-a_1}, \sqrt{a_1}\right),$$

for $N > 2$,

$$O_1(\theta_1, \ldots, \theta_{N-2}) = (O^1_{x_1}, \ldots, O^1_{x_N}) =$$

$$(R(1) - R(a_1))\left(\sqrt{1-a_1}\cos\theta_{N-2}\ldots\cos\theta_1, \sqrt{1-a_1}\cos\theta_{N-2}\ldots\sin\theta_1, \ldots, \sqrt{a_1}\right),$$

for $N \geq 2$,

$$\rho_1(x_1, \ldots, x_N) = \sqrt{(x_1 - O^1_{x_1})^2 + \cdots + (x_N - O^1_{x_N})^2}$$

and

$$\sin\theta^1_{N-1}(x_1, \ldots, x_N) = \frac{x_N - O^1_{x_N}}{\rho_1(x_1, \ldots, x_N)}.$$

Recalling the definition of w^{a_1} in 3), consider

$$\mathcal{D}_1 = \{(x_1, \ldots, x_N) : R(a_1) < \rho_1(x_1, \ldots, x_N) < R(a_1) + r,$$

$$\sqrt{a_2} \leq \sin\theta^1_{N-1}(x_1, \ldots, x_N) \leq \sqrt{a_1}\}$$

and, on $\mathcal{D}_1$, set

$$v_1(x_1, \ldots, x_N) = w^{a_1}(\rho_1(x_1, \ldots, x_N)).$$

The function v_1 is of class $C^2(int(\mathcal{D}_1))$. We claim that v_1 still satisfies $F(v_1) \geq 0$. Remark that the set of the points O^1 is equal to

$$\mathcal{O}^1 = \{(x_1, \ldots, x_N) : \rho = R(1) - R(a_1), \quad \sin\theta_{N-1} = \sqrt{a_1}\}$$

and that for every point $p \in \mathcal{D}_1$, the corresponding point $O^1(p)$ is the projection of p on $\mathcal{O}^1$, while $\rho_1(p) = d(p, \mathcal{O}^1)$. Then we obtain

$$\frac{\partial\rho_1}{\partial\theta_i} = 0 \qquad \text{for every} \quad i = 1, \ldots, N-2,$$

$$\frac{\partial O^1_{x_i}}{\partial x_i} \geq 0 \qquad \text{for every} \quad i = 1, \ldots, N-1,$$

$$\frac{\partial O^1_{x_N}}{\partial x_N} = 0.$$

and

$$\begin{aligned}
\nabla v_1 = \quad & \tfrac{w_\rho^{a_1}(\rho_1)}{\rho_1}\left(x_1 - O^1_{x_1}, \ldots, x_N - O^1_{x_N}\right) = \\
& w_\rho^{a_1}(\rho_1)\left(\cos\theta^1_{N-1}\ldots\cos\theta_1, \cos\theta^1_{N-1}\ldots\sin\theta_1, \ldots, \sin\theta^1_{N-1}\right), \\
(v_1)_{x_ix_i} = \quad & w_{\rho\rho}^{a_1}(\rho_1)\left(\frac{x_i - O^1_{x_i}}{\rho_1}\right)^2 + \frac{w_\rho^{a_1}(\rho_1)}{\rho_1}\left(1 - \left(\frac{x_i - O^1_{x_i}}{\rho_1}\right)^2 - \frac{\partial O^1_{x_i}}{\partial x_i}\right) = \\
& w_{\rho\rho}^{a_1}(\rho_1)\cos^2\theta_{N-1}\ldots\sin^2\theta_{i-1} + \\
& \frac{w_\rho^{a_1}(\rho_1)}{\rho_1}\left(1 - \cos^2\theta_{N-1}\ldots\sin^2\theta_{i-1} - \frac{\partial O^1_{x_i}}{\partial x_i}\right).
\end{aligned}$$

Then

$$\begin{aligned}
F(v_1) = \quad & \sum_{i=1}^{N-1}(v_1)_{x_ix_i} + (v_1)_{x_Nx_N}g((v_1)^2_{x_N}) = \\
& w_{\rho\rho}^{a_1}(\rho_1)\left(\cos^2\theta^1_{N-1} + \sin^2\theta^1_{N-1}g\left((w_\rho^{a_1}(\rho_1))^2\sin^2\theta^1_{N-1}\right)\right) + \\
& \frac{w_\rho^{a_1}(\rho_1)}{\rho_1}\left(N - 2 + \sin^2\theta^1_{N-1} + \cos^2\theta^1_{N-1}g\left((w_\rho^{a_1}(\rho_1))^2\sin^2\theta^1_{N-1}\right)\right) - \\
& \frac{w_\rho^{a_1}(\rho_1)}{\rho_1}\sum_{i=1}^{N-1}\frac{\partial O^1_{x_i}}{\partial x_i} \geq 0
\end{aligned}$$

since $w^{a_1}(\rho_1)$ verifies equation (18).

The sets $\mathcal{D}_0$ and $\mathcal{D}_1$ intersect on $\sin\theta_{N-1}(x_1,\ldots,x_N) = \sin\theta^1_{N-1}(x_1,\ldots,x_N) = \sqrt{a_1}$. For a point $(x_1,\ldots,x_N)$ in this intersection we have

$$\rho_1(x_1,\ldots,x_N) = \rho(x_1,\ldots,x_N) - (R(1) - R(a_1)).$$

Hence, on $\mathcal{D}_0 \cap \mathcal{D}_1$

$$R(a_1) \leq \rho_1(x_1,\ldots,x_N) \leq R(a_1) + r$$

if and only if

$$R(1) \leq \rho(x_1,\ldots,x_N) \leq R(1) + r,$$

and the functions v_0 and v_1 coincide:

$$\begin{aligned}
v_1(x_1,\ldots,x_N) = \quad & w^{a_1}(\rho_1(x_1,\ldots,x_N)) = w(\rho_1(x_1,\ldots,x_N) + R(1) - R(a_1)) = \\
& w(\rho(x_1,\ldots,x_N)) = v_0(x_1,\ldots,x_N).
\end{aligned}$$

The formula

$$\bar{v}(x_1,\ldots,x_N) = \begin{cases} v_0(x_1,\ldots,x_N) & \text{on } \mathcal{D}_0 \\ v_1(x_1,\ldots,x_N) & \text{on } \mathcal{D}_1 \end{cases}$$

defines a function $\bar{v}$ in $C^0(int(\mathcal{D}_0 \cup \mathcal{D}_1))$. We claim that it is also in $C^1(int(\mathcal{D}_0 \cup \mathcal{D}_1))$.

In fact, we have

$$\nabla v_0(x_1,\dots,x_N) = \frac{w_\rho(\rho(x_1,\dots,x_N))}{\rho(x_1,\dots,x_N)}(x_1,\dots,x_N),$$

$$\nabla v_1(x_1,\dots,x_N) = \frac{w_\rho^{a_1}(\rho_1(x_1,\dots,x_N))}{\rho_1(x_1,\dots,x_N)}\left(x_1 - O^1_{x_1},\dots,x_N - O^1_{x_N}\right)$$

and, on $\mathcal{D}_0 \cap \mathcal{D}_1$,

$$\frac{1}{\rho_1(x_1,\dots,x_N)}\left(x_1 - O^1_{x_1},\dots,x_N - O^1_{x_N}\right) = \frac{1}{\rho(x_1,\dots,x_N)}(x_1,\dots,x_N).$$

On $int(\mathcal{D}_0)$ and $int(\mathcal{D}_1)$, the function $\bar v$ is of class C^2 and satisfies, pointwise, the inequality $F(\bar v) \geq 0$. We claim that $\bar v$ is also in $W^{1,2}(int(\mathcal{D}_0 \cup \mathcal{D}_1))$ and that it is a weak solution to $F(\bar v) \geq 0$ on $int(\mathcal{D}_0 \cup \mathcal{D}_1)$. In fact, for every $\eta \in C_0^\infty(int(\mathcal{D}_0 \cup \mathcal{D}_1))$, applying the divergence theorem separately to $int(\mathcal{D}_0)$ and to $int(\mathcal{D}_1)$, we obtain

$$\int_{int(\mathcal{D}_0\cup\mathcal{D}_1)} [\mathrm{div}\nabla_{\nabla v} L(\nabla \bar v(x))\eta(x) + \langle \nabla L(\nabla \bar v(x)), \nabla\eta(x)\rangle]\, dx =$$

$$\int_{int(\mathcal{D}_0)\cup int(\mathcal{D}_1)} [\mathrm{div}\nabla_{\nabla v} L(\nabla \bar v(x))\eta(x) + \langle \nabla L(\nabla \bar v(x)), \nabla\eta(x)\rangle]\, dx =$$

$$\int_{\partial(int(\mathcal{D}_0))} \eta(x)\langle \nabla L(\nabla \bar v(x)), \mathbf{n}_0(x)\rangle dl + \int_{\partial(int(\mathcal{D}_1))} \eta(x)\langle \nabla L(\nabla \bar v(x)), \mathbf{n}_1(x)\rangle dl =$$

$$\int_{\mathcal{D}_0\cap\{\sin\theta_{N-1}=\sqrt{a_1}\}} \eta(x)\langle \nabla L(\nabla \bar v(x)), \mathbf{n}_0(x)\rangle dl +$$

$$\int_{\mathcal{D}_1\cap\{\sin\theta^1_{N-1}=\sqrt{a_1}\}} \eta(x)\langle \nabla L(\nabla \bar v(x)), \mathbf{n}_1(x)\rangle dl,$$

where $\mathbf{n}_0$, $\mathbf{n}_1$ are the outer unit normal respectively to $\partial(int(\mathcal{D}_0))$, $\partial(int(\mathcal{D}_1))$. The last term equals zero, since $\bar v \in C^1(int(\mathcal{D}_0 \cup \mathcal{D}_1))$ and $\mathbf{n}_0 = -\mathbf{n}_1$ on $\mathcal{D}_0 \cap \{\sin\theta_{N-1} = \sqrt{a_1}\} = \mathcal{D}_1 \cap \{\sin\theta^1_{N-1} = \sqrt{a_1}\}$. Hence, when $\eta \geq 0$, we have that

$$\int_{int(\mathcal{D}_0\cup\mathcal{D}_1)} \langle \nabla L(\nabla \bar v(x)), \nabla\eta(x)\rangle dx \leq 0,$$

as we wanted to show.

Assuming defined $O^{n-2}(\theta_1,\dots,\theta_{N-2})$ and a function $v \in C^1(int(\mathcal{D}_0 \cup \dots \cup \mathcal{D}_{n-2}))$, consider

$$O^{n-1}(\theta_1,\dots,\theta_{N-2}) = (O^{n-1}_{x_1},\dots,O^{n-1}_{x_N}) =$$

$$O^{n-2}(\theta_1,\dots,\theta_{N-2}) + (R(a_{n-2}) - R(a_{n-1}))$$

$$\left(\sqrt{1-a_{n-1}}\cos\theta_{N-2}\dots\cos\theta_1, \sqrt{1-a_{n-1}}\cos\theta_{N-2}\dots\sin\theta_1, \dots, \sqrt{a_{n-1}}\right).$$

Set

$$\rho_{n-1}(x_1,\dots,x_N)=\sqrt{(x_1-O_{x_1}^{n-1})^2+\cdots+(x_N-O_{x_N}^{n-1})^2},$$

$$\sin\theta_{N-1}^{n-1}(x_1,\dots,x_N)=\frac{x_N-O_{x_N}^{n-1}}{\rho_{n-1}(x_1,\dots,x_N)}$$

$$\begin{aligned}\mathcal{D}_{n-1}= \quad &\{(x_1,\dots,x_N): R(a_n)<\rho_{n-1}(x_1,\dots,x_N)<R(a_{n-1})+r,\\ &0=\sqrt{a_n}\le\sin\theta_{N-1}^{n-1}(x_1,\dots,x_N)\le\sqrt{a_{n-1}}\}\end{aligned}$$

and define on $\mathcal{D}_{n-1}$ the function

$$v_{n-1}(x_1,\dots,x_N)=w^{a_{n-1}}(\rho_{n-1}(x_1,\dots,x_N)).$$

Set $\mathcal{D}=int(\mathcal{D}_0\cup\cdots\cup\mathcal{D}_{n-1})$, the same considerations as before imply that the function

$$\bar v(x_1,\dots,x_N)=\begin{cases} v_0(x_1,\dots,x_N) & \text{on } \mathcal{D}_0\\ \dots & \dots\\ v_{n-1}(x_1,\dots,x_N) & \text{on } \mathcal{D}_{n-1}\end{cases}$$

is such that $\bar v\in W^{1,2}(\mathcal{D})\cap C^1(\mathcal{D})\cap C(\overline{\mathcal{D}})$ and it is a weak solution to $F(\bar v)\ge 0$ on $\mathcal{D}$. This completes the construction of $\bar v$ as a weak solution to $F(\bar v)\ge 0$ on $\mathcal{D}_0\cup\cdots\cup\mathcal{D}_{n-1}$.

Set $O^*=(O^*_{x_1},\dots,O^*_{x_N})=(0,\dots,0,R(1)-l_N)$. We have that

$$\begin{aligned}\mathcal{D}_0\cup\cdots\cup\mathcal{D}_{n-1}\subset \quad &\{(x_1,\dots,x_N): 0\le x_i\le l+r \text{ for } i=1,\dots,N-1,\\ &O^*_{x_N}\le x_N\le O^*_{x_N}+l_N+r)\}\,.\end{aligned}$$

Define the full domain ω and the solution by symmetry with respect to the point O^*. Figure 2 shows this construction in dimension $N=2$ and for $n-1=2$. Hence the solution will be in $W^{1,2}(\omega)\cap C^1(\omega)\cap C(\overline{\omega})$ and a weak solution of $F(v)\ge 0$ on ω.

7) The previous construction yields a region $\mathcal{A}$ centered in O^*, a corresponding region ω and a function v that solves (15). The change of coordinates $\hat x_1=x_1,\dots,\hat x_{N-1}=x_{N-1}$, $\hat x_N=x_N-O^*_N$, centers $\mathcal{A}$ at the origin and proves the theorem.

□

In order to prove Theorem 3, we need this further lemma.

Lemma 6. *Consider the sets $\mathcal{A}$ and $\mathcal{R}(O^*,l,l_N)$, where $\mathcal{A}$, O^*, l, l_N have been defined in Theorem 3. Then, for every $p\in\partial\mathcal{R}$,*

$$d(p,\overline{\mathcal{A}})<l_N.$$

Proof. Set $q=O^*+(l,\dots,l,l_N)$. We prove that

$$d(q,\overline{\mathcal{A}})<l_N.$$

Set $p_i=O^*+(0,\dots,0,l,0,\dots,0)$ and let Π^{N-1} the hyperplane passing through $p_1,\dots,p_N$. Since $\overline{\mathcal{A}}$ is convex and $p_i\in\overline{\mathcal{A}}$, we obtain that

$$d(q,\overline{\mathcal{A}})<d(q,\Pi^{N-1})<l_N.$$

See Figure 3.

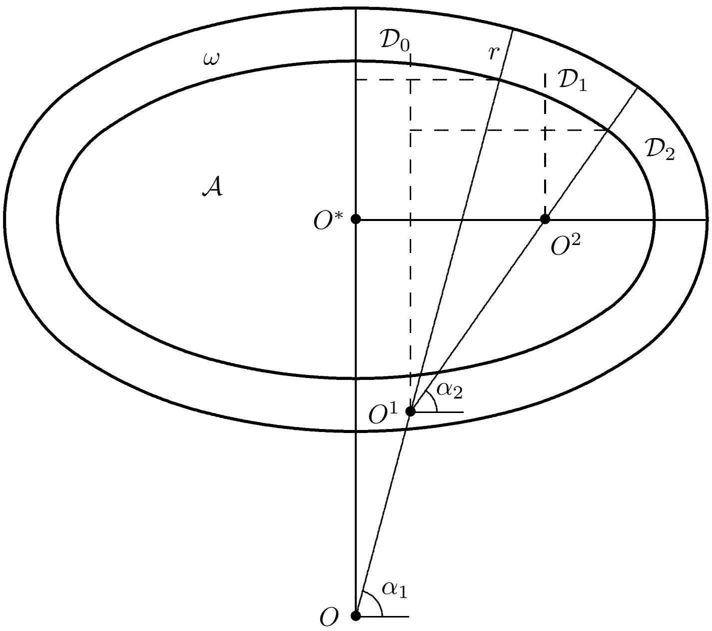

FIGURE 2. The sets ω and $\mathcal{A}$ in the case $N = 2$ and $n = 3$.

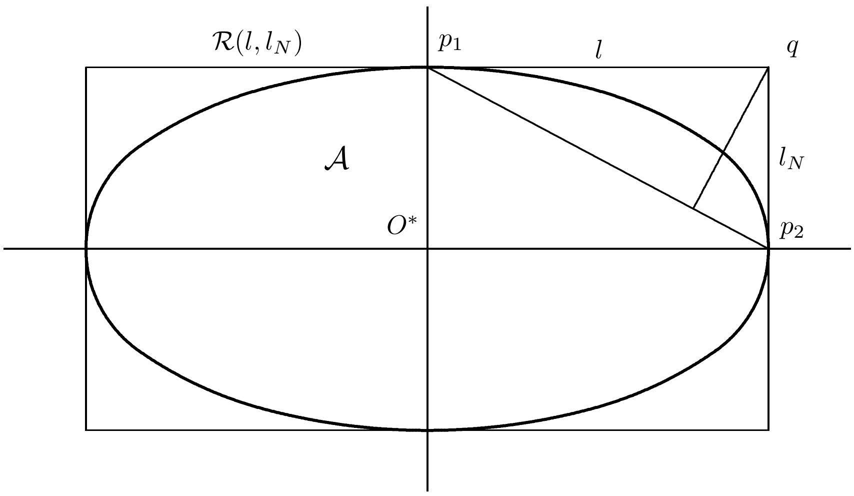

FIGURE 3. $\mathcal{A}$ and $\mathcal{R}(O^*, l, l_N)$ in the case $N = 2$.

□

Proof of Theorem 3. a) Suppose that u attains its minimum in Ω, and assume $\min_\Omega u = 0$ and set $\mathcal{C} = \{x \in \Omega : u(x) = 0\}$. By contradiction, suppose that the open set $\Omega \setminus \mathcal{C} \neq \emptyset$.

b) Since Ω is a connected set, there exist $s \in \mathcal{C}$ and $R > 0$ such that $B(s,R) \subset \Omega$ and $B(s,R) \cap (\Omega \setminus \mathcal{C}) \neq \emptyset$. Let $p \in B(s,R) \cap (\Omega \setminus \mathcal{C})$. Consider the line $\overline{ps}$. Moving p along this line, we can assume that $B(p, d(p,\mathcal{C})) \subset (\Omega \setminus \mathcal{C})$, and that there exists a point $z \in \mathcal{C}$ such that $d(p,\mathcal{C}) = d(p,z)$.

c) Fix r:

$$0 < r < \frac{d(p,\mathcal{C})}{32(N-1)K^2 + \frac{7}{8}}.$$

Set

$$\epsilon(r) = \min\left\{u(z) : z \in \overline{B\left(p, d(p,\mathcal{C}) - \frac{r}{4}\right)}\right\},$$

we have that $\epsilon(r) > 0$, and we set $\epsilon = \min\{\epsilon(r), r\bar{\xi}\}$.

d) For r and ϵ as defined in c), consider: l, l_N, $\mathcal{A}$ and v as defined in Theorem 4. Without loss of generality, since the set $\mathcal{A}$ is symmetric with respect to both coordinate axis, we can suppose that $\overline{pz}$ belongs to the first quadrant, i.e. that, for every $i = 1, \ldots, N$, $z_i \geq p_i$, where $z = (z_1, \ldots, z_N)$ and $p = (p_1, \ldots, p_N)$.

Define the point q on the segment $\overline{pz}$ such that $d(q,p) = d(p,z) - \dfrac{r}{2}$. Set $q^* = q - (l, \ldots, l, l_N)$, $\mathcal{R}(q^*, l, l_N) = q^* + \mathcal{R}(l, l_N)$, $\mathcal{A}^* = q^* + \mathcal{A}$ and $v^*(x + q^*) = v(x)$. We first claim that

$$\mathcal{R}(q^*, l, l_N) \subset B\left(p, d(p,\mathcal{C}) - \frac{r}{4}\right).$$

Let $t \in \mathcal{R}(q^*, l, l_N)$, then t can be written as $(q_1 - 2\alpha_1 l, \ldots, q_{N-1} - 2\alpha_{N-1} l, q_N - 2\alpha_N l_N)$, with $0 \leq \alpha_i \leq 1$, for $i = 1, \ldots, N$. Since $r < \frac{d(p,\mathcal{C})}{32(N-1)K^2 + \frac{7}{8}}$, we have that

$$d(t,p)^2 = \sum_{i=1}^N (q_i - 2\alpha_i l_i - p_i)^2 = d(q,p)^2 + \sum_{i=1}^N 4\alpha_i^2 l_i^2 - \sum_{i=1}^N 4\alpha_i l_i (q_i - p_i) \leq$$

$$\left(d(p,\mathcal{C}) - \frac{r}{2}\right)^2 + \sum_{i=1}^N 4 l_i^2 \leq \left(d(p,\mathcal{C}) - \frac{r}{2}\right)^2 + 16(N-1)K^2 r^2 + \frac{r^2}{4} < \left(d(p,\mathcal{C}) - \frac{r}{4}\right)^2.$$

See Figure 4.

Since $\mathcal{A}^* \subset \mathcal{R}(q^*, l, l_N)$, we have obtained that

$$\mathcal{A}^* \subset B\left(p, d(p,\mathcal{C}) - \frac{r}{4}\right),$$

so that $u \geq \epsilon$ in $\partial\mathcal{A}^*$.

By Lemma 6,

$$d(q, \overline{\mathcal{A}^*}) < l_N \leq \frac{r}{4},$$

we have that

$$d(z, \overline{\mathcal{A}^*}) \leq d(z,q) + d(q, \overline{\mathcal{A}^*}) < \frac{3}{4} r,$$

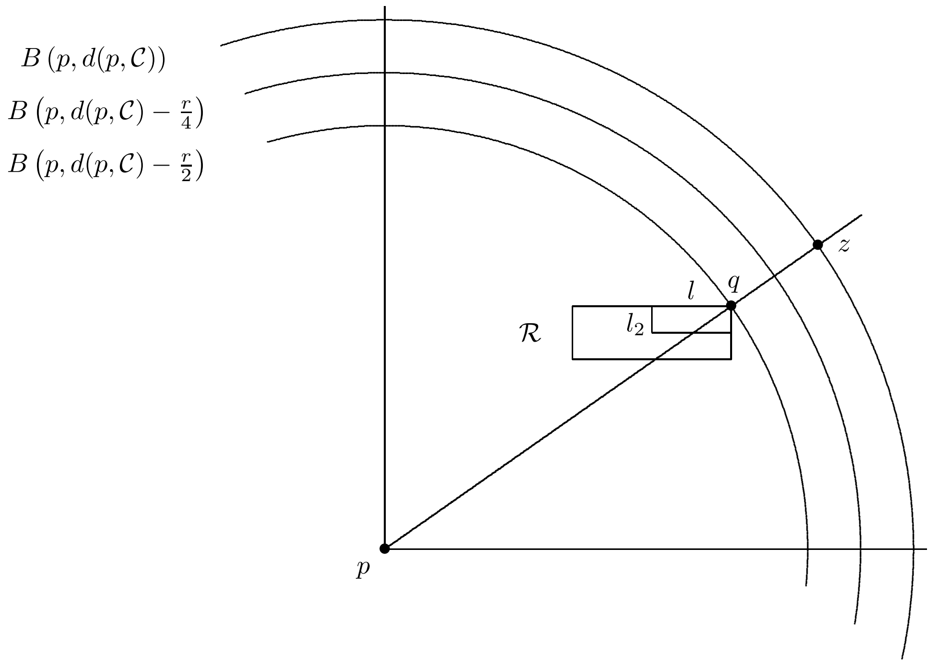

FIGURE 4. The set $\mathcal{R} = \mathcal{R}(q^*, l, l_2)$ in the case $N = 2$.

so that

$$z \in \omega^* = B(\mathcal{A}^*, r) \setminus \overline{\mathcal{A}^*}.$$

e) The function v^* satisfies

$$\begin{cases} v^* \text{ is a weak solution to } F(v) \geq 0 & \text{in } \omega^* \\ v^* > 0 & \text{in } \omega^* \\ v^* = 0 & \text{in } \partial B(\mathcal{A}^*, r) \\ v^* \leq \epsilon & \text{in } \partial\mathcal{A}^*. \end{cases}$$

Since $u, v^* \in W^{1,2}(\omega^*) \cap C(\overline{\omega^*})$, v^* is a weak subsolution and u is a weak solution to $F(u) = 0$, and $v^*_{|\partial\omega^*} \leq u_{|\partial\omega^*}$, applying Lemma 1, we obtain that $u \geq v^*$ in ω^*. But $u(z) = 0 < v^*(z)$, a contradiction.

□

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Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano
E-mail address: simone.bertone@unimib.it

Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano
E-mail address: arrigo.cellina@unimib.it

Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano
E-mail address: elsa.marchini@unimib.it