Geometry of “linearly degenerate” Frobenius manifolds

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WDVV associativity equations for $ F = F(v^1, v^2, \ldots, v^n)$

$$
\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \eta^{\lambda \mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F}{\partial v^\delta \partial v^\beta \partial v^\lambda} \eta^{\lambda \mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\alpha}, \quad \alpha, \beta, \gamma, \delta = 1, \ldots, n
$$

$$
\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^1} = \eta_{\alpha \beta}
$$

$$(\eta_{\alpha \beta})_{1 \leq \alpha, \beta \leq n} \quad \text{and} \quad (\eta^{\alpha \beta})_{1 \leq \alpha, \beta \leq n}$$

are mutually inverse constant symmetric matrices
Equivalently: the family of commutative algebras

\[ e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(v)e_\gamma, \quad \alpha, \beta = 1, \ldots, n \]

depending on \( v = (v^1, v^2, \ldots, v^n) \)

with the structure constants given by the third derivatives

\[ c^\gamma_{\alpha\beta}(v) = \eta^{\gamma\nu} \frac{\partial^3 F(v)}{\partial v^\nu \partial v^\alpha \partial v^\beta} \]

is associative
Main motivations: to clarify a connection between

- bihamiltonian recursion (F. Magri, ’78)

and

- topological recursion (E. Witten, M. Kontsevich et al.)

in integrable PDEs
General scheme: (B.D., Youjin Zhang)

Given a solution to WDVV → Integrable hierarchy of the topological type

Includes many known cases (KdV, NLS, Toda, ADE-Drinfeld-Sokolov) but also yields new integrable hierarchies

Problem: to solve WDVV
For \( n \geq 3 \) one arrives at a system of nonlinear PDEs for \( F \)

How to solve?

Particular solutions from reductions

1. Scaling reduction

\[
F \left( \lambda^{d_1} v^1, \lambda^{d_2} v^2, \ldots, \lambda^{d_n} v^n \right) = \lambda^{d_F} F \left( v^1, v^2, \ldots, v^n \right)
\]

Then, under semisimplicity assumption, WDVV reduces to isomonodromy problem (B.D., ’91)
2. Another reduction: to select solutions to WDVV related to linearly degenerate PDEs

(B.D., M.Pavlov, S.Zykov, 2011)
First: reminder about semisimple solutions to WDVV

By definition the associative algebra

\[ e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma(v)e_\gamma, \quad \alpha, \beta = 1, \ldots, n \]

\[ c_{\alpha\beta}^\gamma(v) = \eta^{\gamma\nu} \frac{\partial^3 F(v)}{\partial v^\nu \partial v^\alpha \partial v^\beta} \]

is semisimple for generic \( v = (v^1, v^2, \ldots, v^n) \)
Theorem (B.D., ’91) There exist **canonical** coordinates \( u_1, \ldots, u_n \) on a semisimple Frobenius manifold in which the multiplication table takes the standard form

\[
\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}
\]

the flat metric becomes diagonal

\[
d s^2 = \eta_{\alpha\beta} d v^\alpha d v^\beta = \sum_{i=1}^{n} h_i^2(u) d u_i^2
\]

and the **rotation coefficients** \( \gamma_{ij}(u) = \frac{1}{h_j} \frac{\partial h_i}{\partial u_j} \) are symmetric \( \gamma_{ji} = \gamma_{ij} \) (**Egorov** metric)
Corollary. The rotation coefficients of a semisimple solution to WDVV satisfy **Darboux - Egorov** equations

\[
\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik} \gamma_{kj}, \quad i, j, k \text{ distinct}
\]

\[
\sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u_k} = 0, \quad i \neq j
\]

Conversely, from any solution to Darboux - Egorov eqs. one can uniquely (up to symmetries) reconstruct a semisimple solution to WDVV
So, we arrive at the problem of selecting particular solutions to the Darboux - Egorov equations

Second: reminder about connection between WDVV and integrable systems of hydrodynamic type

Any solution to WDVV → infinite family of commuting Hamiltonian PDEs of the form \( v_t = A(v)v_x \)
Poisson brackets
\[ \{ v^\alpha(x), v^\beta(y) \} = \eta^{\alpha\beta} \delta'(x - y) \]

Densities of commuting Hamiltonians
recovered from the generating functions
\[ \theta(v, z) = \sum_{p \geq 0} \theta_p(v) z^p \]
that are flat sections of the deformed connection
\[ \tilde{\nabla} d\theta = 0 \]
\[ \tilde{\nabla}_X Y = \nabla_X Y + z X \cdot Y \]
In the semisimple case all these commuting PDEs diagonalize in the canonical coordinates

\[
\frac{\partial u}{\partial t} = \Lambda(u) \frac{\partial u}{\partial x}, \quad \Lambda(u) = \text{diag} (\lambda_1(u), \ldots, \lambda_n(u))
\]

\[u = (u_1, \ldots, u_n) \quad (\text{Riemann invariants!})\]
Definition. The diagonal system

\[ u_t = \Lambda(u)u_x, \quad \Lambda(u) = \text{diag}(\lambda_1(u), \ldots, \lambda_n(u)) \]

is called **linearly degenerate** if

\[ \frac{\partial \lambda_i(u)}{\partial u_i} = 0, \quad i = 1, \ldots, n \]
Definition. A semisimple solution to WDVV is called linearly degenerate if there is at least one linearly degenerate PDE in the linear span of the commuting family

\[ \gamma_{ij}(u) = \left[ G \left( 1 - \frac{1}{\rho} \tanh \rho U \cdot G \right)^{-1} \right]_{ij}, \quad i, j = 1, \ldots, n, \quad i \neq j \]

\[ U = \text{diag}(u_1, \ldots, u_n) \]

\[ G \quad \text{a symmetric matrix satisfying} \quad G^2 = \rho^2 \cdot 1 \]
Proof. Step 1: the matrix of rotation coefficients

\[ \Gamma = (\gamma_{ij}(u))_{1 \leq i, j \leq n} \]

satisfies

\[ \frac{\partial \Gamma}{\partial u_k} = \Gamma E_k \Gamma + \sigma_k(u_k) E_k, \quad k = 1, \ldots, n \]

with some functions \( \sigma_1(u_1), \ldots, \sigma_n(u_n) \)

\[ (E_k)_{i,j} = \delta_{ik} \delta_{jk} \]
Step 2: changes of Riemann invariants

\[ \tilde{u}_k = f_k(u_k), \quad k = 1, \ldots, n \]

\[ \tilde{\gamma}_{ij} = \frac{\gamma_{ij}}{\sqrt{f'_i(u_i)f'_j(u_j)}} - \frac{f''_i(u_i)}{2[f'_i(u_i)]^2} \delta_{ij}, \quad i, j = 1, \ldots, n \]

keeps the form of the equations

\[ \frac{\partial \tilde{\Gamma}}{\partial \tilde{u}_k} = \tilde{\Gamma} E_k \tilde{\Gamma} + \tilde{\sigma}_k(\tilde{u}_k) E_k \]

with

\[ f'_k^2 \tilde{\sigma}_k = \sigma_k - \frac{1}{2} S_{u_k}(f_k) \]

Here

\[ S_u(f) = \frac{f'''}{f'} - \frac{3}{2} \frac{f''^2}{f'^2} \]

is the Schwarzian
Using these symmetries we reduce the Main System to
\[
\frac{\partial \Gamma}{\partial u_k} = \Gamma E_k \Gamma, \quad k = 1, \ldots, n
\]

General solution \( \Gamma = G \left( 1 - UG \right)^{-1} \) where

\( U = \text{diag}(u_1, \ldots, u_n) \)

\( G \) is an arbitrary constant symmetric matrix
Step 3: to select those solutions to the Main System that yield solutions to the Darboux - Egorov eqs.

Main Lemma. A solution \( \Gamma = G (1 - UG)^{-1} \) to the Main System \( \frac{\partial \Gamma}{\partial u_k} = \Gamma E_k \Gamma \) can be obtained, by a change of Riemann invariants from a solution to the Darboux - Egorov eqs. \textit{iff} the matrix \( G \) satisfies the quadratic equation

\[
G R G + Q G + G Q + P = 0
\]

for some diagonal matrices \( P, Q, R \).
The last step: to solve the matrix quadratic equation

\[ GRG + QG + GQ + P = 0 \]

(so-called *algebraic Riccati equation*)

The clue: use the symmetry group of the Main System

\[ \tilde{u}_k = \frac{a_k u_k + b_k}{c_k u_k + d_k}, \quad \left( \begin{array}{cc} a_k & b_k \\ c_k & d_k \end{array} \right) \in SL_2(\mathbb{R}), \quad k = 1, \ldots, n \]

\[ \tilde{G} = (C + D G)(A + B G)^{-1} \]
Buon compleanno, Franco!