Kowalevski top, Magri method of Syzygies and Discriminant Separability

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Bi-Hamiltonian Systems and All That
In honour of Professor Franco Magri’s 65th Birthday,
29 September 2011
1. Discriminantly separable polynomials - an overview
   • The Magri method of Syzygies
   • Discriminantly separable polynomials - a new view on the Kowalevski
   • Buchstaber-Novikov 2-valued groups

2. Classification of discriminantly separable polynomials

3. From discriminant separability to quad-graph integrability

4. Gauge transformations and deformations

5. Remarks

"... Kowalevski had the astonishing idea of replacing the mechanical variables (...) by the four integrals \((h_1, h_2, c_1, c_2)\) and by the roots \((\lambda_1, \lambda_2)\) of the second-order polynomial

\[
S(\lambda) = (x_1 - x_2)^2(\lambda - 1/6h_1)^2 - R(x_1, x_2)(\lambda - 1/6h_1) - 1/4R_1(x_1, x_2)
\]

..."
"... The discovery of the polynomial $S(\lambda)$ has always been a vexata quaestio. In her paper Kowalevski did not provide a convincing motivation for her choice, but only the evidence, a posteriori, that it actually works. Her choice therefore appears as the outcome of a magical intuition. The purpose of the present paper is to derive the polynomial $S(\lambda)$ directly from the equations of motion by the method of syzygies..."  
F. Magri 2005
Discriminantly separable polynomials - a new view on the Kowalevski top

Pencil of conics

Two conics and tangential pencil

\[ C_1 : a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0 \]
\[ C_2 : w_2^2 - 4w_1 w_3 = 0 \]

\[ F(s, x_1, x_2) = L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2). \]
Theorem [V. D. (2009/2010)]

(i) There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial $F$ in $s$ as a polynomial in variables $x_1$ and $x_2$ separates the variables:

$$D_s(F)(x_1, x_2) = P(x_1)P(x_2). \tag{1}$$

(ii) There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial $F$ in $x_2$ as a polynomial in variables $x_1$ and $s$ separates the variables:

$$D_{x_2}(F)(s, x_1) = J(s)P(x_1). \tag{2}$$

Due to the symmetry between $x_1$ and $x_2$ the last statement remains valid after exchanging the places of $x_1$ and $x_2$. 
## Gauge equivalence

**Gauge transformations**

\[ x \mapsto \frac{a_1 x + b_1}{c_1 x + d_1} \]

\[ y \mapsto \frac{a_2 y + b_2}{c_2 y + d_2} \]

\[ z \mapsto \frac{a_3 z + b_3}{c_3 z + d_3} \]
Discriminantly separable polynomials - a new view on the Kowalevski top

Discrim. separable polynomials – definition [V. D. 2009/2010]

For a polynomial $F(x_1, \ldots, x_n)$ we say that it is **discriminantly separable** if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \ldots, n$

$$D_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is **symmetrically discriminantly separable** if

$$f_2 = f_3 = \cdots = f_n,$$

while it is **strongly discriminantly separable** if

$$f_1 = f_2 = f_3 = \cdots = f_n.$$

It is **weakly discriminantly separable** if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \ldots, n$

$$D_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_i^j(x_j).$$
Discriminantly separable polynomials - an overview

Classification of discriminantly separable polynomials

Geometric interpretation of the Kowalevski fundamental equation

\[ Q(w, x_1, x_2) := (x_1 - x_2)^2 w^2 - 2R(x_1, x_2)w - R_1(x_1, x_2) = 0 \]

\[ R(x_1, x_2) = - x_1^2 x_2^2 + 6\ell_1 x_1 x_2 + 2c(x_1 + x_2) + c^2 - k^2 \]

\[ R_1(x_1, x_2) = - 6\ell_1 x_1^2 x_2^2 - (c^2 - k^2)(x_1 + x_2)^2 - 4c\ell x_1 x_2(x_1 + x_2) + 6\ell_1(c^2 - k^2) - 4c^2\ell^2 \]

\[ a_0 = -2 \quad a_1 = 0 \quad a_5 = 0 \]

\[ a_2 = 3\ell_1 \quad a_3 = -2c\ell \quad a_4 = 2(c^2 - k^2) \]
Geometric interpretation of the Kowalevski fundamental equation

Theorem [V. D. (2009)]

The Kowalevski fundamental equation represents a point pencil of conics given by their tangential equations

\[ \hat{C}_1 : -2w_1^2 + 3l_1 w_2^2 + 2(c^2 - k^2)w_3^2 - 4cw_2w_3 = 0; \]
\[ C_2 : w_2^2 - 4w_1w_3 = 0. \]

The Kowalevski variables \( w, x_1, x_2 \) in this geometric settings are the pencil parameter, and the Darboux coordinates with respect to the conic \( C_2 \) respectively.
Multi-valued Buchstaber-Novikov groups

**$n$-valued group on $X$**

\[ m : X \times X \to (X)^n, \quad m(x, y) = x \ast y = [z_1, \ldots, z_n] \]

$(X)^n$ — symmetric $n$-th power of $X$

**Associativity**

Equality of two $n^2$-sets:

\[
[x \ast (y \ast z)_1, \ldots, x \ast (y \ast z)_n] \quad \text{and} \quad [(x \ast y)_1 \ast z, \ldots, (x \ast y)_n \ast z]
\]

for every triplet $(x, y, z) \in X^3$. 
Unity \( e \)
\[
e \ast x = x \ast e = [x, \ldots, x] \text{ for each } x \in X.
\]

Inverse \( \text{inv} : X \to X \)
\[
e \in \text{inv}(x) \ast x, \ e \in x \ast \text{inv}(x) \text{ for each } x \in X.
\]
Two-valued group on $\mathbb{CP}^1$

The equation of a pencil

$$F(s, x_1, x_2) = 0$$

Isomorphic elliptic curves

$$\Gamma_1 : y^2 = P(x) \quad \text{deg } P = 4 \quad \Gamma_2 : t^2 = J(s) \quad \text{deg } J = 3.$$
There is a group structure on the cubic $\Gamma_2$. Together with its subgroup $\mathbb{Z}_2$, it defines the standard two-valued group structure on $\mathbb{C}P^1$:

$$s_1*cs_2 = \left[ -s_1 - s_2 + \left( \frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left( \frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right],$$

where $t_i = J'(s_i), \ i = 1, 2$.

**Theorem [V. D. (2009/2010)]**

The general pencil equation after fractional-linear transformations

$$F(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

defines the two valued group structure $(\Gamma_2, \mathbb{Z}_2)$ and the Kowalevski change of variables.
Two-valued group $\mathbb{CP}^1$

**Theorem [V. D. (2009/2010)]**

Associativity conditions for the group structure of the two-valued group $(\Gamma_2, \mathbb{Z}_2)$ and for its action on $\Gamma_1$ are equivalent to the great Poncelet theorem for a triangle.
Classification of the strongly discriminantly separable polynomials

Natural question: to classify discriminantly separable polynomials of degree two in each of three variables, up to gauge transformations.

Theorem (V. D. - K. Kukić, 2011)

All strongly discriminantly separable polynomials in three variables of degree two in each variable, with polynomial \( P \) with four simple roots, are gauge equivalent to the two valued group defined by the equation:

\[
(x + y + z + \frac{g_2}{4}xyz)^2 - (4 + g_3 xyz)(xy + yz + zx) = 0.
\]
(B) (1,1,2): two simple zeros and one double zero, for canonical form \( P(x) = x^2 - \epsilon^2 \),

\[
F_B = x_1 x_2 x_3 + \frac{\epsilon}{2} (x_1^2 + x_2^2 + x_3^2 - \epsilon^2),
\]

(C) (2, 2): two double zeros, for canonical form \( P(x) = x^2 \),

\[
F_{C1} = \alpha_1 x_1^2 x_3^2 + \alpha_2 x_1 x_2 x_3 + \alpha_3 x_2^2, \quad \alpha_2^2 - 4 \alpha_1 \alpha_3 = 1,
\]

\[
F_{C2} = \beta_1 x_1^2 x_2^2 x_3^2 + \beta_2 x_1 x_2 x_3 + \beta_3, \quad \beta_2^2 - 4 \beta_1 \beta_3 = 1,
\]

(D) (1,3): one simple and one triple zero, for canonical form \( P(x) = x \),

\[
F_D = -\frac{1}{2} (x_1 x_2 + x_2 x_3 + x_1 x_3) + \frac{1}{4} (x_1^2 + x_2^2 + x_3^2),
\]
Classification-continuation

(E) (4): one quadruple zero, for canonical form $P(x) = 1$,

\[
F_{E1} = \gamma_1(x_1 + x_2 + x_3)^2 + \gamma_2(x_1 + x_2 + x_3) + \gamma_3, \quad \gamma_2^2 - 4\gamma_1\gamma_3 = 1,
\]

\[
F_{E2} = \gamma_1(x_2 + x_3 - x_1)^2 + \gamma_2(x_2 + x_3 - x_1) + \gamma_3, \quad \gamma_2^2 - 4\gamma_1\gamma_3 = 1,
\]

\[
F_{E3} = \gamma_1(x_1 + x_3 - x_2)^2 + \gamma_2(x_1 + x_3 - x_2) + \gamma_3, \quad \gamma_2^2 - 4\gamma_1\gamma_3 = 1,
\]

\[
F_{E4} = \gamma_1(x_1 + x_2 - x_3)^2 + \gamma_2(x_1 + x_2 - x_3) + \gamma_3, \quad \gamma_2^2 - 4\gamma_1\gamma_3 = 1.
\]
Integrable quad-graphs
Toward Adler-Bobenko-Suris quad graphs: \( h \)

**From \( F \) to \( \hat{h} \)**

\[
\hat{h}(x_1, x_2, \alpha) = \frac{F(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}
\]

**The system for \( h_B \)**

\[
\begin{align*}
    h_{22} &= 0, \ h_{21} = h_{12} = 0, \ h_{01} = h_{10} = 0 \\
    h_{02} &= h_{20}, \ h_{11} = \pm \sqrt{1 + 4b_{20}^2}, \ h_{00} = \frac{e^2}{4b_{20}}. \\
    h_{20} \text{ arbitrary function of } \alpha. \ \text{ABS2009: } h_{20} = \alpha/(1 - \alpha^2).
\end{align*}
\]
\( \hat{h}_B \) and \( \hat{Q}_B \)

**\( \hat{h}_B \)**

\[ \hat{h}_{20} = \frac{e}{2} \sqrt{\alpha^2 - e^2} \]

\[ \hat{h}_B(x_1, x_2, \alpha) = \left( \frac{e}{2} (x_1^2 + x_2^2 + \alpha^2) + \alpha x_1 x_2 - \frac{e^3}{2} \right) / \sqrt{\alpha^2 - e^2} \]

\[ = F_B(x_1, x_2, \alpha) / \sqrt{\alpha^2 - e^2}. \]

**\( \hat{Q}_B \)**

\[ \hat{Q}_B = \sqrt{\beta_1^2 - e^2 (x_1 x_4 + x_2 x_3)} + \sqrt{\alpha_1^2 - e^2 (x_1 x_2 + x_3 x_4)} + \]

\[ \frac{\alpha_1 \sqrt{\beta_1^2 - e^2} + \beta_1 \sqrt{\alpha_1^2 - e^2}}{e} (x_1 x_3 + x_2 x_4) \]

\[ - \sqrt{\beta_1^2 - e^2} \sqrt{\alpha_1^2 - e^2} \frac{(\alpha_1 \sqrt{\beta_1^2 - e^2} + \beta_1 \sqrt{\alpha_1^2 - e^2})}{e}. \]
Types of pencils of conics: A, B
Types of pencils of conics: C, D, E
Deformation

Deformation of the Kowalevski top

\[ F(x_1, x_2, s) := s^2A + sB + C = 0. \]

A gauge transformation

\[ s \mapsto t + \alpha. \]

\[ F_\alpha(x_1, x_2, t) = t^2A + t(B + 2\alpha A) + (C + \alpha B + \alpha^2 A) = 0. \]

\[ C = F^2 - EG, \quad A = (x_1 - x_2)^2 \]
\[ A_\alpha = A \]
\[ B_\alpha = B + 2\alpha A \]  \hspace{1cm} (3)
\[ C_\alpha = C + \alpha B + \alpha^2 A \]
\[ F_\alpha = F + \alpha F_1 \]
\[ E_\alpha = E + \alpha E_1 \]  \hspace{1cm} (4)
\[ G_\alpha = G + \alpha G_1 \]
\[ B = 2FF_1 - E_1 G - EG_1 \]
\[ A = F_1^2 - E_1 G_1 \]  \hspace{1cm} (5)
From

\[ B = -2(Ex_1x_2 + F(x_1 + x_2) + G) \]

we get

\[ F_1 = -(x_1 + x_2) \]
\[ G_1 = 2x_1x_2 \]
\[ E_1 = 2 \]

One easily checks

\[ F_1^2 - E_1G_1 = A. \]
$E_\alpha = 6l_1 - (x_1 + x_2)^2 + 2\alpha$

$F_\alpha = 2cl + x_1x_2(x_1 + x_2) - \alpha(x_1 + x_2)$

$G_\alpha = c^2 - k^2 - x_1^2x_2^2 + 2\alpha x_1x_2$
Elastic deformation

Jurdjevic considered a deformation of the Kowalevski case associated to a Kirchhoff elastic problem. The systems are defined by the Hamiltonians

$$H = M_1^2 + M_2^2 + 2M_3^2 + \gamma_1$$

where deformed Poisson structures \(\{\cdot, \cdot\}_\tau\) are defined by

$$\{M_i, M_j\}_\tau = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\}_\tau = \epsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\}_\tau = \tau \epsilon_{ijk} M_k,$$

the deformation parameter takes values \(\tau = 0, 1, -1\). The classical Kowalevski case corresponds to the case \(\tau = 0\).
Denote

\[ e_1 = x_1^2 - (\gamma_1 + i\gamma_2) + \tau \]
\[ e_2 = x_2^2 - (\gamma_1 - i\gamma_2) + \tau, \]

where

\[ x_{1,2} = \frac{M_1 \pm iM_2}{2}. \]
Integrals of motion

The integrals of motion

\[ l_1 = e_1 e_2 \]
\[ l_2 = H \]
\[ l_3 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 \]
\[ l_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \tau (M_1^2 + M_2^2 + M_3^2) \]

may be rewritten in the form:

\[ k^2 = l_1 = e_1 \cdot e_2 \]
\[ M_3^2 = e_1 + e_2 + \hat{E}(x_1, x_2) \]
\[ M_3 \gamma_3 = -x_2 e_1 - x_1 e_2 + \hat{F}(x_1, x_2) \]
\[ \gamma_3^2 = x_2^2 e_1 + x_1^2 e_2 + \hat{G}(x_1, x_2) , \]
where

\[
\begin{align*}
\hat{G}(x_1, x_2) &= -x_1^2x_2^2 - 2\tau x_1x_2 - 2\tau(l_1 - \tau) + \tau^2 - l_2 \\
\hat{F}(x_1, x_2) &= (x_1x_2 + \tau)(x_1 + x_2) + l_3 \\
\hat{E}(x_1, x_2) &= -(x_1 + x_2)^2 + 2(l_1 - \tau).
\end{align*}
\]
A gauge transformation

\[ s \mapsto t + \alpha \]

transforms the Kowalevski top to Jurdjevic elasticae according to the formulae

\[ \tau = -\alpha \]
\[ l_1 = 3l_1 \]
\[ l_3 = 2cl \]
\[ l_2 = c^2 - k^2 + 2\alpha (3l_1 + \alpha) + \alpha^2 \]
<table>
<thead>
<tr>
<th>Classification</th>
<th>Time Period</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petrov classification</td>
<td>1950’-60’</td>
<td>General relativity, classical differential geometry, Weyl curvature tensor.</td>
</tr>
<tr>
<td>Krichever - V. D. classification</td>
<td>1980’-90’</td>
<td>$4 \times 4$ solutions of the Yang-Baxter equation, lattice statistical mechanics.</td>
</tr>
<tr>
<td>Adler-Bobenko-Suris classification</td>
<td>2009</td>
<td>ABS quad graphs.</td>
</tr>
<tr>
<td>V. D.- Kukic classification</td>
<td>2011</td>
<td>Strongly discriminantly separable polynomials of second degree in each of three variables.</td>
</tr>
</tbody>
</table>
Experimental Math

Other \( n \)-valued groups

\[
p_3 = s_1^3 - 3^3 s_3
\]
\[
Dp_3 = y^2 x^2 (x - y)^2.
\]
\[
p_4 = s_1^4 - 2^3 s_1^2 s_2 + 2^4 s_2^2 - 2^7 s_1 s_3
\]
\[
Dp_4 = y^3 x^3 (x - y)^2 (y + 4x)^2 (4y + x)^2.
\]
\[
p_5 = s_1^5 - 5^4 s_1^2 s_3 + 5^5 s_2 s_3
\]
\[
Dp_5 = y^4 x^4 (x - y)^4 (-y^2 - 11xy + x^2)^2 (-y^2 + 11xy + x^2)^2.
\]
Our book: V. D, M. Radnovic, Poncelet Porisms and Beyond, Springer 2011, Russian version RCD 2010
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**Billiard algebra, integrable line congruences and DR-nets** in progress
Dear Professor Magri,
Auguri! Many happy returns!