Strict Convexity, Comparison Results and Existence of Solutions to Variational Problems

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Abstract. The aim of this paper is to discuss the assumption of strict convexity in problems of the the Calculus of Variations, and to present some results that avoid introducing this assumption.

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1. Introduction

We consider variational problems of the kind

$$\text{Minimize } \int_\Omega [f(\nabla u(x)) + g(u(x))]dx \text{ on } u - u_0 \in W_0^{1,1}(\Omega). \tag{1.1}$$

Here f is a convex function, possibly extended valued, whose effective domain, the set of points where f takes finite values, is denoted by $Dom(f)$ and whose epigraph is denoted $epi(f)$. Since this paper is concern with the implications of strict convexity, or of the lack of it, let us state precisely what we mean by a strictly convex function.

Definition 1.1. We shall call a convex function f *strictly convex* if for every $x \in Dom(f)$, the point $(x, f(x))$ is an extremal point of $epi(f)$.

We shall call a function u a solution to a variational problem if it yields a *finite* minimum to the problem.

The use of strict convexity appears in its most immediate form when we consider a minimization problem depending only on ∇u: consider the problem of minimizing

$$\int_{\Omega} f(\nabla u(x))dx \tag{1.2}$$

with appropriate boundary conditions, and assume to have two solutions v and w, so that

$$\int_{\Omega} f(\nabla v(x))dx = \int_{\Omega} f(\nabla w(x))dx = min.$$

Then the function $u = \frac{1}{2}v + \frac{1}{2}w$ satisfies the same boundary conditions and is such that $\nabla u(x) = \frac{1}{2}\nabla v(x) + \frac{1}{2}\nabla w(x)$ a.e. in Ω. By convexity, u is also a solution, since

$$\int_{\Omega} f(\nabla u(x))dx \leq \frac{1}{2}\int_{\Omega} f(\nabla v(x))dx + \frac{1}{2}\int_{\Omega} f(\nabla w(x))dx = min.$$

Then, we must have that, a.e. in Ω, $f(\nabla u(x)) = \frac{1}{2}f(\nabla v(x)) + \frac{1}{2}f(\nabla w(x))$ and the point $(\nabla(u(x)), f(\nabla u(x)))$ is not an extreme point of $epi(f)$, unless $\nabla u(x) = \nabla v(x) = \nabla w(x)$ a.e. in Ω. Hence strict convexity yields *uniqueness* of the solution. The remainder of the paper will discuss other implications of strict convexity, or of the lack of it.

2. Existence of solutions

The main method for proving the existence of solutions is, as it is well known, the Direct Method. This method demands the assumption of convexity for f, to prove the weak lower semicontinuity of the integral functional, but no use is made of strict convexity. The proof is a combination of growth assumptions (superlinear growth, to have weak compactness), and of convexity, to pass to the limit with the values of the functional. Here the assumption of strict convexity not only is of no use, but would actually prevent proving the existence of solutions to those minimum problems whose Lagrangeans are generated from the convexification of Lagrangeans that were not, originally, convex, as sometimes it happens in the applications. To the other extreme of the range of the existence theorems, there is the result for the existence of solutions to the *least area problem*, in the parametric case. Here again a region Ω is given (in general, it suffices to consider $\Omega \subset \mathbb{R}^2$); on Ω a function ϕ is defined, the boundary condition, and one wishes to minimize

$$\int_{\Omega} \sqrt{1 + \|\nabla u(x)\|^2}dx \text{ on } u - \phi \in W_0^{1,1}(\Omega).$$

The map $\sqrt{1 + \|\xi\|^2}$ does not grow superlinearly and, due to the lack of weak compactness, the Direct Method cannot be applied. Still, this famous problem, a special case of Plateau's Problem, has been thoroughly investigated and a nice existence theorem has been proved for it. This theorem takes advantage of the special properties of the function $\sqrt{1 + \|\xi\|^2}$ and, above all, of the fact that it is strictly convex. The (rather delicate) existence proof can be found, e.g. in [4]. Our purpose here is simply to point out the parts of this proof that are related to the strict convexity and to comparison results.

The first step is to derive *a priori* estimates for the solution. One has

Proposition 2.1. *Let f be strictly convex, let ϕ be continuous on $\overline{\Omega}$. Let u be a solution to the minimization problem (1.2) and let $C = \sup_{\xi \in \partial\Omega} \phi(\xi)$. Then, a.e. in Ω, one has*

$$u(x) \leq C.$$

Proof. In fact, assume that there exists a set $E \subset \Omega$ and a constant $\delta > 0$, such that $u(x) \geq C + \delta$ on E. Consider $v(x) = C$ and $w(x) = C + (u(x) - C)^+$: they both are solutions to (1.2) with boundary data C, and they are different on E. This is a contradiction to the uniqueness provided by the strict convexity.

□

Corollary 2.2. *Let u be a solution to problem* (1.2)*, where f is strictly convex and assume that u is enough regular. Then* $\sup u$ *and* $\inf u$ *are assumed at $\partial\Omega$.*

The next step in the existence proof is to consider two solutions, u and v (regular, continuous on $\overline{\Omega}$)), each with its own boundary data. Again as a consequence of strict convexity, we obtain

Proposition 2.3. *Let u and v solutions, such that at $\partial\Omega$, $u(x) \leq v(x)$. Then, $u(x) \leq v(x)$ on Ω*

Corollary 2.4. $\sup(u - v)$ *and* $\inf(u - v)$ *are attained on $\partial\Omega$.*

Proof. Let $C_b = \sup_{x \in \partial\Omega} u(x) - v(x)$ so that, at $\partial\Omega$, $u(x) \leq v(x) + C_b$. Since $w(x) = v(x) + C_b$ is also a solution, the previous Proposition applies. □

Corollary 2.5. *The supremum of the Lipschitz constant of a solution*

$$\sup\{\frac{|u(x) - u(y)|}{\|x - y\|} : x, y \in \Omega; x \neq y\}$$

is attained when one of the points x or y is at the boundary of Ω.

Proof. Fix a vector k and consider $u(x + k)$. It is defined whenever $x \in \Omega - k$, so that the two functions $u(x)$ and $u_k(x) = u(x + k)$ have $\Omega_k = \Omega \cap (\Omega - k)$ as a common domain of definition. Both are solutions for the minimization problem on Ω_k, each with its own boundary data at $\partial\Omega_k$. So, $\sup(u - u_k)$ and $\inf(u - u_k)$ are attained on $\partial(\Omega - k)$. One of the two points is on $\partial\Omega$. □

Having established this fundamental fact, then the existence proof proceeds by showing that one can buid a "barrier" for the solution. Our purpose here was to point out that:

Remark 2.6. The previous steps, needed to obtain the property of a solution stated in the Proposition above, all depend only on f being strictly convex, and not on other properties of f.

However, consider the following example

Example. Let Ω be the interval $[-1, 1]$; let

$$f(\xi) = \begin{cases} 0 & |\xi| \leq 1 \\ (|\xi| - 1)^2 & |\xi| > 1 \end{cases}$$

and consider the problem

$$\text{minimize} \int_{[-1,1]} f(u'(x))\, dx; \qquad u \in W_0^{1,1}([-1, 1]).$$

The function $v(x) \equiv 0$ and $w(x) = -|x| + 1$ are solutions, $w \leq v$ at $\partial\Omega$ but it is not true that $w \leq v$ on Ω.

Moreover, it is not true that $\sup u$ and $\inf u$ are assumed at $\partial\Omega$. In addition, again considering the two solutions v and w, it is not true that the $\sup(w - v)$ is attained at the boundary.

Hence, without the assumption of strict convexity for f, the whole approach to this existence ought to be carefully checked.

Remark 2.7. Sometimes, one finds the suggestion: in case f is not strictly convex, simply add $\varepsilon\|\xi\|^2$ to it. This will make the new f strictly convex; you can use the results for strict convexity and pass these results to the limit.

Let us add $\varepsilon\|\xi\|^2$ to the Lagrangean of the previous Example: now we wish to minimize

$$\int_\Omega f_\varepsilon(\nabla u(x))dx \tag{2.1}$$

where $f_\varepsilon(\xi) = \varepsilon\|\xi\|^2$ when $\|\xi\| \leq 1$, $= +\infty$ otherwise. When the boundary condition is such that $u_0 \leq 0$ at $\partial\Omega$, it is true that *every* solution v to (2.1) is such that $v \leq 0$ a.e. in Ω, but it is *not* true that *every* solution v to the limit problem is such that $v \leq 0$ a.e. in Ω. At most, one can say that, for the limit problem, *there exists a solution* v satisfying $v \leq 0$ a.e. in Ω. However, for the purpose of obtaining *a priori* estimates, this statement is not strong enough. In fact, consider the least area problem of minimizing

$$\int_\Omega \sqrt{1 + \|\nabla u(x)\|^2} dx. \tag{2.2}$$

By adding $\varepsilon\|\nabla u(x)\|^2$ to the Lagrangean, we obtain a strictly convex coercive functional, so that solutions exist no matter what the boundary data and Ω are (sufficiently regular). However, it is not true that for the limiting problem (the least area problem), one has existence of solutions no matter what the boundary data and Ω are.

3. Comparison results without strict convexity

If one carefully examines the line of thought followed in the existence proof in the previous paragraph, one finds that strict convexity has been used to ensure a comparison result between solutions; in some cases, while one of the solutions was unknown, the other was simply a constant. It might be possible to have existence theorems based on the construction of barriers without the assumption of strict convexity; one should carefully reconsider the comparison results. For some purposes, it would be enough to obtain results when one of the two belongs to a special class of solutions, for instance, the affine functions, and one might aim at results for this more restricted class of problems. In [1] a comparison result was proved. This result concerns a special class of solutions to the minimum problem 1.2).

The class of solutions to be considered is described below. It is a generalization, of the class of affine maps. Before introducing it, let us revisit the affine maps. Consider

$$\langle a, x - x^0 \rangle + r :$$

we have

$$\langle a, x-x^0 \rangle + r = \langle x-x^0, a \rangle + r = \sup_z \{ \langle x-x^0, z \rangle - I_{\{a\}}(x) \} + r = \left(I_{\{a\}} \right)^* (x-x^0) + r.$$

When a function f is not strictly convex, its graph contains, so to say, some "flat" parts; in turn, this is reflected on the polar of f, whose graph contains angles, i.e., the subdifferential of the polar is not single-valued whenever f is not strictly convex. This is essential in the definition of the class of solutions we want to consider.

Definition 3.1. For $z \in Dom(f^*)$, $x^0 \in \mathbb{R}^N$ and $r \in \mathbb{R}$, set

$$h^+_{z,x^0,r}(x) = \left(I_{f^*(z)} \right)^* (x - x^0) + r \text{ and } h^-_{z,x^0,r}(x) = - \left(I_{f^*(z)} \right)^* (x - x^0) + r.$$

Next Theorem shows that the maps just introduced are solutions to the minimum problem; we recall that by saying that a function v is a solution, we mean that it is a solution among all those functions that have the same boundary data as v. In it, we assume the following growth assumption: $Dom(f^*)$ is open. This assumption is very general; in particular we have

Proposition 3.2. *Let f be an extended valued, convex, lower semicontinuous function with superlinear growth; then $Dom(f^*) = \mathbb{R}^N$.*

However, a Lagrangean as $f(t) = |t| - \sqrt{|t|}$, whose polar is $f^*(p) = \frac{1}{4}\frac{1}{1-|p|}$ for $p \in (-1, 1)$, satisfies the condition $Dom(f^*)$ open, without being of superlinear growth.

The only case of Lagrangean (among those usually encounterd in Variational Problems!), that does not satisfy the assumption that $Dom(f^*)$ is open, is the least area Lagrangean $f(t) = \sqrt{1 + t^2}$, whose polar is $f^*(p) = -\sqrt{1 - p^2}$, so that $Dom(f^*) = \overline{B(0, 1)}$.

Theorem 3.3. *Let $Dom(f^*)$ be open. For $z \in Dom(f^*)$, $x^0 \in \mathbb{R}^N$ and $r \in \mathbb{R}$, the maps $h^+_{z,x^0,r}(x)$ and $h^-_{z,x^0,r}(x)$ are solutions to the minimum problem* 1.2*), among those u in*

$$\mathcal{S}^+_{z,x^0,r} = \left\{ u \in W^{1,1}(\Omega) : u - h^+_{z,x^0,r} \in W^{1,1}_0(\Omega) \right\}$$

and

$$\mathcal{S}^-_{z,x^0,r} = \left\{ u \in W^{1,1}(\Omega) : u - h^-_{z,x^0,r} \in W^{1,1}_0(\Omega) \right\}$$

respectively.

Notice that, in the case f is strictly convex, $\partial f^*(z)$ is single-valued and the maps h^+_z and h^-_z are affine maps. Hence the above class of functions can be seen as the natural generalization of the affine maps when f is not strictly convex.

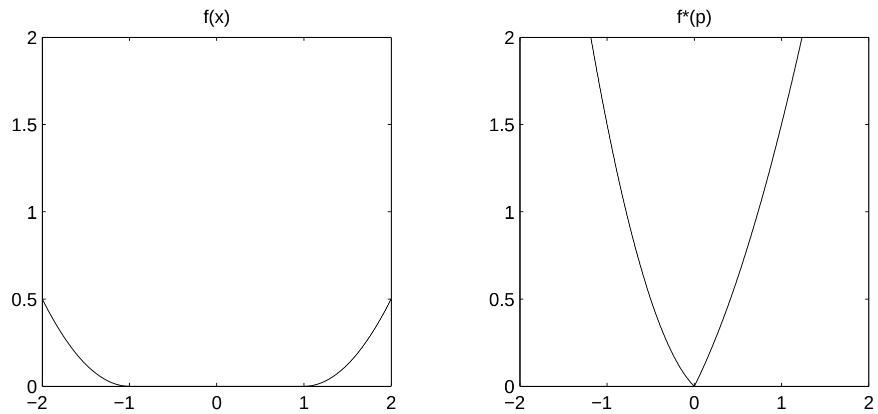
Let us consider the following examples.

Example.

$$f(\xi) = \begin{cases} 0 & \|\xi\| \le 1 \\ \frac{1}{2}(\|\xi\| - 1)^2 & \|\xi\| > 1 \end{cases}$$

whose polar is

$$f^*(p) = \frac{1}{2}\|p\|^2 + \|p\|$$

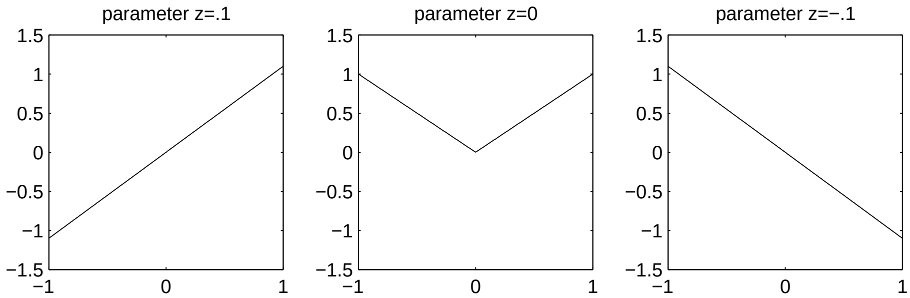


we have

$$\partial f^*(p) = \begin{cases} B(0,1) & p = 0 \\ p + \frac{p}{\|p\|} & p \neq 0 \end{cases} \tag{3.1}$$

and we obtain that $\left(I_{\partial f^*(z)}\right)^*(x)$ is the family of maps

$$\left(I_{\partial f^*(z)}\right)^*(x) = \begin{cases} \|x\| & z = 0 \\ \langle z + \frac{z}{\|z\|}, x \rangle & z \neq 0 \end{cases}$$



Example. for the Lagrangean (see [5])

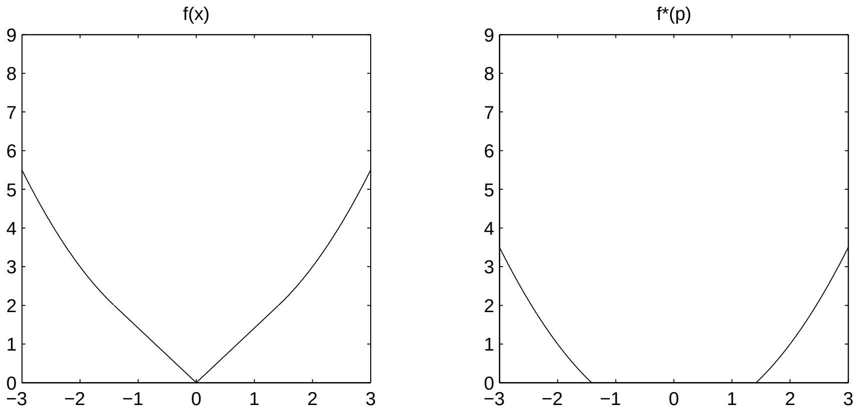
$$f(\xi) = \begin{cases} \sqrt{2}\|\xi\| & \|\xi\| \leq \sqrt{2} \\ 1 + \frac{1}{2}\|\xi\|^2 & \|\xi\| \geq \sqrt{2} \end{cases} \tag{3.2}$$

whose polar is

$$f^*(p) = \begin{cases} 0 & \|p\| \leq \sqrt{2} \\ \frac{1}{2}\|p\|^2 - 1 & \|p\| \geq \sqrt{2} \end{cases} \tag{3.3}$$

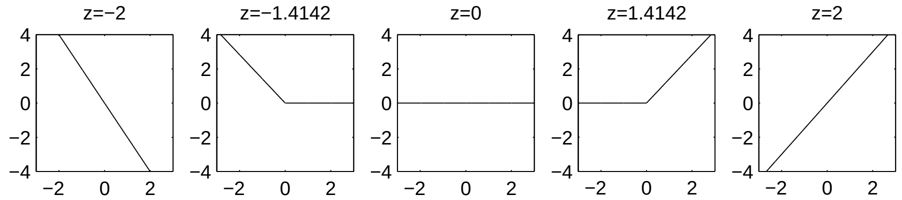
we have

$$\partial f^*(p) = \begin{cases} 0 & \|p\| \leq \sqrt{2} \\ \{\alpha p\} & 0 \leq \alpha \leq 1, \|p\| = \sqrt{2} \\ \frac{1}{2}\|p\|^2 - 1 & \|p\| \geq \sqrt{2} \end{cases} \tag{3.4}$$



and we obtain that $\left(I_{\partial f^*(z)}\right)^*(x)$ is the family of maps

$$\left(I_{\partial f^*(z)}\right)^*(x) = \begin{cases} 0 & \|z\| \leq \sqrt{2} \\ \sqrt{2}\langle \frac{z}{\|z\|}, x\rangle \chi_{\{x:\langle z,x\rangle \geq 0\}} & \|z\| = \sqrt{2} \\ \langle z, x\rangle & \|z\| > \sqrt{2} \end{cases}$$



Besides being solutions, the maps introduced above enjoy some kind of "maximality" condition, that yields comparison results,as in the theorem that follows. In it, the assumption of convexity of the domain Ω appears: this assumption seems to be needed for technical reasons only.

Theorem 3.4. *Let Ω be convex, let f be a (possibly extended valued) lower semicontinuous, convex function such that $Dom(f^*)$ is open. Let w be a solution to the problem of minimizing the functional*

$$\mathcal{J}(u) = \int_\Omega f\left(\nabla u(x)\right) dx$$

on $\{u : u - u^0 \in W_0^{1,1}(\Omega)\}$.

Assume that, for $z \in Dom(f^*)$, $x^0 \in (\Omega)^c$ and $r \in \mathbb{R}$, we have $h^+_{z,x^0,r} \geq w$ on $\partial\Omega$. Then, $h^+_{z,x^0,r} \geq w$ on Ω.

Remarks. Notice that any affine function $\ell(x) = \langle a, x\rangle + b$ can be written as $\ell(x) = \langle a, x - x^0\rangle + r$ with $x^0 \notin \Omega$ and $r = b + \langle a, x^0\rangle$.

Notice also that Example 1 shows that the analogous Theorem, where we had an affine function ℓ (in particular, $\ell(x) \equiv 0$) instead of the convex function $h^+_{z,x^0,r}$, would be false.

Finally notice that the functions $u(x) \equiv 0$ and $h^+_{0,0,-1}(x) = -1 + |x|$ are solutions to the problem of Example 1; still, $h^+_{0,0,-1} \geq u$ on $\partial\Omega$, but it is not true that $h^+_{0,0,-1} \geq u$ on Ω. Here, the point $x^0 = 0 \in \Omega$, opposite to our assumptions.

4. Another class of minimization problems

Another class of minimization problems, connected to the class previously considered, are the problems of the kind

$$\text{Minimize } \int_\Omega [f(\nabla u(x)) + \alpha u(x)]dx \text{ on } u - u_0 \in W_0^{1,1}(\Omega) \tag{4.1}$$

where α is a given parameter. We emphasize this point, since some similar problems occur where the parameter is a Lagrange multiplier (expressing, for instance, a problem under volume constraints), and this multiplier is to be determined by the solution to the problem itself. A case where this arises is the problem of capillary surfaces without gravity, as in [3]: in this case $f(\xi) = \sqrt{1 + |\xi|^2}$.

In the previous section we have introduced a class of solutions to problem (1.2) that is explicit, i.e., such that it can be computed directly from the Lagrangean; in this section we will show that the same can be done for problem (4.1). At the end of it we shall examine the behaviour of these solutions as the parameter α tends to zero, obtaining a result that, to this author, was rather surprising. In [2] the following result is proved.

Theorem 4.1. *Let Ω be an open bounded set, enough regular so that the Divergence Theorem holds, and let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be an extended valued, convex, lower semicontinuous function. For x_0 and z in $\mathbb{R}^N$ and $c \in \mathbb{R}$, consider the function*

$$\omega_\alpha(x) = \frac{N}{\alpha} f^* \left(z + \frac{x - x_0}{N} \alpha \right) + c.$$

If ω_α is defined on Ω and belongs to $W^{1,1}(\Omega)$, then it is the only minimum of the functional

$$\mathcal{J}(u) = \int_\Omega [f(\nabla u(x)) + \alpha u(x)]\, dx,$$

in the class of functions

$$\mathcal{S} = \left\{ u \in W^{1,1}(\Omega), u - \omega_\alpha \in W_0^{1,1}(\Omega) \right\}.$$

For instance, for $N = 1$, a solution to the problem of minimizing

$$\int_\Omega [f(u'(x)) + u(x)]\, dx$$

where

$$f(\xi) = \begin{cases} 0 & \text{if } \|\xi\| \le 1 \\ \|\xi\| - 1 & \text{if } 1 \le \|\xi\| \le 2 \\ +\infty & \text{elsewhere} \end{cases}$$

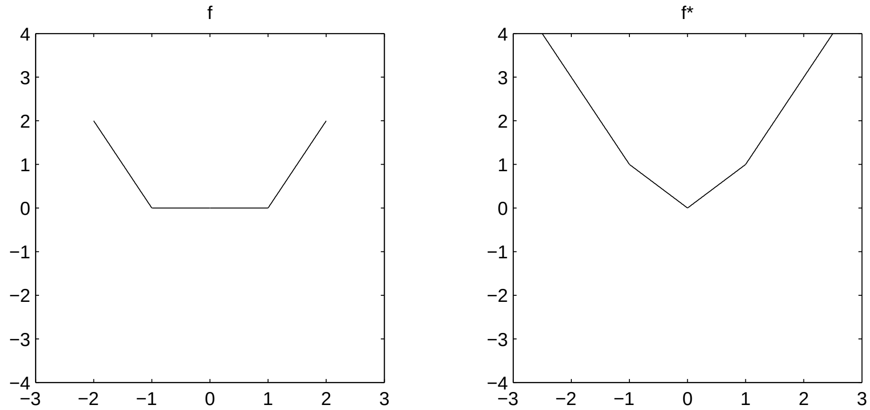
is the map ω_1 defined by

$$\omega_1 = f^*(x) = \begin{cases} |x| & \text{if } |x| \le 1 \\ 2|x| - 1 & \text{if } 1 \le |x| \le 2 \end{cases}$$

So the previous result yields at once an explicit formula for a solution to a variational problem and an uniqueness result, under very few assumptions on f, mainly that it is a convex, lower semicontinuous function. There is *some* uniqueness even though the map f, as in the example above, can be far from being strictly convex.

Moreover, assume that the domain of f^* is open. Then, being a convex function Lipschitzian in the interior of its domain, there exists some open ball $B(x^0, r)$ strictly contained in the domain of f^*, where f^* is lipschitzian; hence, taking $\Omega = \{x : \|Nz + (x - x^0)\| < r\}$, we have that ω_α belongs to $W^{1,\infty}(\Omega)$. This remark proves the following Corollary.

Corollary 4.2. *Let $Dom(f^*)$ contain an open set. Then there exists a region Ω and suitable boundary conditions, such that the problem (4.1) admits a solution.*



The condition that the effective domain of f contains an open set is very weak, and is satisfied by any reasonable problem; for instance, when $f(\xi) = \sqrt{1 + |\xi|^2}$, as in the capillarity problem, $f^*(p) = -\sqrt{1 - p^2}$, so that $Dom(f^*) = \overline{B(0,1)}$. In this case, whenever $\overline{\Omega} \subset B(0,1)$, problem (4.1) admits some solution. This might suggest that in general, problem (4.1) admits solutions when Ω is sufficiently small (and, may be, the mean curvature of $\partial\Omega$ is positive). The author does not know of any such result.

A class of problems clearly excluded from the application of Theorem (4.1) is when $f(x) = \langle a, x\rangle + b$: in this case, the domain of the polar to f reduces to one point.

5. Passing to the limit

The formula for a solution presented in Theorem 4.1 is not defined for $\alpha = 0$, and does not hold for this case. Still, as the parameter α tends to 0, a rather surprising connection, among the different classes of solutions we have presented, arises. It is enough to write the solution ω_α in the form

$$\frac{N}{\alpha} f^*\left(z + \frac{x - x_0}{N}\alpha\right) - \frac{N}{\alpha} f^*(z)$$

to realize that, in fact, the solution ω_α is a differential quotient. It not surprising, then, that the following result holds, as proved in [2].

Theorem 5.1. *Let f be an extended valued, convex, lower semicontinuous function with superlinear growth. Then:*

a) when f^* is differentiable at z, as α tends to 0, the function $\omega_{(\alpha,z,\beta)}$ converges to the affine map $\langle \nabla f^*(z), x - x_0\rangle + \beta$, a solution to Problem 1.2).

b) in general, as α tends to 0^+, the function $\omega_{(\alpha,z,\beta)}$ converges to $h^+_{z,x_0,\beta}$, the solution to Problem 1.2*, presented in Theorem 1 of* [1]*; as α tends to 0^-, $\omega_{(\alpha,z,\beta)}$ converges to $h^-_{z,x_0,\beta}$.*

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