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On the existence of solutions to a class of minimum time control problems and applications to Fermat's Principle and to the Brachystocrone

A. Cellina*, A. Ferriero, E.M. Marchini

Dipartimento di Matematica e Applicazioni, Universitá Degli Studi di Milano-Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy

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Abstract

We prove a theorem for the existence of solutions to minimum time control problems, under assumptions that do not require the convexity of the images and that weaken the assumption of upper semicontinuity. Our result applies to Fermat's Principle and to the Brachystocrone.

Keywords: Minimum time; Fermat's Principle; Brachystocrone

1. Introduction

Around 1650, Fermat stated that the light, to pass from a point to a second point in space, follows the path (among all the possible paths), that reaches the second point in minimum time.

In 1696, Jakob Bernoulli raised the following question: find the path from an initial point x^0 to a target point x^f such that a body, subject to gravity only, starting from x^0 with initial velocity zero, would reach x^f in minimum time.

In 1959, Filippov [1] proved the first general theorem on the existence of solutions to minimum time control problems of the form

$$x'(t) = f(x(t), u(t)), \quad u(t) \in U(x(t))$$

requiring that the set-valued map $x \to U(x)$ be upper semicontinuous (with respect to the inclusion) and that the values $F(x) = f(x, U(x))$ be compact and convex. In Theorem 2.2 of the present paper, we prove the existence of solutions to minimum time problems for differential inclusions, under assumptions that do not require the convexity of the images $F(x) = f(x, U(x))$ and, at the same time, weaken the assumption of upper semicontinuity of F. In Sections 3 and 4, we show that our result applies to Fermat's Principle and to the Brachystocrone.

2. The existence of solutions to minimum time problems

For a compact subset $A \subset \mathbb{R}^d$, set co(A) be its convex hull. For basic results relating to solutions to differential inclusions, measurable selections and properties of set-valued maps we refer to any standard text on the subject. The proof of the existence theorem is based on the following lemma. In it, no assumptions on F are required.

Lemma 2.1. *Let $x : [0, t^*] \to \mathbb{R}^d$ be absolutely continuous and such that $x'(t) = 0$ on a subset E of $[0, t^*]$ of positive measure; let $X = \{x(t) : t \in [0, t^*]\}$ and let F, defined on X, be such that, for almost every $t \in [0, t^*]$,*

$$x'(t) \in F(x(t)).$$

Then, there exist τ^*, $0 < \tau^* < t^*$ and an absolutely continuous function $\tilde{x} : [0, \tau^*] \to X$, such that $\tilde{x}(0) = x(0)$, $\tilde{x}(\tau^*) = x(t^*)$ and

$$\tilde{x}'(t) \in F(\tilde{x}(t))$$

for almost every $\tau \in [0, \tau^*]$.

* Corresponding author. Tel.: +39 02 6448 7724; fax: +39 02 6448 7705.
E-mail addresses: cellina@matapp.unimib.it (A. Cellina), ferriero@matapp.unimib.it (A. Ferriero), elsa@matapp.unimib.it (E.M. Marchini).

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Proof. (a) It follows from the assumptions that there exists a closed subset K of E of positive measure. The complement of K consists of at most countably many open non-overlapping intervals (a_i, b_i), $i \in I$. Since the intervals (a_i, b_i) are disjoint, we must have that $\tau^* = \sum_{i\in I}(b_i - a_i) < t^*$. For each $i \in I$, set

$$\tau(b_i) = \sum_{j\in I: b_j<b_i} (b_j - a_j).$$

From this definition we infer that, when $b_l > b_m$, we have that $\tau(b_l) \geqslant \tau(b_m) + (b_m - a_m)$.

Consider the family of intervals $(\tau(b_i), \tau(b_i) + b_i - a_i)$; they are disjoint, since in case we had $(\tau(b_l), \tau(b_l) + b_l - a_l) \cap (\tau(b_m), \tau(b_m) + b_m - a_m) \neq \emptyset$ with $b_l > b_m$, we would obtain $\tau(b_m) + (b_m - a_m) > \tau(b_l)$, a contradiction to the inequality obtained above. Consider T, the open subset of $[0, t^*]$ defined by $T = \bigcup_{i\in I}(\tau(b_i), \tau(b_i) + b_i - a_i)$; since, for every $\tau \in T$, we have $\tau \leqslant \sum_{i\in I}(b_j - a_j) = \tau^*$, we obtain that $T \subset [0, \tau^*]$. Moreover, the measure of T equals τ^*, in fact

$$m(T) = \sum_{i\in I}(b_i - a_i) = \tau^*.$$

(b) Define the absolutely continuous function $\tilde{x} : [0, \tau^*] \to \mathbb{R}^d$ by $\tilde{x}(\tau) = x(0) + \int_0^\tau \tilde{x}'(s)\,ds$, where

$$\tilde{x}'(s) = \begin{cases} x'(s + a_i - \tau(b_i)), & s \in (\tau(b_i), \tau(b_i) \\ & \quad +b_i - a_i), i \in I, \\ 0, & s \in [0, \tau^*]\setminus T, \end{cases}$$

in particular, we have $\tilde{x}(0) = x(0)$.

Fix $\tau \in T$, there exists a unique $i \in I$ such that $\tau \in (\tau(b_i), \tau(b_i)+b_i-a_i)$. Notice that $\tau \in (\tau(b_i), \tau(b_i)+b_i-a_i)$ if and only if $\tau + a_i - \tau(b_i) \in (a_i, b_i)$. We have

$$\begin{aligned}\tilde{x}(\tau) - x(0) &= \int_0^{\tau(b_i)} \tilde{x}'(s)\,ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)\,ds \\ &= \sum_{j:\tau(b_j)+b_j-a_j\leqslant\tau(b_i)} \int_{\tau(b_j)}^{\tau(b_j)+b_j-a_j} \\ &\quad \times \tilde{x}'(s)\,ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)\,ds \\ &= \sum_{j:\tau(b_j)+b_j-a_j\leqslant\tau(b_i)} \\ &\quad \times \int_{\tau(b_j)}^{\tau(b_j)+b_j-a_j} x'(s + a_j - \tau(b_j))\,ds \\ &\quad + \int_{\tau(b_i)}^{\tau} x'(s + a_i - \tau(b_i))\,ds \\ &= \sum_{j:\tau(b_j)+b_j-a_j\leqslant\tau(b_i)} \int_{a_j}^{b_j} x'(s)\,ds \\ &\quad + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)\,ds.\end{aligned}$$

Notice that we have

$$\sum_{j:\tau(b_j)+b_j-a_j\leqslant\tau(b_i)} \int_{a_j}^{b_j} x'(s)\,ds = \sum_{j:b_j<b_i} \int_{a_j}^{b_j} x'(s)\,ds.$$

in fact, by the definition, $\tau(b_j) + b_j - a_j \leqslant \tau(b_i)$ if and only if $\tau(b_j) < \tau(b_i)$. Hence

$$\begin{aligned}\tilde{x}(\tau) - x(0) &= \sum_{j:b_j<b_i} \int_{a_j}^{b_j} x'(s)\,ds \\ &\quad + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)\,ds \\ &= \int_0^{a_i} x'(s)\chi_{\{[0,t^*]\setminus K\}}(s)\,ds \\ &\quad + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)\,ds \\ &= x(\tau + a_i - \tau(b_i)) - x(0).\end{aligned}$$

The previous equality implies that the function $\tilde{x}$ is a solution to the differential inclusion. In fact, we have that, almost everywhere in $[0, \tau^*]$,

$$\begin{aligned}\tilde{x}'(\tau) &= x'(\tau + a_i - \tau(b_i)) \in F(x(\tau + a_i - \tau(b_i)) \\ &= F(\tilde{x}(\tau)).\end{aligned}$$

(c) Set $B = \sup\{b_i\}$. Then either the supremum is attained or it is not. In the first case, for some $\tilde{j}$, $B = b_{\tilde{j}}$ and $\tau(b_{\tilde{j}}) + (b_{\tilde{j}} - a_{\tilde{j}}) = \tau^*$. From (b), for every $t \in [a_{\tilde{j}}, b_{\tilde{j}}]$, we have that $x(t) = \tilde{x}(t - a_{\tilde{j}} + \tau(b_{\tilde{j}}))$, in particular $x(B) = x(b_{\tilde{j}}) = \tilde{x}(b_{\tilde{j}} - a_{j_{\tilde{k}}} + \tau(b_{\tilde{j}})) = \tilde{x}(\tau^*)$. On the other hand, since $x'(t) = 0$ on $[B, t^*]$, we have that $x(t^*) = x(B)$, so that $x(t^*) = \tilde{x}(\tau^*)$.

In the second case, let $\{b_{j_k}\}$ be an increasing sequence, converging to B. From

$$\begin{aligned}|x(t^*) - x(a_{j_k})| &= \left| \int_{a_{j_k}}^{B} x'(s)\,ds \right| \\ &\leqslant \left| \sum_{\{j\in I: b_j > b_{j_{k-1}}\}} \int_{(a_j,b_j)} x'(s)\,ds \right|\end{aligned}$$

it follows that $x(a_{j_k}) \to x(t^*)$, while from

$$\sum_{\{j\in I: b_j > b_{j_k}\}} b_j - a_j < B - b_{j_k} \quad \text{and}$$

$$\tau^* = \sum_{j\in I} b_j - a_j = \tau(b_{j_k}) + \sum_{j\in I: b_j \geqslant b_{j_k}} b_j - a_j$$

we obtain that $\tau(b_{j_k}) \to \tau^*$. By the previous point (b) we have that $\tilde{x}(\tau(b_{j_k})) = x(a_{j_k})$ and by continuity we infer that $x(t^*) = \tilde{x}(\tau^*)$. $\square$

In what follows we shall consider the following *minimum time problem* for solutions to a differential inclusion: X and

S are closed subset of $\mathbb{R}^d$, $S \subset X$, $x^0 \in X \backslash S$ and F is a set-valued map. Consider the problem of reaching the target set S from x^0, satisfying the constraint $x(t) \in X$, where $x(\cdot)$ is a solution to the differential inclusion

$$x'(t) \in F(x(t)).$$

Theorem 2.2. *Let $X \subset \mathbb{R}^d$ be closed and let F be a set-valued map defined on X with compact non-empty images, linearly bounded, i.e. such that, for some α and β, for every $x \in X$ and for every $\xi \in F(x)$, we have $\|\xi\| \leqslant \alpha\|x\| + \beta$. In addition, assume that*

(i) *$x \to \mathrm{co}(F(x))$ is upper semicontinuous, and*
(ii) *for every $x \in X$, for every $\xi \in \mathrm{co}(F(x))$, with $\xi \neq 0$, there exists $\lambda \geqslant 1$ such that $\lambda\xi \in F(x)$.*

Assume that there exists $\tilde{t} > 0$ and a solution x to

$$x'(t) \in \mathrm{co}(F(x(t))), \quad x(0) = x^0$$

such that $x(t) \in X$ for every $t \in [0, \tilde{t}]$ and that $x(\tilde{t}) \in S$. Then the minimum time problem for

$$x'(t) \in F(x(t))$$

admits a solution.

Proof. (a) Let $A^{\mathrm{co}}_{x^0}(t)$ be the attainable set at time t of the Cauchy problem for the convexified inclusion, and set $t^* = \inf\{t : A^{\mathrm{co}}_0(t) \cap S \neq \emptyset\}$. We notice that $t^* > 0$, since $x^0 \notin S$ and $\mathrm{co}(F(x))$ is bounded in a neighborhood of x^0. Let (t_n) be decreasing to t^* and le x_n be solutions to the differential inclusion

$$x'(t) \in \mathrm{co}(F(x(t)))$$

such that $x_n(0) = x^0$ and $x(t_n) \in S$, $x(t) \in X$ for $t \in [0, t_n]$. A subsequence of this sequence converges uniformly, on $[0, t^*]$, to x_*. Clearly, $x_*(0) = x^0$ and $x_*(t) \in X$ for $t \in [0, t^*]$. It is known that, under the assumptions of the theorem, x_* is again a solution to

$$x'(t) \in \mathrm{co}(F(x(t))).$$

Hence, x_* is a solution to the convexified problem that reaches S in minimum time, and t^* is the value of the minimum time for the convexified problem.

(b) It cannot be that $m\{t : x'_*(t) = 0\} = t^*$, since $x^0 \notin S$. Then, applying Lemma 1, we infer that $x'_*(t) \neq 0$ for a.e. $t \in [0, t^*]$. In fact, otherwise, we could define a different solution to the convexified differential inclusion, defined on an interval $[0, \tau^*]$ with $\tau^* < t^*$, having the same initial and final point: hence t^* would not be the value of the minimum time for the convexified problem.

(c) By the previous point and the assumption on $F(x)$, for almost every t there exists a non-empty set $\Lambda(t)$ such that $\lambda \in \Lambda(t)$ implies $\lambda x'_*(t) \in F(x_*(t))$ and $\lambda \geqslant 1$. Reasoning as in [2], we obtain that $\Lambda(\cdot)$ is measurable on $[0, t^*]$, hence, by standard arguments, that there exists a measurable selection $\lambda(\cdot)$ from $\Lambda(\cdot)$. Define the absolutely continuous map s by $s(0) = 0$ and $s'(t) = 1/\lambda(t)$: s is an increasing map and maps $[0, t^*]$ onto $[0, s^*]$, where $s^* \leqslant t^*$. Let $t = t(s)$ be its inverse and consider the map $\tilde{x}(s) = x_*(t(s))$. We obtain in particular that $\tilde{x}(0) = x_*(0)$ and that $\tilde{x}(s^*) = x_*(t(s^*)) = x_*(t^*)$. We also have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\tilde{x}(s) &= x'_*(t(s))t'(s) = x'_*(t(s))\frac{1}{s'(t(s))}\\ &= x'_*(t(s))\lambda_1(t(s)) \in F(x_*(t(s)))\\ &= F(\tilde{x}(s)).\end{aligned}$$

Hence, we have obtained that $\tilde{x}$ is at once a solution to the original differential inclusion and a minimum time solution to the convexified inclusion. Since every solution to the original problem is also a solution to the convexified problem, the infimum of the times needed to reach S along the solutions to the original problem cannot be lesser than the minimum time for the convexified problem. Hence $\tilde{x}$ is a solution to the minimum time problem for the original differential inclusion. □

The following is a result on the existence of solutions to initial value problems for differential inclusions with non-convex right-hand side. Besides being non-convex-valued, the map F need not be upper semicontinuous: this assumption is replaced by the weaker assumption that the map $\mathrm{co}(F)$ be upper semicontinuous.

Theorem 2.3. *Let Ω be open, $x^0 \in \Omega$ and let F be as in Theorem 2.2. Assume that there exists $t^* > 0$ such that, on $[0, t^*]$, the Cauchy Problem*

$$x'(t) \in \mathrm{co}\, F(x(t)), \quad x(0) = x^0$$

admits a solution $x \not\equiv x^0$. Then the Cauchy Problem

$$x'(t) \in F(x(t)), \quad x(0) = x^0$$

admits a solution on some interval $[0, \tau^*]$.

Proof. Since $x \not\equiv x^0$, there exists $t^1 \in [0, t^*]$ such that $x(t^1) \neq x^0$. Consider the minimum time problem for the convexified inclusion, with target set $S = \{x(t^1)\}$. This problem has a solution with minimum time τ^*, where $0 < \tau^* \leqslant t^1$. By Theorem 2.2, the original non-convexified problem has a solution on $[0, \tau^*]$. □

3. Fermat's Principle

Fermat's Principle states that, among the virtual paths the light could follow to pass from point P_1 to point P_2 in a medium with speed $\rho(x)$, it actually follows the one that minimizes the time needed to pass from P_1 to P_2. In mathematical terms, set $\partial B = \{x : \|x\| = 1\}$: a virtual path followed by the light is a solution to the differential inclusion

$$x'(t) \in \rho(x(t))\partial B, \text{ or } \|x'(t)\| = \rho(x(t)).$$

In these authors' opinion, the remarkable interest of the problem consists in the fact that Fermat's aim was to explain the phenomena occurring when ρ, the velocity of light, is discontinuous, as in passing from air to water. A differential inclusion with discontinuous right hand side is *essential* to describe the problem.

Theorem 3.1. *Let X be closed and convex, let the scalar-valued function ρ be upper semicontinuous, linear bounded and such that, on each compact $C \subset X$, there is $\alpha > 0$ such that $\rho(x) \geqslant \alpha$ on C. Let P_1 and P_2 be in X, $P_1 \neq P_2$. Then there exists a path followed by the light to travel from P_1 to P_2 in minimum time.*

Proof. The assumptions on ρ imply that the set-valued map $\rho(x)B = \text{co}(\rho(x)\partial B)$ is upper semicontinuous: notice however that, under the same assumptions, in general the map $\rho(x)\partial B$ is *not* upper semicontinuous. Consider $\text{co}\{P_1, P_2\}$, the (compact) segment joining P_1 and P_2. Since an upper semicontinuous map attains its maximum on a compact set, there exists R such that $\rho(x) \leqslant R$ on $\text{co}\{P_1, P_2\}$, so that, on this segment, $\alpha \leqslant \rho(x) \leqslant R$; in turn, this implies that there is $\tilde{t} > 0$ and a solution x to

$$x'(t) \in \text{co}(\rho(x(t))\partial B), \quad x(0) = P_1$$

such that $x(\tilde{t}) = P_2$. Assumption (ii) of Theorem 2.2 is clearly satisfied, thus an application of Theorem 2.2 proves the present theorem. $\square$

The assumption $\rho(x) \geqslant \alpha > 0$ cannot be removed, otherwise, first, an opaque barrier could completely prevent the light to travel from P_1 to P_2 or, second, by allowing ρ to go to zero fast enough, there could be no solution reaching P_2 in finite time.

Remark. Assume we have two media, A and B, medium A in the plane region $y > 0$ and B in the region $y < 0$, with velocities $\rho_A = 1$ and $\rho_B = 2$. Assigning the boundary surface $y = 0$ to either medium is probably a physically meaningless operation. Mathematically, the choice of assigning it to A, makes the velocity ρ lower semicontinuous while, assigning it to B, means making it upper semicontinuous. Our theorem above assures the existence of a minimum time solution, no matter what P_1 and P_2 are, when we make the second choice. We claim that, to the opposite, the first choice would make our result false.

In fact, let us define $\rho_A = 1$ on the line $y = 0$, choose the points $P_1 = (0, 0)$ and $P_2 = (2, 0)$, and assume that there exists a solution $\tilde{\xi} = (\tilde{x}, \tilde{y})$ to the corresponding minimum time problem. We claim that $\tilde{\xi}$ cannot take all of its values in the half plane $y \geqslant 0$: in this case the time needed to pass from P_1 to P_2, is at least 2. Fix an angle $\alpha < 0$; set $T = 1/\cos\alpha$. The trajectory

$$\xi_\alpha(t) = (t\cos\alpha, t\sin\alpha)2\chi_{[0,T/2]} + (t\cos\alpha, t\sin\alpha - T\sin\alpha)2\chi_{[T/2,T]}$$

satisfies $\|\xi'_\alpha(t)\| = 2$ a.e., and passes from P_1 to P_2 in time

$$T = 1/\cos\alpha$$

that is less than 2 for α small, a contradiction. So the solution $\tilde{\xi}$ has to enter the half plane $y < 0$. Let (a, b) a maximal open interval where $\tilde{y}(t) < 0$, so that $\tilde{y}(a) = \tilde{y}(b) = 0$, and we can assume that $x(a) > x(b)$; at a certain time $\tilde{t} \in (a, b)$, $\tilde{x}$ will take the value $(x(a) + x(b))/2$; define the angle $\tilde{\alpha}$ by $\tan\tilde{\alpha} = \tilde{y}(\tilde{t})/\tilde{x}(\tilde{t})$. By the previous reasoning, the time needed to pass from $(x(a), 0)$ to $(x(b), 0)$ through the point $\tilde{\xi}(\tilde{t})$ is at least $(x(b) - x(a))/2\cos\tilde{\alpha}$, so that

$$b - a \geqslant \frac{x(b) - x(a)}{2\cos\tilde{\alpha}}.$$

However, by chosing a smaller (in absolute value) angle α, we have a trajectory that passes from $(x(a), 0)$ to $(x(b), 0)$ in less than $b - a$. We have reached a contradiction, hence, a solution cannot exist.

4. The Brachystocrone

Bernoulli's problem of the Brachystocrone can be stated as follows: in the plane, an initial condition (ξ_1^0, ξ_2^0) is given; consider all the possible oriented rectifiable curves passing through it. Each such curve is defined by assigning $(u_1(.), u_2(.))$, a unit vector describing the direction of its (oriented) tangent. In this way, the parameter t is the arc-length parametrization of the curve.

The system of equations

$$\begin{aligned} \xi_1' &= u_1\xi_3, \\ \xi_2' &= u_2\xi_3, \\ \xi_3' &= -gu_2, \end{aligned} \tag{B}$$

where the maps $u_1(.)$ and $u_2(.)$ are measurable and $u_1^2(t) + u_2^2(t) = 1$ a.e., describes the motion of a body in the plane, defined by the coordinates $(\xi_1(t), \xi_2(t))$, with (scalar) velocity $\xi_3(t) = \sqrt{(\xi_1'(t))^2 + (\xi_2'(t))^2}$, along a curve identified assigning the direction of its tangent vector $(u_1(.), u_2(.))$, subject to the gravity g.

Hence, the problem raised by Bernoulli can be stated as the following minimum time control problem:

The Brachystocrone Minimum Time Problem: find a solution to the control system

$$\begin{aligned} \xi_1' &= u_1\xi_3, \\ \xi_2' &= u_2\xi_3, \\ \xi_3' &= -gu_2 \end{aligned}$$

$(\xi_1(0), \xi_2(0), \xi_3(0)) = (\xi_1^0, \xi_2^0, 0)$, subject to the constraint $\xi_3 \geqslant 0$, with control set

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 = 1)\},$$

that would reach the target set $S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)$, where $\xi_2^f \leqslant \xi_2^0$, in minimum time.

Theorem 4.1. *The Brachystocrone Minimum Time Problem admits a solution.*

Proof. Let $x = (\xi_1, \xi_2, \xi_3)$, $X = \{\xi_3 \geqslant 0\}$, $S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)$; let $f(x, u)$ be the right-hand side of (B) and set $F(x) = f(x, U)$. Then, as one can check, $F(x)$ satisfies the assumptions of Theorem 2.2. Hence, to apply the Theorem, it is enough to show that there are solutions $x(t)$ issuing from $x^0 = (\xi_1^0, \xi_2^0, 0)$ with $\xi_3(t) \geqslant 0$ and such that, at some finite time t^*, $\xi_1(t^*) = \xi_1^f$, $\xi_2(t^*) = \xi_2^f$.

This is so in the case where $\xi_2^f < \xi_2^0$. In fact, in this case we have that, by choosing the constant control

$$u_1 = \frac{\xi_1^f - \xi_1^0}{\sqrt{(\xi_1^f - \xi_1^0)^2 + (\xi_2^f - \xi_2^0)^2}},$$
$$u_2 = \frac{\xi_2^f - \xi_2^0}{\sqrt{(\xi_1^f - \xi_1^0)^2 + (\xi_2^f - \xi_2^0)^2}},$$

we obtain that

$$\xi_1(t) = u_1\left(-gu_2\frac{t^2}{2}\right) + \xi_1^0,$$
$$\xi_2(t) = u_2\left(-gu_2\frac{t^2}{2}\right) + \xi_2^0,$$

so that $\xi_1(t^f) = \xi_1^f$ and $\xi_2(t^f) = \xi_2^f$ for $t^f = \sqrt{2/g}\sqrt{(\xi_1^f - \xi_1^0)^2 + (\xi_2^f - \xi_2^0)^2 / -(\xi_2^f - \xi_2^0)}$. We also obtain that $\xi_3(t) > 0$ on $(0, t^f)$ and that $\xi_3(t^f) = \sqrt{-2g(\xi_2^f - \xi_2^0)}$.

Consider the case $\xi_2^f = \xi_2^0$. We can assume that $\xi_1^f \neq 0$, otherwise $t^*=0$ is the solution to the minimum time problem. In the case $\xi_1^f > \xi_1^0$, consider the solution with the constant control $u_1 = 1/\sqrt{2}$, $u_2 = -1/\sqrt{2}$ on $\left[0, \sqrt{2(\xi_1^f - \xi_1^0)/g}\right]$. At time $\tilde{t} = \sqrt{2(\xi_1^f - \xi_1^0)/g}$, we have that $\xi_1(\tilde{t}) - \xi_1^0 = (\xi_1^f - \xi_1^0)/2$, $\xi_2(\tilde{t}) - \xi_2^0 = -(\xi_1^f - \xi_1^0)/2$ and $\xi_3(\tilde{t}) = \sqrt{g(\xi_1^f - \xi_1^0)}$. The solution with constant control $u_1 = 1/\sqrt{2}$, $u_2 = 1/\sqrt{2}$ on theinterval $(\tilde{t}, 2\tilde{t})$, with initial conditions $\xi_1(\tilde{t}) = (\xi_1^f + \xi_1^0)/2$, $\xi_2(\tilde{t}) = \xi_2^0 - (\xi_1^f - \xi_1^0)/2$ and $\xi_3(\tilde{t}) = \sqrt{g(\xi_1^f - \xi_1^0)}$ is such that at $t = 2\tilde{t}$, $\xi_1(2\tilde{t}) = \xi_1^f$ and $\xi_2(2\tilde{t}) = \xi_2^f$. Hence $t^f = 2\tilde{t}$ and $\xi_3(t) > 0$ on $(0, t^f)$. Analogously for the case $\xi_1^f < \xi_1^0$. □

The Brachystocrone as a minimum time control problem has already been discussed in [3–5]. The model presented in these papers has a right-hand side that is not Lipschitz in all its variables, to the opposite of the model proposed here.

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