Commuting vector fields, integrable PDEs of hydrodynamic type, and the gradient catastrophe of multidimensional waves

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Outline

Introduction

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IST for vector fields

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The Cauchy problem for small data in n+1 dims
INTEGRABLE SOLITON PDEs
Waves propagating in weakly nonlinear and (strongly and weakly) dispersive media are described by integrable soliton equations. The Spectral Transform is the spectral method allowing one to solve the Cauchy problem for such PDEs, predicting that a localized disturbance evolves into a number of soliton pulses + radiation. Soliton = balance between nonlinearity and dispersion.

INTEGRABLE PDEs of HYDRODYNAMIC TYPE
There is another class of integrable PDEs, the so-called dispersionless PDEs (or PDEs of hydrodynamic type), whose Lax pair is made of vector fields ⇒ PDEs of hydrodynamic type can be in arbitrary dimensions. A novel ST for vector fields has been recently developed, to solve the Cauchy problem, allowing, in particular, to establish if, due to the lack of dispersion, the nonlinearity of the PDE is “strong enough” to cause the gradient catastrophe of localized multidimensional disturbances.
Nonlinear wave breaking and shock wave propagation in multidimensions are very frequent natural phenomena:

1. colour print by Hokusai, 1826; 2. micro-explosion in air

3. breaking waves in clouds; 4. Lagoon Nebula by Hubble

but difficult to be treated mathematically.
How does a water wave break?

The first breaking takes place in a point of spacetime, rapidly propagating in the transversal direction, and slowly propagating longitudinally. Can we describe analytically this phenomenon?
Equations describing such phenomena (equations of acoustics, hydrodynamics, plasma physics, etc.) are systems of PDEs, in general too complicated for extracting satisfactory informations on the phenomenon.

Simplifying hypotheses, physically relevant, are useful, in the search for a simpler model equation, but able to capture the essence of the phenomenon:

**Universal properties of the model.** Consider ANY system of PDEs:

i) characterized by a nonlinearity of hydrodynamic type (more generally, a nonlinearity containing quadratic terms)

ii) whose linear limit, at least in some approximation, is described by the wave equation. Then,

iii) studying the propagation of weakly nonlinear and quasi-one-dimensional waves and

iv) neglecting dispersion and dissipation, one obtains the **dKP model in** $n + 1$ **dimensions**, arising from many physical contexts, like acoustics, hydrodynamics, plasma physics, etc.::

$$
(u_t + uu_x)_x + \Delta_\perp u = 0, \quad u = u(x, \vec{y}, t), \quad \vec{y} = (y_1, \ldots, y_{n-1})
$$

$$
\Delta_\perp = \sum_{i=1}^{n-1} \partial_{y_i}^2, \quad n \geq 2,
$$

(1)
By the proper change of variables:

\[ x = X_1 - T = O(1), \quad y_j = \epsilon^{1/2} X_{j+1}, \quad j = 1, \ldots, n - 1, \quad t = \epsilon T \quad (2) \]

the spherical wave front \((\sum_{k=1}^{n} X_k^2)^{1/2} - T\) becomes, approximately, its second order contact, its osculating ellipsoidal paraboloid:

\[
(\sum_{k=1}^{n} X_k^2)^{1/2} - T \sim x + \frac{1}{2t} \sum_{k=1}^{n-1} y_k^2 = O(1). \quad (3)
\]
Basic example: Acoustics. $n = 3$. Due to the above rescalings, $\vec{k}_\perp = \epsilon^{1/2} \vec{\kappa}_\perp$, the dispersion relation $\omega(\vec{k})$ of the wave equation reduces to that of the linearized $dKP_n$ (6): $u_{tx} + \Delta_\perp u = 0$.

$$\omega(\vec{k}) = \sqrt{\sum_{j=1}^n k_j^2} \sim k_1 + \epsilon \frac{\kappa_\perp^2}{2k_1}, \quad \kappa_\perp^2 = \sum_{j=2}^n \kappa_j^2,$$

$$\theta(\vec{X}, T) = \vec{k} \cdot \vec{X} - \omega(\vec{k}) T \sim k_1 x + \vec{\kappa}_\perp \cdot \vec{y} - \frac{\kappa_\perp^2}{2k_1} t,$$ (4)

The nonlinear part $(uu_x)_x$ arises at the second order in the multiscale $\epsilon$-expansion of the equs of acoustics:

$$\rho_T + \nabla \cdot (\rho \vec{v}) = 0, \quad \vec{v}_T + (\vec{v} \cdot \nabla) \vec{v} + (\nabla p)/\rho = a \Delta \vec{v}, \quad p = P(\rho, S), \quad a = 0$$

$$\rho = \rho_0 + \epsilon \rho_1 + O(\epsilon^2), \quad p = p_0 + \epsilon p_1 + O(\epsilon^2), \quad \vec{v} = \epsilon \vec{v}^{(1)} + O(\epsilon^2)$$

$O(\epsilon)$: wave equation $f_{TT} - c^2 \Delta f = 0$ for $\rho_1 = (\rho_0/c) \nu_1^{(1)} = c^{-2} \rho_1$

$O(\epsilon^2)$: $dKP_3$ equation $(u_t + uu_x)_x + \Delta_\perp u = 0$ for $u = \rho_1$
Universality vs Integrability
The dKP equation arises as a result of a physically motivated multiscale expansion from a very large class of systems of nonlinear PDEs; it is an example of universal model equation.
a) For this reason a universal equation is clearly an applicable equation.
b) For the same reason, it is also a very distinguished mathematical equation, expected to possess, say, a lot of symmetries. Why? Because “it is enough that in the large class of PDEs generating the model equation through multiscale expansion there exists at least an example possessing lots of symmetries, since this property is inherited by the model equation through the multiscale expansion”. If, in particular, this large class of PDEs contains an integrable model, then the model equation inherits such property, being integrable too (Calogero).
Therefore it is not a surprising coincidence if the $dKP_n$ equations turn out to be integrable for $n = 1, 2$ and, for $n > 2$, the existence of distinguished symmetries allows one to solve, at least asymptotically, the Cauchy problem for small data.
\[
(u_t + uu_x)_x + \Delta \perp u = 0,
\]

\(n = 1\) the (Riemann-Burgers-) Hopf equation

\[u_t + uu_x = 0, \Rightarrow u = F(x - ut)\]

prototype model in the description of wave breaking of 1D waves. Each characteristic curve carries its own portion of profile \(u\). When characteristics meet first, the solution becomes multivalued and the profile has a vertical slope.

\[t_B = \min \zeta \left( -\frac{1}{F'(\zeta)} \right) = -\frac{1}{F'(\zeta_b)} \quad \Rightarrow \quad x_b = \zeta_b + F(\zeta_b) t_b\]
Dissipative regularization and shock formation. If \( u_t + uu_x = 0 \) is regularized by the Burgers equation \( u_t + uu_x = \nu u_{xx}, \ 0 < \nu << 1 \), after the first breaking, formation of a shock: two characteristics \( \zeta_1 \) and \( \zeta_2 \) meet at the shock front \( \xi_S(t) \), whose position is determined by an equal area condition:

\[
\frac{d\xi_S(t)}{dt} = \frac{F(\zeta_1) + F(\zeta_2)}{2},
\]

with the initial conditions:

\[
\xi_S(t_b) = \xi_b, \ \zeta_1(t_b) = \zeta_2(t_b) = \zeta_b.
\]
For $n = 2$ one obtains the integrable dKP equation

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t)$$

(9)

**NATURAL QUESTIONS:**
Can dKP be viewed as the prototype model for describing the gradient catastrophe of 2D waves?

1) Do localized waves evolving according to dKP break?
2) If yes, does a small initial datum also break?
3) If yes, does breaking take place in a point of the $(x, y)$ plane or on a line?
4) Do the geometric and analytic aspects of breaking exhibit universal feature, as in the (1+1)-dimensional case?
5) How are these features connected with the dKP initial data?
Commuting vector fields generate integrable PDEs in arbitrary dimensions [Zakharov Shabat ’79]

IST for VECTOR FIELDS [SV Manakov and PMS ’06]

Integrability scheme:

\[(u_t + uu_x)_x + u_{yy} = 0 \iff [\hat{L}_1, \hat{L}_2] = 0\]  \hspace{1cm} (10)

\[
\hat{L}_1 = \partial_y + \{H_1, \cdot\}_(\lambda, x) = \partial_y + \lambda \partial_x - u_x \partial_\lambda,
\]

\[
\hat{L}_2 = \partial_t + \{H_2, \cdot\}_(\lambda, x), = \partial_t + (\lambda^2 + u) \partial_x + (-\lambda u_x + u_y) \partial_\lambda
\]

\[H_1 \equiv \frac{\lambda^2}{2} + u(x, y, t), \quad H_2 \equiv \frac{\lambda^3}{3} + \lambda u - \partial_x^{-1} u_y,\]  \hspace{1cm} (11)

[Zakharov ’94]

Novel features of the IST for vector fields

Since the Lax pair is made of vector fields, Hamiltonian in the heavenly and dKP reductions:

1) The space of eigenfunctions is a ring: if \(f_1, f_2\) are two eigenfunctions, then an arbitrary differentiable function \(F(f_1, f_2)\) of them is also an eigenfunction.

2) In the heavenly and dKP (Hamiltonian) reductions, the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket: if \(f_1, f_2\) are two eigenfunctions, then their Poisson bracket \(\{f_1, f_2\}\) is also an eigenfunction.

Cauchy problem within the class of rapidly decreasing real 2D waves
If $f$ is a solution of $\hat{L}_1 f = 0$, then

$$f(x, y, \lambda) \to f_\pm(\xi, \lambda), \ y \to \pm \infty,$$
$$\xi := x - \lambda y;$$

i.e., asymptotically, $f$ is an arbitrary function of $\xi = (x - \lambda y)$, and $\lambda$.

Jost eigenfunctions $\vec{\varphi}(\vec{x}, z, \lambda)$:

$$\vec{\varphi}(\vec{x}, z, \lambda) \equiv \begin{pmatrix} \varphi_1(\vec{x}, z, \lambda) \\ \varphi_2(\vec{x}, z, \lambda) \end{pmatrix} \to \begin{pmatrix} -\lambda^2 t - \lambda y + x \\ \lambda \end{pmatrix} \equiv \vec{\xi}, \ y \to -\infty.$$

(13)

$$G_J(x, y; \lambda) = \theta(y) \delta(x - \lambda y).$$

(14)

Analytic eigenfunctions $\vec{\psi}_\pm(x, y, \lambda)$ (existence and uniqueness, under a small norm assumption, for $u \in H^q(\mathbb{R}^2), \ q \geq 2$, by Grinevich) normalized as:

$$\vec{\psi}_\pm(\lambda; x, y, t) \sim \begin{pmatrix} -\lambda^2 t - \lambda y + x - 2ut \\ \lambda \end{pmatrix}, \ |\lambda| >> 1$$

(15)

$$G_{\pm}(x, y; \lambda) = \pm \frac{1}{2\pi i [x - (\lambda \pm i\epsilon)y]}.$$

(16)
SPECTRAL DATA The $y = +\infty$ limit of $\varphi$ defines the scattering vector $\tilde{\sigma}$ of $\hat{L}_1$:

$$\lim_{y \to +\infty} \varphi(x, y; \lambda) \equiv \tilde{S}(\xi) = \xi + \tilde{\sigma}(\xi).$$  \hspace{1cm} (17)

DIRECT PROBLEM: $(u(x, y), v(x, y)) \rightarrow \tilde{\sigma}(\xi, \lambda)$

The counting is consistent.

The Jost solutions $\varphi_{1,2}$ and $\lambda$ form a basis in the ring of the eigenfunctions of $\hat{L}_1$. The representation of the analytic eigenfunctions $\tilde{\psi}_\pm$ in terms of $\varphi$ defines the spectral data $\tilde{\chi}_\pm$:

$$\tilde{\psi}_\pm(x, y, \lambda) = \tilde{K}_\pm(\varphi(x, y, \lambda)) = \varphi(x, y, \lambda) + \tilde{\chi}_\pm(\varphi(x, y, \lambda)),$$  \hspace{1cm} (18)

The step: $\tilde{\sigma}(\xi) \rightarrow \tilde{\chi}_\pm(\xi)$ is made solving the linear integral equations:

$$\tilde{\chi}_\pm(\tilde{\omega}) + \theta(\pm \omega_1) \left(\tilde{\sigma}(\tilde{\omega}) + \int_{\mathbb{R}^2} d\eta \tilde{\chi}_\pm(\eta) Q(\eta, \tilde{\omega})\right) = \tilde{0},$$  \hspace{1cm} (19)

for the Fourier transforms:

$$\tilde{\sigma}(\tilde{\omega}) \equiv \int_{\mathbb{R}^2} d\xi \tilde{\sigma}(\xi) e^{-i\tilde{\omega} \cdot \xi}, \quad \tilde{\chi}_\pm(\tilde{\omega}) \equiv \int_{\mathbb{R}^2} d\xi \tilde{\chi}_\pm(\xi) e^{-i\tilde{\omega} \cdot \xi}$$

$$Q(\tilde{\eta}, \tilde{\omega}) \equiv \int_{\mathbb{R}^2} \frac{d\xi}{(2\pi)^2} e^{i(\tilde{\eta} - \tilde{\omega}) \cdot \xi} \left[e^{i\tilde{\eta} \cdot \tilde{\sigma}(\xi)} - 1\right].$$  \hspace{1cm} (20)

The step $\tilde{\chi}_\pm(\xi) \rightarrow \tilde{R}(\xi)$ leads to a RH problem:

$$\tilde{\psi}^+(\lambda) = \tilde{R}(\tilde{\psi}^-(\lambda)), \quad \lambda \in \mathbb{R},$$  \hspace{1cm} (21)
INVERSE PROBLEM: \( R(\zeta_1, \zeta_2) \Rightarrow u(x, y, t) \)

\( \vec{\psi}^+ \), \( \vec{\psi}^- \) are good bases of the ring of solutions of \( \hat{L}_{1,2} \vec{g} = 0 \), generating a vector nonlinear Riemann-Hilbert (RH) problem:

\[
\psi_{\pm}^1(\lambda) = -\lambda^2 t - \lambda y + x - 2ut + \sum_{n \geq 1} \frac{q_1^{(n)}}{\lambda^n},
\]

\[
\psi_{\pm}^2(\lambda) = \lambda + \frac{u}{\lambda} + \sum_{n \geq 2} \frac{q_2^{(n)}}{\lambda^n}, \quad (q_2^{(1)} = u)
\]

where \( \vec{R}(\vec{\zeta}) = (R_1(\zeta_1, \zeta_2), R_2(\zeta_1, \zeta_2)) \in \mathbb{C}^2, \vec{\zeta} \in \mathbb{C}^2 \). In the dKP reduction:

\[
\vec{R}(\vec{R}(\vec{\zeta})) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2, \quad \{R_1, R_2\}_{\vec{\zeta}} = 0,
\]

Then \( u = F(x - 2ut, y, t) \in \mathbb{R} \) is solution of the dKP equation, where

\[
F(\zeta, y, t) = -\int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2 \left( \psi^-_1(\lambda; \zeta, y, t), \psi^-_2(\lambda; \zeta, y, t) \right).
\]

a) Existence and uniqueness of the RH problem proven by iteration, under a small norm assumption, in \( H^1(\mathbb{R}) \) wrt \( \lambda \). (D. Wu)
b) Parametric dependence on \((x - 2ut, y, t)\) through normalization at \( \infty \). Inverse formula is implicit equation similar to \( u = F(x - ut) \Rightarrow \) localized solutions of dKP are expected to break at finite time.
The longtime behavior of dKP solutions

Let \( t \gg 1 \) and

\[
x = \tilde{x} + v_1 t, \quad y = v_2 t, \\
\tilde{x} - 2ut, \ v_1, \ v_2 = O(1), \quad v_2 \neq 0, \quad t \gg 1.
\]

(25)

On the parabola

\[
x + \frac{y^2}{4t} = \tilde{x} \quad (v_1 = -\frac{v_2^2}{4}),
\]

(26)

the longtime behaviour of the solutions of the dKP equation is given by

\[
u = \frac{1}{\sqrt{t}} G\left(x - 2ut + \frac{y^2}{4t}, \frac{y}{2t}\right) \left(1 + o\left(\frac{1}{\sqrt{t}}\right)\right),
\]

\[
G(\xi, \eta) = -\frac{1}{2\pi i} \int_\mathbb{R} d\mu R_2\left(\xi + \mu^2 + a_1(\mu; \xi, \eta), \eta + a_2(\mu; \xi, \eta)\right),
\]

(27)

where \( a_j(\mu: \xi, \eta) \) solve “asymptotic” RH problem:

\[
\vec{A}^+(\mu; \xi, \eta) = \vec{A}^-(\mu; \xi, \eta) + \vec{R}(\vec{A}^-(\mu; \xi, \eta)), \quad \mu \in \mathbb{R},
\]

\[
\vec{A}^\pm(\mu; \xi, \eta) = \begin{pmatrix} \xi + \mu^2 \\ \eta \end{pmatrix} + \vec{a}(\mu; \xi, \eta).
\]

(28)

Small initial data start evolving according to \( u_{tx} + u_{yy} = 0 \). Only in the longtime regime the nonlinear term becomes relevant, causing the breaking of the small localized initial wave in a point of the parabola.
Solvable nonlinear RH problems Consider an autonomous Hamiltonian two-dimensional dynamical system with Hamiltonian

\[ H(x) = \mathcal{H}(E(x)) \]  

(29)

where \( x \equiv (q, p) \) are canonically conjugated coordinates, \( E(x) \) is a polynomial function of the coordinates and \( \mathcal{H}(\cdot) \) is an arbitrary function of a single argument, corresponding to the equations of motion

\[ \frac{dx}{d\tau} = \mathcal{H}'(E) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_x E(x). \]  

(30)

Introducing action-angle variables in the usual way:

\[ J \equiv \frac{1}{2\pi} \oint p(q, H) dq, \quad \Rightarrow \quad H = H(J), \]

\[ \theta - \theta_0 \equiv \omega(J) \mathcal{H}'(E)(\tau - \tau_0) = \mathcal{H}'(E) \int_{q_0}^{q} \frac{\partial p(q', H(J))}{\partial J} dq', \]  

(31)

\[ \omega(J) \equiv \frac{\partial H(J)}{\partial J}, \]

the solution can be found inverting the quadrature (31):

\[ \vec{x} = \vec{D}(\theta - \theta_0; \vec{x}_0, J), \]

\[ \{\mathcal{D}_1, \mathcal{D}_2\}_{(q_0, p_0)} = 1 \text{ symplectic} \]  

(32)
Identifying $\vec{x}(\tau) \rightarrow \vec{\psi}^+(\lambda)$, $\vec{x}(\tau_0) \rightarrow \vec{\psi}^-(\lambda)$, equation (32) becomes the two-dimensional vector NRH problem

$$\vec{\psi}^+ = \vec{D} \left( \omega(J(\vec{\psi}^-))\mathcal{H}'(E(\vec{\psi}^-)); \vec{\psi}^-, J(\vec{\psi}^-) \right) \equiv \vec{R} \left( \vec{\psi}^- \right)$$  (33)

connecting the (−) and (+) vector functions through the canonical transformation. $E(\vec{x}_0) = E(\vec{x}) \rightarrow E(\vec{\psi}^-) = E(\vec{\psi}^+)$, “invariant” of the NRH problem (33). Since $E(\vec{\psi})$ is a polynomial function of its arguments, equation $E(\vec{\psi}^-) = E(\vec{\psi}^+)$ define a polynomial in $\lambda$:

$$E(\vec{\psi}^-(\lambda)) = E(\vec{\psi}^+(\lambda)) \equiv W(\lambda; \vec{q}_N^1, \vec{q}_N^2),$$  (34)

given by the polynomial part of the asymptotic expansion of $E(\vec{\psi}^\pm)$ for large $\lambda$, depending on a finite number of coefficients $\vec{q}_1^N = (q_1^1, \ldots, q_1^N)$, $\vec{q}_2^N = (q_2^1, \ldots, q_2^N)$ of the expansion. Since $E$ is a real function of its arguments $\Rightarrow \overline{W(\lambda)} = W(\lambda) \Rightarrow$

$$\mathcal{H}'(\cdot) = i f(\cdot).$$  (35)
Define

\[
\theta^{\pm}(\lambda) \equiv \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda'-(\lambda^{\pm}0)} \left( i\omega(J(\bar{\psi}^- (\lambda'))f(W(\lambda'))) \right) (\lambda'),
\]

\[
\theta^{\pm}(\lambda; \bar{q}^{N_1}_1, \bar{q}^{N_2}_2) \equiv - \sum_{n \geq 1} \frac{\langle \lambda^{-1}\omega f \rangle}{\lambda^n}, \quad |\lambda| >> 1,
\]

\[
\langle \lambda^n g \rangle \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^n g(\lambda)d\lambda
\]

so that \( i\omega f = \theta^+ - \theta^- \), the RH problem becomes

\[
\vec{D}(-\theta^+; \psi^+, J(\psi^+)) = \vec{D}(-\theta^-; \psi^-, J(\psi^-)),
\]

(37)

and provides the solution of the problem if \( \vec{D} \) is formally expandible, for large \( \lambda \), in Laurent series with a finite number of positive powers:

\[
\vec{D}(-\theta^{\pm}(\lambda; \bar{q}^{N_1}_1, \bar{q}^{N_2}_2); \psi^{\pm}, J(\bar{\psi}^{\pm})) = \bar{A}(\lambda)
\]

\[
\bar{A}(\lambda; \bar{q}^{N_1}_1, \bar{q}^{N_2}_2) = \left( \vec{D}(-\theta^{\pm}(\lambda; \bar{q}^{N_1}_1, \bar{q}^{N_2}_2); \bar{\psi}^{\pm}, J(\bar{\psi}^{\pm})) \right)_+ + \text{polynomial}
\]

(38)

Since the negative power part of such expansion is absent, the corresponding coefficients are zero; the first \( N_1 + N_2 \) of such equations for the first and second component of \( \vec{D} \) define a closed system of algebraic equations for the unknown fields \( (\bar{q}^{N_1}_1, \bar{q}^{N_2}_2) \), providing the wanted integration of the target nonlinear PDE.
Increasing the richness of the solution space

Let $\psi_{1,2}^{\pm}$ be the solutions of the above solvable NRH problem, satisfying the usual asymptotics:

$$
\psi_1^{\pm}(\lambda) = -\lambda^2 t - \lambda y + x - 2tq_2^{(1)} + \sum_{n \geq 1} \frac{q_{1}^{(n)}}{\lambda^n}, \\
\psi_2^{\pm}(\lambda) = \lambda + \frac{q_{2}^{(1)}}{\lambda} + \sum_{n \geq 2} \frac{q_{2}^{(n)}}{\lambda^n}, \quad q_{2}^{(1)} = u.
$$

(39)

then arbitrary differentiable functions of $(\psi_1^{\pm}, \psi_2^{\pm})$ are also solutions.

The first transformation: $\psi_1^{(1)\pm} \equiv \psi_1^{\pm} + a_1 f^{(1)}(\psi_2^{\pm})$, $\psi_2^{(1)\pm} \equiv \psi_2^{\pm}$

where $a_1 \in \mathbb{R}$ and $f^{(1)}$ is an arbitrary real function of one variable, is triangular, invertible, symplectic and preserves the reality constraint $\Rightarrow \psi_1^{(1)\pm}$ are also canonically conjugated solutions satisfying the reality constraint.

The second transformation:

$\psi_1^{(2)\pm} \equiv \psi_1^{(1)\pm}$, $\psi_2^{(2)\pm} \equiv \psi_2^{(1)\pm} + a_2 f^{(2)}(\psi_1^{(1)\pm})$

has the same properties. Alternating the two transformations, at the $m^{th}$ step, one constructs canonically conjugated solutions $\psi_{1,2}^{(m)\pm}$ satisfying the reality constraint, and parametrized by $(m + 1)$ arbitrary real functions $f, f^{(1)}, \ldots, f^{(m)}$ of a single argument.
Examples: the invariant $\psi_1^+ + a(\psi_2^+)^n$, $n \in \mathbb{N}^+$

If $E(q, p) = q + ap^n$, where $a$ is a real parameter and $n \in \mathbb{N}^+$, then equation (32) read

$$q(\tau) = q_0 + ap_0^n - a(p_0 - \mathcal{H}'(E)(\tau - \tau_0))^n, \quad p(\tau) = p_0 - \mathcal{H}'(E)(\tau - \tau_0), \Rightarrow$$

$$\psi_1^+ = \psi_1^- + a\psi_2^- n - a\left(\psi_2^- - if(\psi_1^- + a\psi_2^- n)\right)^n,$$

$$\psi_2^+ = \psi_2^- - if(\psi_1^- + a\psi_2^- n), \quad \lambda \in \mathbb{R},$$

Invariance equation $\psi_1^+ + a\psi_2^+ n = \psi_1^- + a\psi_2^- n = -t\lambda^2 - y\lambda + x - 2ut + a\left((\psi_2^-)^n\right)^+ \equiv W(\lambda)$, the NRH problem linearizes and decouples: $\psi_2^+(\lambda) + if^+(\lambda) = \psi_2^- + if^-(\lambda) = \lambda$, and it is solved by:

$$\psi_1^\pm = W(\lambda) - a\psi_2^\pm n, \quad \psi_2^\pm = \lambda - if^\pm(\lambda),$$

$$f^\pm(\lambda) \equiv \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{(\lambda - (\lambda \pm i0))^n} f(W(\lambda')),$$

$$f^\pm(\lambda) \sim i \sum_{n \geq 1} \langle \lambda^{n-1} f \rangle \lambda^{-n}, \quad \langle \lambda^n f \rangle \equiv \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^n f(W(\lambda))d\lambda,$$

Equation (42a) for $|\lambda| \gg 1$ yields $q_2^{(n)} = \langle \lambda^{n-1} f \rangle$, $n \geq 1$.

Since $W$ depends on the $(n - 1)$ unknowns $u, q_2^{(n)}$, $n = 2, \ldots, n - 1$, it is an algebraic system of $(n - 1)$ equ.s characterizing a family of implicit solutions of dKP parametrized by the arbitrary real function $f(\cdot)$.
The corresponding differential constraint:
\[
\left( \left( \psi_1^\pm + a\psi_2^\pm \right)^n \right)_{-1} = yu_x - 2tu_y + anu_{tn} = 0
\]

where \( u_{tn} \) is the \( n^{th} \) flow of the dKP hierarchy.

If \( n = 1 \): invariance: \( W(\lambda) = \psi_1^\pm + a\psi_2^\pm = -t\lambda^2 - (y - a)\lambda + x - 2ut \),

NRH problem linearizes and decouples:
\[
\psi_1^+ = \psi_1^- + iaf(W), \quad \psi_2^+ = \psi_2^- - if(W), \quad \lambda \in \mathbb{R},
\]
its solution:
\[
\psi_1^\pm = -t\lambda^2 - y\lambda + x - 2ut + iaf^\pm(\lambda), \quad \psi_2^\pm = \lambda - if^\pm(\lambda) \quad (43)
\]

Since \( W \) is function of \( q_2^{(1)} = u \) only:
\[
\begin{align*}
    u &= q_2^{(1)} = \frac{1}{\sqrt{t}} F \left( x + \frac{(y-a)^2}{4t} - 2ut \right), \\
    F(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} f(-\mu^2 + z) d\mu 
\end{align*} \quad (44)
\]

describing a family of solutions of dKP, constant on the parabola \( x + \frac{(y-a)^2}{4t} = \xi \), and breaking simultaneously in all points of it. The corresponding differential constraint reads
\[
(y - a)u_x - 2tu_y = 0. \quad (45)
\]

If \( n = 2 \) the solution is the same up to shifts.
If \( n = 3 \), in the longtime regime and for \(|a| \ll 1\):

\[
\begin{align*}
    u & \sim \frac{1}{\sqrt{\tau_3}} F(\xi - a\eta^3 - 2u\tau_3), \\
    \tau_3 & = t + 3a\eta, \quad \eta = y/2t, \\
    \eta & = O(1), \quad \xi - a\eta^3 - 2u\tau_3 = O(1), \quad t \gg 1, \quad |a| \ll 1,
\end{align*}
\]

(46)

Known the first breaking time \( \tau_b \) from the well known formula

\[
\tau_b = \frac{1}{4F'(\zeta_b)^2} = \min_{\zeta \in \mathbb{R}} \frac{1}{4F'(\zeta)^2}, \quad F'(\zeta_b) < 0,
\]

(47)

equation (46b) suggests that, if \( a > 0 \), the first breaking takes place when \( t_b \sim -\infty \) at \( y_b \sim -\infty \), outside the asymptotic region (46c) of validity of our approximation, travelling towards the inner region (46) along the wave front. Now let \( t \) be close to \( \tau_b \); then, from (46b),

\[
y = \frac{2}{3a} t(\tau_b - t) \sim \frac{2}{3a} \tau_b(\tau_b - t),
\]

(48)

implying that, in the asymptotic region (46), the breaking point moves approximately with the constant speed \( 2\tau_b/(3a) \) along the wave front.
If \( n = 4 \), then

\[
\begin{align*}
    u & \sim \frac{1}{\sqrt{\tau_4}} F(\xi + a\eta^4 - 2u\tau_4), \\
    \tau_4 & = t - 6a\eta^2, \\
    \eta & = O(1), \quad \xi + a\eta^4 - 2u\tau_4 = O(1), \quad t \gg 1, \quad 0 < a \ll 1.
\end{align*}
\]

(49)

If the graph of \( F(z) \) is a single positive hump, it describes, before breaking, a saddle wave front with saddle point \((\zeta_0 + 2F(\zeta_0)/\sqrt{t}, 0)\), where \( \zeta_0 \) is the maximum of the hump: \( F'(\zeta_0) = 0 \).

Known the first breaking time \( \tau_b \) as before, the first (physical) breaking time \( t_b \) is achieved at \( y_b = 0 \) (\( \eta_b = y_b/2t_b = 0 \)) and coincides with \( \tau_b \), while \( x_b \) follows from \( x_b = \zeta_b + 2F(\zeta_b)\sqrt{t_b} \).
Using the same approach, we have solved other multidimensional nonlinear PDEs of physical and mathematical significance, like i) dKP system [Manakov-PMS’06]

\[
\begin{align*}
    u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\
    v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} &= 0,
\end{align*}
\]

(50)

(general Einstein-Weyl metric). If \( u = 0 \): \( v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0 \) (MA-S-P equ.) (here no spectral mechanism for breaking was found). ii) the second heavenly equation of Plebanski:

\[
\theta_{tz} - \theta_{xy} + \theta_{xx} \theta_{yy} - \theta_{xy}^2 = 0
\]

(51)

integrable exact reduction of the Einstein equations (here no spectral mechanism for breaking was found). iii) the two dimensional dispersionless Toda lattice:

\[
\varphi_{z_1 z_2} = (e^\varphi)_{tt}, \quad \text{or} \quad \phi_{xx} + \phi_{yy} = (e^\varphi)_{tt}
\]

(52)

relevant in Differential Geometry (continuous limit of the Laplace transformation of a conjugate net), integrable Einstein - Weyl geometries, Hele - Shaw problem (here spectral mechanism for breaking as for dKP)
Exact solutions in $n+1$ dimensions

The universal properties of the $dKP_n$ equation suggest its invariance under motions on the associated paraboloid. Indeed (6) admits the following Lie point symmetry group of transformations

$$\tilde{x} = x + \sum_{i=1}^{n-1} \left( \delta_i y_i - \delta_i^2 t \right),$$

$$\tilde{y}_j = y_j - 2\delta_j t, \quad j = 1, \ldots, n-1,$$

where $\delta_j, \quad j = 1, \ldots, n-1$ are the parameters of the group, leaving invariant the paraboloid

$$x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 = \xi.$$  \hspace{2cm} (54)

Correspondingly, $dKP_n$ possesses the following exact implicit solution:

$$u = \begin{cases} 
  t^{\frac{n-1}{2}} F \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 - \frac{2ut}{3-n} \right), & n \neq 3, \\
  t^{-1} F \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 - u t \ln t \right), & n = 3,
\end{cases}$$  \hspace{2cm} (55)

where $F$ is an arbitrary function of one argument;
characterized by the differential constraint $\sigma = 0$, where $\sigma$ is the corresponding “characteristic symmetry” of equation (6):

$$\sigma = \left( \sum_{i=1}^{n-1} \delta_i y_i \right) u_x - 2t \sum_{i=1}^{n-1} \delta_i u_{y_i} \tag{56}$$

Indeed, looking for solutions in the form

$$u = v(\xi, t), \quad \xi = x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2, \tag{57}$$

one obtains $v_t + \frac{n-1}{2t} v + vv_{\xi} = 0$. Its $v/t$ term can be eliminated by the change of variables

$$v(\xi, t) = t^{-\frac{n-1}{2}} q(\xi, \tau), \tag{58}$$

where

$$\tau(t) = \begin{cases} \frac{2}{3-n} t^{\frac{3-n}{2}}, & n \neq 3, \\ \ln t, & n = 3, \end{cases} \tag{59}$$

leading to the Hopf equation $q_\tau + qq_{\xi} = 0$, whose general solution is implicitly given by $q = F(\xi - q\tau)$, where $F$ is an arbitrary function of one argument. □
The Cauchy problem for small and localized data
Since the paraboloid (54) plays an important role in the asymptotics of the $dKP_n$ equation (see §2), the exact solution (55) is physically relevant.

Basic idea. If the initial condition is small:

$$u(x, \vec{y}, 0) = \epsilon u_0(x, \vec{y}), \quad 0 < \epsilon << 1,$$

(60)

the solution of the Cauchy problem for $dKP_n$ is well approximated by the corresponding solution for the linearized $dKP_n$ until one enters the nonlinear regime, in which the Riemann-Hopf equation becomes relevant, eventually causing wave breaking. Since the breaking time of $O(\epsilon)$ initial data evolving according to it is $\tau = O(\epsilon^{-1})$, the nonlinear regime for $dKP_n$ is characterized by the condition $t = O(\tau^{-1}(\epsilon^{-1}))$, where $\tau^{-1}$ is the inverse of (59); so that:

$$t = O\left(\tau^{-1}(\epsilon^{-1})\right) = \begin{cases} O\left(\epsilon^{-\frac{2}{3-n}}\right) & \text{if } 1 \leq n < 3, \\ O\left(e^{\epsilon^{-1}}\right) & \text{if } n = 3, \end{cases}$$

(61)

and a proper matching has to be made between the solution of the linearized $KZ_n$, valid for $t \ll O(\tau^{-1}(\epsilon^{-1}))$, and the exact solution of the previous section, valid in the nonlinear regime $t = O(\tau^{-1}(\epsilon^{-1}))$. 


Linear regime
Since the initial condition (60) is small, the solution of $dKP_n$ is well approximated, *for finite times*, by the solution of the linearized $dKP_n$ equation:

$$u(x, y, t) \sim \epsilon \int_{\mathbb{R}^n} \frac{d\vec{k}}{(2\pi)^n} \hat{u}_0(\vec{k}) e^{i(k_1 x + \vec{k}_\perp \cdot \vec{y} - \frac{k_2^2}{k_1} t)}$$  \hspace{1cm} (62)

where $\vec{k} = (k_1, \vec{k}_\perp)$ and $\hat{u}_0(\vec{k})$ is the Fourier transform of $u_0(x, y)$

$$\hat{u}_0(\vec{k}) = \int_{\mathbb{R}^n} u_0(x, y) e^{-i(k_1 x + \vec{k}_\perp \cdot \vec{y})} dxd\vec{y}$$  \hspace{1cm} (63)

approximation valid also far away from the nonlinear regime, in the longtime interval $1 \ll t \ll O\left(\tau^{-1} (\epsilon^{-1})\right)$, in which:

$$u(x, y, t) \sim t^{-\frac{n-1}{2}} \epsilon G \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2, \frac{\vec{y}}{2t} \right),$$  \hspace{1cm} (64)

where

$$G(\xi, \eta) := 2^{-n} \pi^{-\frac{n+1}{2}} \int_{\mathbb{R}} d\lambda |\lambda|^{-\frac{n-1}{2}} \hat{u}_0(\lambda, \lambda\eta) e^{i\lambda \xi - i\frac{\pi}{4} (n-1) \lambda}.$$  \hspace{1cm} (65)

valid on the paraboloid, in the space-time region

$$(x - \xi)/t, \quad y_i/t = O(1), \quad i = 1, \ldots, n,$$  \hspace{1cm} (66)

on the paraboloid (54).
The nonlinear regime
The approximate solution of $dKP_n$ in the nonlinear regime $t = O(\tau^{-1}(\epsilon^{-1}))$, obtained matching equations (64) and (55), reads as follows:

$$u \sim u^\text{as}_n(x, \bar{y}, t) \equiv \begin{cases} 
    t^{-\frac{n-1}{2}} \epsilon G \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 - \frac{2ut}{3-n}, \frac{\bar{y}}{2t} \right), & n \neq 3, \\
    t^{-1} \epsilon G \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 - u t \ln t, \frac{\bar{y}}{2t} \right), & n = 3.
\end{cases}$$

(67)

Since the $u$-term inside the first argument of function $G$, responsible for the wave breaking, is $O(t^{\frac{3-n}{2}})$ for $n \neq 3$, and $O(\ln t)$, for $n = 3$, then it is large for $n = 1, 2, 3$ and infinitesimal for $n \geq 4$. It follows that wave breaking takes place only for $n = 1, 2, 3$; for $n \geq 4$ the solution (67) coincides with the linearized solution (64), and no breaking takes place. We remark that, for $n = 2$ (the integrable case), we recover the results obtained using the IST for vector fields.

Estimate of the error: $u = u^\text{as}_n(x, \bar{y}, t)(1 + O(t^{-1})), \ n = 2, 3$.

If $n = 1$ (the Riemann - Hopf case), waves break at $t = O(\epsilon^{-1})$; if $n = 2$, waves break at $t = O(\epsilon^{-2})$; if $n = 3$, small waves break, but at $t = O(e^{1/\epsilon})$; if $n \geq 4$, small waves do not break in the longtime regime.
Explanation

This result has a clear physical meaning: the steepening of the profile, due to the term $uu_x$ of the $dKP_n$ equation

$$(u_t + uu_x)_x + \Delta_\perp u = 0,$$  \hspace{1cm} (68)

is opposed by the transversal diffraction channels, represented by the transversal Laplacian.

Increasing the dimensionality of the transversal space, the number of diffraction channels of the wave increases, until such diffraction, acting for a long time, is strong enough to prevent the breaking of small $n$ dimensional waves.

It is remarkable that small initial data break, in the longtime regime, only in $1 + 1$, $2 + 1$ and $3 + 1$ dimensions; i.e., only in physical space!
Explicit example.

If \( u_0(x, \vec{y}) = d_n \exp(-\frac{x^2 + |\vec{y}|^2}{4}) \), where \( d_n \) is a constant, then:

\[
G(\xi, \vec{\eta}) = \frac{d_n}{\sqrt{\pi}} \frac{1}{(1+|\vec{\eta}|^2)^{n+1/4}} \left[ \cos \frac{\pi(n-1)}{4} \Gamma \left( \frac{n+1}{4} \right) \, _1F_1 \left( \frac{n+1}{4}, \frac{1}{2}; \frac{-Y^2}{4} \right) + \right.
\]

\[
\sin \frac{\pi(n-1)}{4} \Gamma \left( \frac{n+3}{4} \right) \, Y \, _1F_1 \left( \frac{n+3}{4}, \frac{3}{2}; \frac{-Y^2}{4} \right) \right],
\]

\[
Y := \frac{\xi}{\sqrt{1+|\vec{\eta}|^2}},
\]

where \( \Gamma \) is the Gamma function and \( _1F_1 \) is the Kummer confluent hypergeometric function:

\[
_1F_1(a; b; z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},
\]

\[
(a)_n \equiv a(a + 1)(a + 2) \ldots (a + n - 1).
\]
If, in particular, \( n = 2 \) with \( d_2 = \sqrt{2\pi} \):

\[
G(\xi, \eta) = (1 + \eta^2)^{-\frac{3}{4}} \left[ \Gamma \left( \frac{3}{4} \right) \ \mathbf{1}_F_1 \left( \frac{3}{4}, \frac{1}{2}; -\frac{\eta^2}{4} \right) + \right.
\]

\[
Y \ \Gamma \left( \frac{5}{4} \right) \ \mathbf{1}_F_1 \left( \frac{5}{4}, \frac{3}{2}; -\frac{\eta^2}{4} \right) \right].
\]

(71)

If \( n = 3 \), with \( d_3 = 2 \):

\[
G(\xi, \eta_1, \eta_2) = \frac{\xi}{(1+\eta_1^2+\eta_2^2)^{\frac{3}{2}}} e^{-\frac{\xi^2}{4(1+\eta_1^2+\eta_2^2)}}.
\]

(72)
Geometric and analytic aspects of wave breaking

We first rewrite (67) in the more convenient form

\[ w \sim \epsilon G(\zeta, \vec{\eta}), \quad \xi = \epsilon G(\zeta, \vec{\eta}) \tau + \zeta, \]  \hfill (73)

where

\[ n = 2 : \quad w = \sqrt{tu}, \quad \tau = 2\sqrt{t}, \quad \xi = x + \frac{1}{4t} y^2, \quad \eta = \frac{y}{2t} \]  \hfill (74)

\[ n = 3 : \quad w = tu, \quad \tau = \ln t, \quad \xi = x + \frac{1}{4t} (y_1^2 + y_2^2), \quad \vec{\eta} = \frac{\vec{y}}{2t} \]  \hfill (75)

describing the evolution of an \( n \)-dimensional wave according to the \( 1 + 1 \) dimensional Riemann - Hopf equation \( w_\tau + ww_\xi = 0 \). In the following, we mainly concentrate on the case \( n = 3 \).

One solves the second of equations (73) with respect to the parameter \( \zeta \), obtaining \( \zeta(\xi, \vec{\eta}, \tau) \), and replaces it into the first, to obtain the solution \( w \sim \epsilon G(\zeta(\xi, \vec{\eta}, \tau), \vec{\eta}) \). Therefore the singularity manifold (SM) of the solution is the \( n \) - dimensional manifold

\[ S(\zeta, \vec{\eta}, \tau) \equiv 1 + \epsilon G_\zeta(\zeta, \vec{\eta}) \tau = 0 \quad \Rightarrow \quad \tau = -\frac{1}{G_\zeta(\zeta, \vec{\eta})}. \]  \hfill (76)

Since

\[ \nabla_{(\xi, \vec{\eta})} w = \frac{\epsilon \nabla_{(\zeta, \vec{\eta})} G(\zeta, \vec{\eta})}{1 + \epsilon G_\zeta(\zeta, \vec{\eta}) \tau}, \]  \hfill (77)

breaking takes place on the SM.
The first breaking time $\tau_b$ and the corresponding characteristic parameters $\vec{\zeta}_b = (\zeta_b, \vec{\eta}_b)$ through the global minimum of a function of $n$ variables:

$$\tau_b = -\frac{1}{\epsilon G_\zeta(\vec{\zeta}_b)} = \text{global min} \left( -\frac{1}{\epsilon G_\zeta \zeta, \vec{\eta}} \right) > 0,$$

characterized by the conditions:

i) $G_\zeta(\vec{\zeta}_b) < 0$, ii) $(z, Hz) > 0$, $\forall z \in \mathbb{R}^n$, where $H$ is the Hessian matrix of function $G_\zeta(\zeta, \vec{\eta})$, evaluated at $\vec{\zeta}_b$.

The corresponding point in which the first wave breaking takes place is, from (73), $\vec{\xi}_b = (\xi_b, \vec{\eta}_b) \in \mathbb{R}^n$, where:

$$\xi_b = \zeta_b + \epsilon G(\vec{\zeta}_b) \tau_b.$$  \hfill (79)

Now we evaluate equations (73) and (76) near breaking, in the regime:

$$\xi = \xi_b + \xi', \quad \vec{\eta} = \vec{\eta}_b + \vec{\eta}', \quad \tau = \tau_b + \tau', \quad \zeta = \zeta_b + \zeta',$$

where $\xi', \vec{\eta}', \tau', \zeta'$ are small.
At the leading order, we obtain a cubic equation in $\zeta'$:

$$\zeta'^3 + a(\bar{\eta}')\zeta'^2 + b(\bar{\eta}', \bar{\tau})\zeta' - \gamma X(\xi', \bar{\eta}', \bar{\tau}) = 0,$$

(81)

where

$$a(\bar{\eta}') = \frac{3}{G_{\zeta\zeta\zeta}} (G_{\zeta\zeta\eta_1} \eta_1' + G_{\zeta\zeta\eta_2} \eta_2'),$$

$$b(\bar{\eta}', \bar{\tau}) = \frac{3}{G_{\zeta\zeta\zeta}} \left[ 2G_{\zeta} \bar{\tau} + G_{\zeta\eta_1} \eta_1' \eta_1' + 2G_{\zeta\eta_1} \eta_1' \eta_2' + G_{\zeta\eta_2} \eta_2' \right],$$

$$X(\xi', \bar{\eta}', \bar{\tau}) = \xi' - \varepsilon G(\zeta_b, \bar{\eta}_b + \bar{\eta}') \tau' - \varepsilon \left[ G(\zeta_b, \bar{\eta}_b + \bar{\eta}') - G \right] \tau_b \sim \xi' + (\frac{G_{\eta_1}}{G_{\zeta}} \eta_1' + \frac{G_{\eta_2}}{G_{\zeta}} \eta_2') - \frac{G}{|G_{\zeta}|} \bar{\tau} + \frac{1}{2G_{\zeta}} \left( G_{\eta_1} \eta_1' \eta_1' + 2G_{\eta_1} \eta_1' \eta_2' + G_{\eta_2} \eta_2' \right) - \frac{1}{|G_{\zeta}|} (G_{\eta_1} \eta_1' + G_{\eta_2} \eta_2') \bar{\tau} + \frac{1}{6G_{\zeta}} (G_{\eta_1} \eta_1' \eta_1' \eta_1' + 3G_{\eta_1} \eta_1' \eta_2' \eta_2' + 3G_{\eta_1} \eta_2' \eta_1' \eta_2' + G_{\eta_2} \eta_2' \eta_2' \eta_2'), \quad \gamma = \frac{6|G_{\zeta}|}{G_{\zeta\zeta\zeta}},$$

(82)

and

$$\bar{\tau} \equiv \frac{\tau'}{\tau_b},$$

(83)

corresponding to the maximal balance

$$|\zeta'|, |\eta_1'|, |\eta_2'| = O(|\bar{\tau}|^{1/2}), \quad |X| = O(|\bar{\tau}|^{3/2}).$$

(84)
The three roots $\zeta'_1, \zeta'_\pm$ of the cubic are given by the well-known Cardano-Tartaglia formula.

The discriminant $\Delta$ of the cubic is $\Delta = R^2 + Q^3$, where

$$Q(\bar{\eta}', \bar{\tau}) = \frac{3b-a^2}{9} = -\frac{2|G_\zeta|}{G_\zeta\zeta\zeta} \bar{\tau} + \frac{1}{G_\zeta\zeta\zeta^2} [(G_\zeta\zeta\zeta G_\zeta\eta_1 - G_\zeta^2 \zeta \eta_1) \eta_1'^2 + 2(G_\zeta\zeta\zeta G_\zeta\eta_2 - G_\zeta\zeta\zeta G_\zeta\zeta\eta_2) \eta_1' \eta_2' + (G_\zeta\zeta\zeta G_\zeta\zeta\zeta - G_\zeta^2 \zeta \eta_2) \eta_2'^2],$$

$$R(\bar{\xi}', \bar{\eta}', \bar{\tau}) = \frac{\gamma}{2} X(\bar{x}', \bar{y}', \bar{t}') + \frac{ab}{18} + \frac{a}{3} Q(\bar{\eta}', \bar{\tau}).$$

At the same order, function $S$ in (76) becomes

$$S(\zeta', \bar{\eta}', \bar{\tau}) = -\ddot{\tau} + \frac{1}{2|G_\zeta|} [G_\zeta\zeta' \zeta'^2 + G_\zeta' \eta_1 \eta_1'^2 + G_\zeta' \eta_2 \eta_2'^2 + 2G_\zeta' \eta_1 + 2G_\zeta' \eta_2 + 2G_\zeta \eta_1 \eta_2] .$$

Known $\zeta'$ as function of $(\bar{\xi}', \bar{\eta}', \bar{\tau})$ from the cubic (81), the solution $w$ and its gradient are then approximated, near breaking, by the formulae:

$$w(\bar{\xi}, \bar{\eta}, \tau) \sim \epsilon G(\bar{\zeta}_b + \zeta', \bar{\eta}_b + \bar{\eta}'),$$

$$\nabla(\bar{\xi}, \bar{\eta}) w \sim \epsilon \frac{\nabla(\zeta', \bar{\eta}') G(\bar{\zeta}_b + \zeta', \bar{\eta}_b + \bar{\eta}')} {S(\zeta', \bar{\eta}', \bar{\tau})}.$$
Before breaking
If \( \tau < \tau_b \), \( \Delta = R^2 + Q^3 \) is strictly positive and only the root \( \zeta' \) is real. Correspondingly, the real solution \( w \) is single valued.
Two simplifications of the cubic
1. In a narrower volume, the cubic (81) reduces to the linear equation
   \( b\zeta' = \gamma X \), and the solution coincides with the following exact similarity solution of the equation \( w_{\tau} + w w_{\xi} = 0 \):

   \[
   w \sim \frac{\xi - \xi_b + (G_{n1}/G_\zeta)(\eta_1 - \eta_1 b) + (G_{n2}/G_\zeta)(\eta_2 - \eta_2 b)}{\tau - \tau_b} = \frac{\bar{\nu} \cdot (\xi - \xi_b)}{\tau - \tau_b},
   \]

   describing the hyperplane tangent to the wave, and

   \[
   \nabla (\xi, \eta) w \sim \frac{1}{\tau} \bar{\nu}, \quad \bar{\nu} = \left(1, \frac{G_{n1}}{G_{\zeta}}, \frac{G_{n2}}{G_{\zeta}} \right)
   \]

   (89)
2. Different balance: $\xi', \eta'_j$ of the same order, and $\tilde{\tau} \leq O(\eta'_j)$, suitable for taking the $\tau \uparrow \tau_b$ limit. In this case the cubic simplifies to $\zeta'^3 \sim \gamma X$ and

$$w \sim \epsilon G \left( \zeta_b + \sqrt[3]{\gamma \left( \xi' + \frac{G_{\eta_1}}{G_{\zeta}} \eta'_1 + \frac{G_{\eta_2}}{G_{\zeta}} \eta'_2 + \frac{G}{G_{\zeta}} \tilde{\tau} \right), \tilde{\eta}_b \right) \Rightarrow$$

$$\nabla_{\xi, \eta} w \sim \frac{1}{3} \sqrt[3]{\frac{6|G_{\zeta}|}{G_{\zeta}^2 \zeta \zeta}} \frac{\epsilon \nabla_{\zeta, \eta} G}{\sqrt[3]{\left( \xi' + \frac{G_{\eta_1}}{G_{\zeta}} \eta'_1 + \frac{G_{\eta_2}}{G_{\zeta}} \eta'_2 + \frac{G}{G_{\zeta}} \tilde{\tau} \right)^2}}.$$ (90)
At breaking
As $\tau \uparrow \tau_b$: the cubic (81) simplifies to $\zeta'^3 = \gamma X$ and

$$w \sim \epsilon G \left( \zeta_b + 3\sqrt{\gamma} \tilde{X}, \tilde{\eta}_b \right) \Rightarrow \nabla_{\xi, \tilde{\eta}} w \sim \frac{3\sqrt{\gamma}}{3} \frac{\epsilon \nabla_{\zeta, \tilde{\eta}} G}{\sqrt{\tilde{X}^2}}, \quad \tilde{X} \neq 0$$

$$\tilde{X} \equiv \xi' + (G_{\eta_1}/G_\zeta)\eta_1' + (G_{\eta_2}/G_\zeta)\eta_2' = 0.$$  \hspace{1cm} (91)

$n = 2$: only the derivative $(G_{\eta_1} \partial_\zeta - G_\zeta \partial_{\eta_1}) w$ in the transversal direction does not blow up at the breaking point.

$n = 3$, only derivatives in any direction defined by the transversal plane (i.e., f.i., $(G_{\eta_1} \partial_\zeta - G_\zeta \partial_{\eta_1}) w$ and $(G_{\eta_2} \partial_\zeta - G_\zeta \partial_{\eta_2}) w$) do not blow up at the breaking point.
After breaking, the solution becomes three-valued in a compact region of the \((\xi, \vec{\eta})\) - space, and does not describe any physics; nevertheless a detailed study of the multivalued region is important, in view of a proper regularization of the model.

If \(\tau > \tau_b\) \((\tilde{\tau} > 0)\), the SM equation \(S = 0\):

\[
2|G_\zeta|\tilde{\tau} = G_{\zeta\zeta\zeta}\zeta'^2 + G_{\zeta\eta_1\eta_1}\eta_1'^2 + G_{\zeta\eta_2\eta_2}\eta_2'^2 + 2G_{\zeta\zeta\eta_1}\zeta'\eta_1' + 2G_{\zeta\zeta\eta_2}\zeta'\eta_2' + 2G_{\zeta\eta_1\eta_2}\eta_1'\eta_2'
\]  

(92)

describes an ellipsoidal paraboloid in the \((\zeta', \vec{\eta}', \tilde{\tau})\) parameter space, with minimum at the breaking point \((\vec{\xi}_b, \tilde{\tau}_b)\). Eliminating \(\zeta'\), one obtains the SM equation in space-time coordinates,

\[
\begin{align*}
\Delta & = 0 \\
\end{align*}
\]

coinciding with the \(\Delta = 0\) condition, describing a closed surface in the \((\xi, \vec{\eta})\) - space made of two surfaces having the same boundary: the transversal ellipse \(Q = 0\):

\[
2|G_\zeta|G_{\zeta\zeta\zeta}\tilde{\tau} = \alpha_1\eta_1'^2 + 2\alpha_2\eta_1'\eta_2' + \alpha_3\eta_2'^2
\]
For $n = 2$, the SM is a closed curve with two cusps in the $(\xi, \eta)$ - plane, whose transversal and longitudinal widths are respectively $O(\tilde{\tau}^{1/2})$ and $O(\tilde{\tau}^{3/2})$. Therefore this closed curve develops, at $\tau = \tau_b$, from the breaking point $\vec{\xi}_b$, with an infinite speed in the tranversal direction, and with zero speed in the longitudinal direction, recovering the results obtained using the IST.
For \( n = 3 \), the SM is a closed surface in the \((\xi, \vec{\eta})\) - space made of two surfaces having the same boundary: the transversal ellipse \( Q = 0 \). While the axes of the \( Q = 0 \) ellipse are of \( O(\tilde{\tau}^{1/2}) \), the thickness of the longitudinal region between the two surfaces is of \( O(\tilde{\tau}^{3/2}) \). Therefore this closed surface develops, at \( \tau = \tau_b \), from the breaking point \( \vec{\xi}_b \), with an infinite speed in the transversal plane of the ellipse, and with zero speed in the longitudinal direction. Intersecting this closed surface with any plane containing the \( \xi \) - axis, one obtains a closed curve with two cusps as in Fig.2; therefore the \( Q = 0 \) ellipse is made of all these cusps.
Since the transformations (74),(75) are invertible:

\[
\begin{align*}
n = 2 & : \quad u = \frac{1}{\sqrt{t}} w(\xi, \eta), \quad t = \frac{\tau^2}{4}, \quad x = \xi - \frac{\eta^2 \tau^2}{4}, \quad y = \frac{\eta \tau^2}{2}, \\
n = 3 & : \quad u = \frac{1}{t} w(\xi, \vec{\eta}), \quad t = e^\tau, \quad x = \xi - e^\tau (\eta_1^2 + \eta_2^2), \quad \vec{y} = 2e^\tau \vec{\eta},
\end{align*}
\]  

(93) (94)

all the above considerations transfer to the \textit{dKP}_n case: small and localized initial data evolving according to the \textit{dKP}_n equation (6) break, at \(t_b = \frac{\tau_b^2}{4}\) in the point \((\xi_b - \frac{\tau_b^2 \eta_b^2}{4}, \eta_b \tau_b^2/2)\) of the parabolic wave front (54) if \(n = 2\), and at \(t_b = e^{\tau_b}\) in the point \((\xi_b - e^{\tau_b} |\vec{\eta}_b|^2, 2e^{\tau_b} \vec{\eta}_b)\) of the paraboloidal wave front (54) if \(n = 3\).
For instance, at $t = t_b$ and in the space region $|x' + \eta_1 b y'_1 + \eta_2 b y'_2| = O(|y'_j|/(2 t_b))$ (the transformed of $|\xi'| = O(|\eta'_j|)$) equations (91) become

$$u \sim \epsilon t_b^{-\frac{n-1}{2}} G \left( \zeta_b + 3 \sqrt{\gamma} \tilde{X}_b(x', y'), \eta_b + \frac{1}{2t_b} \tilde{y}' \right) \Rightarrow$$

$$\nabla_{(x,\tilde{y})} u \sim \epsilon t_b^{-\frac{n-1}{2}} \frac{3/\gamma}{3 \sqrt{X_b^2}} \left( G_\zeta, G_\eta \eta_b + \frac{1}{2t_b} \nabla_\eta G \right),$$

where $x' = x - x_b$, $\tilde{y}' = \tilde{y} - \tilde{y}_b$ and $\tilde{X}_b(x', \tilde{y}') = X_b(\xi', \eta')$. Again, if $n = 2$, all derivatives of $u$ at the breaking point $(x_b, y_b)$ blow up, except that along the transversal line $\tilde{X}_b(x', y') = 0$, for which:

$$(\tilde{X}_y u_x - u_y)|_{(x_b, y_b)} = - \frac{\epsilon}{2 t_b^{3/2}} G_\eta. \tag{96}$$

Analogously, if $n = 3$, all derivatives of $u$ at the breaking point $(x_b, \tilde{y}_b)$ blow up, except those along the transversal surface $\tilde{X}_b(x', \tilde{y}') = 0$, represented by the basis vector fields $\hat{X}_j = \tilde{X}_{by_j} \partial_x - \partial_y$, $j = 1, 2$, for which:

$$\hat{X}_j u|_{(x_b, \tilde{y}_b)} = - \frac{\epsilon}{2 t_b^2} G_{\eta_j}, \ j = 1, 2. \tag{97}$$
References.

