ON THE STRONG CONVERGENCE OF DERIVATIVES IN A TIME OPTIMAL PROBLEM.

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Abstract. We consider a time optimal problem for a system described by a Differential Inclusion, whose right hand side is upper semicontinuous but not necessarily convex valued. Under the assumption of strict convexity of the map obtained by convexifying the original, non convex valued map, we obtain the strong convergence of the derivatives of any minimizing sequence. The assumptions required by this result are satisfied, for instance, by the classical Brachystocrone problem and by Fermat's principle.

1. Introduction

In 1984, A. Visintin [4] proved a result implying strong convergence in L^1 of a sequence, u_n, weakly convergent to a function u_*, based on the assumption that the weak limit u_* belongs, for a.e. t, to the set of extreme points of a given set valued map $U(t)$ (related, but different results, were obtained by C. Olech, see [3]). In [1], the authors prove a result on the existence of minimum time solutions to a differential inclusions of the form

$$x'(t) \in F(x(t))$$

under suitable assumptions on the set valued map F, and apply this result to some well known problems, as the Brachistocrone Problem and Fermat's Principle. A closer examination of these problems shows that, in fact, the corresponding right hand sides satisfy conditions that are stricter than those needed simply for the validity of the main existence theorem of minimum time solutions. The purpose of the present Note is to show that, by assuming these stricter conditions (those naturally satisfied by the Brachistocrone Problem and by Fermat's Principle), one can obtain a stronger result, namely that, given any minimizing sequence $x_n(\cdot)$, the sequence of derivatives $x'_n(\cdot)$ converges *strongly* to the derivative of the (unique) minimum time solution. Besides its theoretical interest, this result is also of practical interest, since it stabilizes any approximations method used in a numerical construction. The proof of the present result combines the techniques of [1] with those of Visintin [4].

2. Main results

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In what follows we shall consider the following *minimum time problem* for solutions to a differential inclusion: X and S are closed subset of $\mathbb{R}^N$, $S \subset X$, $x^0 \in X$ and F is a set valued map. Consider the problem of reaching the target set S from x^0, satisfying the constraint $x(t) \in X$, where $x(\cdot)$ is a solution to the differential inclusion

$$x'(t) \in F(x(t)) \tag{1}$$

For this problem, we shall assume that the set of solutions reaching S is non-empty, and we shall call t_* the infimum of the times needed to pass from x^0 to S. We shall call *minimizing* a sequence (x_n) of solutions to the differential inclusion (1) such that: $x_n(0) = x^0$, $x_n(t_n) \in S$ and $t_n \downarrow t_*$.

It is our purpose to prove the following result.

Theorem 1. *Let F be a (set valued) map, defined on a closed set $X \subset \mathbb{R}^N$, with values that are non-empty and compact subsets of $\mathbb{R}^N$, such that:*

- *F is linearly bounded, i.e., there exist α and β such that, for $x \in X$ and $\xi \in F(x)$, one has $\|\xi\| \le \alpha\|x\| + \beta$*
- *For $x \in X$, the set $co(F(x))$ is strictly convex*
- *The map $x \longrightarrow co(F(x))$ is upper semicontinuous.*

Assume that there exits $\widehat{t} > 0$ and a solution $x(\cdot)$ of:

$$x'(t) \in co(F(x(t))), \qquad x(0) = x_0 \tag{2}$$

such that $x(\widehat{t}) \in S$. Then every time-minimizing sequence (x_n) converges uniformly on $I_* = [0, t_*]$, to x_*, a solution to the minimum time problem, and

$$x'_n \longrightarrow x'_* \text{ strongly in } L^1(I_*).$$

Notice that, in general without the strict convexity assumption, it is not even true that, from a time minimizing sequence, one can extract a subsequence such that the sequence of its derivatives strongly converges to the derivative of the limit; the following is an example.

Example We shall consider the costant map $F : \mathbb{R}^2 \longrightarrow P(\mathbb{R}^2)$ defined as:

$$F(x) := \{(x, y) \ : \ |x| \le 1, \ |y| \le 1\}$$

and the related minimum time problem with boundary conditions:

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is simple to see that the derivative of an optimum solution assumes only the values $\begin{pmatrix} x \\ 1 \end{pmatrix}$ reaching the final point in $t_f = 1$. Let us consider a minimizing sequence composed by minimum time solutions whose derivatives take only the two values $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and uniformly converging to the function $x(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$.

It is easy to see that we have weak convergence of the derivatives, but not strong convergence.

In the proof of our result we shall need the following lemmas:

Lemma 1. [1] *Let* $x : [0, t_*] \longrightarrow \mathbb{R}^N$ *be absolutely continuous and such that* $x'(t) = 0$ *on a subset* E *of* $[0, t_*]$ *of positive measure; let* $X = \{x(t) : t \in [0, t_*]\}$ *and let* F*, defined on* X*, be such that, for almost every* $t \in [0, t_*]$*,*

$$x'(t) \in F(x(t))$$

then, there exist τ_**,* $0 < \tau_* < t_*$ *and an absolutely continuous function* $\tilde{x} : [0, \tau_*] \longrightarrow X$*, such that* $\tilde{x}(0) = x(0)$*,* $\tilde{x}(\tau_*) = x(t_*)$ *and*

$$\tilde{x}'(t) \in F(\tilde{x}(t))$$

for almost every $t \in [0, \tau_*]$*.*

As a consequence of this lemma, a minimum time solution x is such that $x'(t) \neq 0$ a.e..

Lemma 2. *Let* Φ *be a closed and strictly convex set,* $y_* \in \Phi$ *an extremal point and* π *an hyperplane that supports* Φ *in* y_**; then* $\forall \varepsilon > 0$ $\exists \delta > 0$ *such that* $y \in [B(\pi, \delta) \cap B(\Phi, \delta)] \Longrightarrow \mathrm{dist}(y, y_*) < \varepsilon$*.*

Proof. We suppose, by contradiction, that there exists $\varepsilon > 0$ and a sequence (y_n) such that: $\mathrm{dist}(y_n, \pi) \longrightarrow 0$ and $\mathrm{dist}(y_n, \Phi) \longrightarrow 0$, but $y_n \notin B(y_*, \varepsilon)$.
If there is a converging subsequence, denoted by $\overline{y}$ its limit, we can conclude that: $\overline{y} \in (\Phi \cap \pi)$ and $\mathrm{dist}(\overline{y}, y_*) \geq \varepsilon$, a contradiction.
Otherwise we have that $(y_n - y_*) \longrightarrow \infty$.
Set $w_n := y_* + \frac{y_n - y_*}{\|y_n - y_*\|} = y_n \frac{1}{\|y_n - y_*\|} + (1 - \frac{1}{\|y_n - y_*\|}) y_* \in [B(\pi, \delta) \cap B(\Phi, \delta)]$ for large n. We have $\|w_n - y_*\| = 1$. Then, from the convexity of $B(\pi, \delta)$ and $B(\Phi, \delta)$, we apply the previous arguments to the limit of a subsequence of w_n. □

Lemma 3. *Let* $\Phi(\cdot)$ *be upper semicontinuous with closed and strictly convex values,* $x'_*(\cdot)$ *be measurable and, for a.e.* $t \in I_*$*, let* $x'_*(t)$ *be an extremal point of* $\Phi(t)$*. Set* $L(t)$ *be the set of supporting functionals at* $x'_*(t)$*, i.e.,* $L(t) := \{l :\| l \|= 1; \forall v \in \Phi(t), \langle v - x'_*(t), l \rangle \geq 0\}$ *Then, there exists a measurable selection* $\overline{l}(\cdot)$ *from* $L(\cdot)$*.*

Proof. In fact, when $\Phi(\cdot)$ and $x'(\cdot)$ are both continuous, it is easy to see that *Graph(L)* is closed. Applying Lusin's theorem we infer the existence of a sequence (E_n) of compact sets such that: $m\left([0, t_*] \setminus \bigcup E_n\right) = 0$ and the restrictions of $\Phi(\cdot)$ and of $x'(\cdot)$ to each E_n are both continuous, so that the restriction of the map L to each E_n has closed graph. Since the image of L is contained in $B(0, 1)$, it follows that $L(\cdot)$ is upper semicontinuous on each E_n. Then, we can apply the selection theorem of Kuratowski and Ryll-Nardzewski [2] to obtain a measurable selection $\overline{l}(\cdot)$ on the whole $[0, t_*]$. □

Lemma 4. *Let* $\Phi(\cdot)$ *be upper semicontinuous with closed and strictly convex values,* $x'_*(\cdot)$ *be measurable and, for a.e.* $t \in I$*, let* $x'_*(t)$ *be an extremal point of* Φ*. Let* $\overline{l}(\cdot)$ *be a measurable selection from the set of supporting functionals at* $x'_*(t)$*. Then, for every* $\varepsilon > 0$*, there exists a measurable and positive function* $\delta_\varepsilon(\cdot)$ *such that*

$$|\langle y - x'_*(t), \overline{l}(t) \rangle| \leq \delta_\varepsilon(t) \text{ and } y \in B(\Phi, \delta_\varepsilon(t)) \text{ imply } dist(y, x'_*(t)) < \varepsilon.$$

Proof. Let $\pi(t)$ the unique plane orthogonal to $\overline{l}(t)$ through $x'_*(t)$, so that $|\langle y - x'_*(t), \overline{l}(t) \rangle| = \mathrm{dist}(y, \pi(t))$ and apply Lemma 3 □

Lemma 5. *Let $\delta : [0, t_*] \longmapsto R$ be a measurable function such that $\delta(t) > 0$ for almost every $t \in [0, t_*]$. For every $\sigma > 0$, there exists $\eta > 0$ such that: $E \subset [0, t_*]$ and $m(E) \geq \sigma$ imply $\int_E f(t)\, dt > \eta$.*

Proof. Set $S_n := \{t \in [0, t_*] : \frac{1}{n+1} \leq f(t) < \frac{1}{n}\}$. Since $f(t) > 0$ on $[0, t_*]$, there exists N such that:

$$\sum_{n \geq N} m(S_n) < \frac{\sigma}{2}.$$

Let $S := \bigcup_{n \geq N} S_n$ and $\eta = \frac{\sigma}{2N}$. Then:

$$\int_E f(t)\, dt = \int_{E \cap S} f(t)\, dt + \int_{(E \cap S)^C} f(t)\, dt > \int_{(E \cap S)^C} f(t)\, dt > \frac{\sigma}{2N} = \eta.$$

□

Proof of Theorem 1. **(a)** Under our assumptions there exists a minimum time solution to the convexified problem (2) (see [1]) defined on $[0, t_*]$. As in [1], we want to show that this function is also a minimum time solution to our problem.
We define, for all $t \in [0, t_*]$:

$$R(t) := \{r \geq 1 \mid r \cdot x_*'(t) \in co(F(x_*(t)))\} \text{ and } r(t) := \sup\ R(t).$$

by Lemma 1, $x_*'(t) \neq 0$ for almost every t so that $r(t)$ is well defined. Moreover, by the compactness of $co(F(x_*(t)))$, $r(t)$ is actually a maximum for every t.
Hence $r(t)x_*'(t) \in \partial(co(F(x_*(t))))$. From the assumption of strictly convexity, we have $r(t)x_*'(t) \in extr(co(F(x_*(t))))$, so that:

$$r(t)x_*'(t) \in F(x_*(t)).$$

Define the absolutely continuous function $s : [0, t_*] \longrightarrow [0, s_*]$ by setting $s(0) = 0$, $s'(t) = 1/r(t)$. It is clearly an increasing map and $s_* \leq t_*$.
Let $t = t(s)$ be its inverse and consider the map $\widehat{x}(s) := x_*(t(s))$. We obtain, in particular, that:

$$\widehat{x}(0) = x_*(t(0)) = x_*(0) = x_0$$
$$\widehat{x}(s_*) = x_*(t(s_*)) = x_*(t_*) \in S$$

We have also that:

$$\frac{d}{ds}\widehat{x}(s) = x_*'(t(s))t'(s) = x_*'(t(s))\frac{1}{s'(t(s))} = x_*'(t(s))r(t(s)) \in F(x_*(t(s))) = F(\widehat{x}(s)),$$

Hence $\widehat{x}(\cdot)$ is a solution to the original differential inclusion.
Since $s_* \leq t_*$, $\widehat{x}(\cdot)$ is also a minimum time solution to the convexified problem; from the definition of t_* we obtain that $s_* = t_*$, so that $s'(t) = 1$ for a.e.t. Hence

$$\widehat{x}(\cdot) \equiv x_*(\cdot). \tag{3}$$

(b) Let $(x_n)_n$ be a minimizing sequence of solutions to the problem, such that $x_n(t_n) \in S$. We suppose that the sequence $\{t_n\}_n$ is decreasing to t_*. An application of the Ascoli-Arzelà's theorem yields the uniform convergence of a subsequence, still denoted x_n, on $[0, t_*]$ to x_{**}, a minimum time solution to the convexified problem. By (3), x_{**} is also a minimum time solution to the original problem. Hence, by the uniqueness of solutions to the minimum time problem, $x_{**} = x_*$, and the whole sequence converges uniformly to x_*. Since the x_n' are equibounded, we also have that:

$$x_n' \rightharpoonup x_*' \text{ weakly in } \quad L^1.$$

From point **(a)**: $x'_*(t) \in extr(co(F(x_*(t))))$ for a.e.$t \in [0, t_*]$.
Since the map $x \longrightarrow co(F(x))$ is upper semicontinuous, we have that

$$\forall \varepsilon > 0 \; \exists \; \delta > 0 \;\; \text{such that } y \in B(x, \delta) \text{ implies } co(F(y)) \subset B(co(F(x)), \varepsilon)$$

For every fixed t, from the uniform convergence of the sequence x_n to x_*, we have that $x_n(t) \in B(x_*(t), \delta)$ for large n, so that:

$$x'_n(t) \in co(F(x_n(t))) \subset B(co(F(x_*(t))), \varepsilon)$$

this proves that, for every fixed t:

$$\text{dist}(x'_n(t), co(F(x_*(t))) := \min_{w \in co(F(x_*(t)))} (\text{dist}(x'_n(t), w)) \longrightarrow 0 \tag{4}$$

Whenever $co(F(x_*(t)) = \{x'_*(t)\}$, (3) shows that $x'_n(t)$ converges to $x'_*(t)$. Otherwise, applying Lemma 3, let $\bar{l}(\cdot)$ be a measurable selection from $L(\cdot)$. For every t such that $x'_n(t)$ and $x'_*(t)$ exist, set $u_n(t) := \langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle$ and denote by $v_n(t)$ the vector of $co(F(x_*(t))$ such that: $\text{dist}(x'_n(t), co(F(x_*(t))) = \text{dist}(x'_n(t), v_n(t))$ and by $(u_n)^-(t) := \min\{0, \langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle\}$. We have:

$$|(u_n)^-(t)| = |(\langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle)^-| = |(\langle x'_n(t) - v_n(t), \bar{l}(t) \rangle + \langle v_n(t) - x'_*(t), \bar{l}(t) \rangle)^-|.$$

We notice that $\langle v_n(t) - x'_*(t), \bar{l}(t) \rangle > 0$, hence:

$$|(\langle x'_n(t) - v_n(t), \bar{l}(t) \rangle + \langle v_n(t) - x'_*(t), \bar{l}(t) \rangle)^-| \leq |(\langle x'_n(t) - v_n(t), \bar{l}(t) \rangle)^-| \leq$$

$$\leq |\langle x'_n(t) - v_n(t), \bar{l}(t) \rangle| \leq \text{dist}(x'_n(t), co(F(x_*(t))).$$

Then, from (3), we can conclude that $(u_n)^-(t) \longrightarrow 0$ for almost every t.
The sequence (x'_n) is bounded in L^∞ and so it is also the sequence $(u_n)^-$, so that $\int_0^{t_*} |u_n^-(t)| \, dt \longrightarrow 0$. Hence:

$$\int_0^{t_*} |u_n(t)| \, dt = \int_0^{t_*} (u_n^+ - u_n^-)(t) \, dt = \int_0^{t_*} (u_n^+ + u_n^-)(t) \, dt - 2 \int_0^{t_*} u_n^-(t) \, dt =$$

$$\int_0^{t_*} u_n(t) \, dt + 2 \int_0^{t_*} |u_n^-(t)| \, dt.$$

On the other hand, from the weak convergence of x'_n to x'_*, we have that $\int_0^{t_*} u_n(t) \, dt \longrightarrow 0$, so that we conclude that

$$\int_0^{t_*} |u_n(t)| \, dt = \int_0^{t_*} |\langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle| \, dt \longrightarrow 0.$$

Let M be an upper bound for $|\langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle|$ and for $\text{dist}(x'_n(t), co(F(x_*(t))))$. From 4, we infer that $\text{dist}(x'_n(\cdot), co(F(x_*(t)))$ converges to zero in L^1 as well. Fix ε. Consider the function $\hat{\delta}(\cdot) = \delta_{\frac{\varepsilon}{2t_*}}(\cdot)$ as supplied by the Corollary to Lemma 2, so that

$$|\langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle| \leq \hat{\delta}(t) \text{ and } \text{dist}(y, co(F(x_*(t)))) \leq \hat{\delta}(t) \Longrightarrow \text{dist}(y, x'_*(t)) \leq \frac{\varepsilon}{2t_*}.$$

Set

$$I_n^1 = \{t : |\langle x'_n(t) - x'_*(t), \bar{l}(t) \rangle| \geq \hat{\delta}(t)\}$$

and

$$I_n^2 = \{t : \text{dist}(x'_n(t), co(F(x_*(t))) \geq \hat{\delta}(t)\}.$$

Since

$$\int_0^{t_*} \operatorname{dist}(x_n'(t), co(F(x_*(t)))\, dt \geq \int_{I_n^2} \operatorname{dist}(x_n'(t), co(F(x_*(t)))\, dt \geq \int_{I_n^2} \hat{\delta}(t)\, dt$$

by Lemma 5 we infer that $m(I_n^2) \to 0$, and similarly for I_n^1. Hence, there exists N such that, for $n \geq N$, $m(I_n^1) \leq \frac{M\varepsilon}{4}$ and $m(I_n^2) \leq \frac{M\varepsilon}{4}$. Applying Lemma 4 to $\Phi(t) = co(F(x_*(t))$ and $y = x_n'(t)$, we obtain

$$\int_0^{t_*} \operatorname{dist}(x_n'(t), x_*'(t))\, dt =$$

$$\int_{[0,t_*]\setminus(I_n^1 \cup I_n^2)} \operatorname{dist}(x_n'(t), x_*'(t))\, dt + \int_{(I_n^1 \cup I_n^2)} \operatorname{dist}(x_n'(t), x_*'(t))\, dt$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

□

In the theory of existence of solutions to differential inclusions, the convexity of the right hand side is used to pass, through Mazur's Lemma, from the weak convergence of derivatives to the strong convergence of a sequence of convex combinations. The previous result shows that in some cases, it might be possible to obtain the existence of solutions using the strong convergence of the derivatives.

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