Bispectrality and Bihamiltonian Systems

J.P.Zubelli (IMPA - Brazil)

Thanks to the organizers: B. Konopelchenko, G. Falqui, M.Pedroni
September 30, 2011
Apparently Unrelated Topics

1. Virasoro Algebra
2. KdV, KP, AKNS and other integrable systems
3. Bispectrality
4. Huygens’ principle in Hadamard’s strict sense
Plan

- Definition and Motivation of Bispectrality
- Bihamiltonian Formulation of KdV
- Extended Symmetries and Virasoro
- Connection with Bispectrality
- Further Results
- Huygens’ property (a la Hadamard)
- Recent progress in Huygens for Dirac
Motivations

1. The time-band limiting operators (work of F.A. Grünbaum)
2. Asymptotics of special functions

Other motivations:
1. Parallel Computing (see A. Edelman work on the subject)
2. Random matrices (recall P.V. Moerbeke’s talk)
Time-Band Limiting

Slepian-Landau-Pollak (ATT-Bell Labs ‘60s)

Notation: \( \hat{g}(\omega) = \mathcal{F}[g](\omega) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t) e^{-i\omega t} \, dt \).

Physical Channels:
band limited to \([-\Omega, \Omega]\).

time limited to \([-T, T]\):

\[ \chi[-T,T] \]

Question: How close are we from our original signal \( g \in L^2 \) after going through the following process:

\[
g \mapsto \chi[-T,T]g \mapsto \mathcal{F}[\chi[-T,T]g] \xrightarrow{\downarrow} \mathcal{F}^{-1} (\chi[-\Omega,\Omega] \mathcal{F}[\chi[-T,T]g]) \mapsto \chi[-\Omega,\Omega] \mathcal{F}[\chi[-T,T]g] \]
This corresponds to getting the SVD of the following operator:

\[( Ag ) ( \lambda ) = \begin{cases} \int_{-T}^{T} e^{i\lambda t} g(t) dt & \lambda \in [-\Omega, \Omega] \\ 0 & \lambda \in \mathbb{R} \setminus [-\Omega, \Omega] \end{cases} \]

\[( A^* A g ) ( t ) = \begin{cases} \int_{-T}^{T} \frac{\sin \Omega (t-s)}{\pi (t-s)} g(s) ds & t \in [-T, T] \\ 0 & t \in \mathbb{R} \setminus [-T, T] \end{cases} \quad (1) \]

Notes:

- \( T \overset{\text{def}}{=} A^* A \) defines a compact s.a. operator in \( L^2(\mathbb{R}) \).
- Even numerically, it is not a simple task to get \( \text{spec} T \).
Slepian, Landau & Pollak found an ord. diff. op. \( L \) w/ *simple* spectrum s.t.

\[
[T, L] = 0.
\]

Namely:

\[
L : g \mapsto \frac{d}{dt} \left( (T^2 - t^2) \frac{d}{dt} g \right) - \Omega^2 t^2 g.
\]

Moreover, Slepian, Landau & Pollak found many other discrete and continuous instances of this.
Recall Shannon’s thm: The Nyquist rate is $2\Omega/2\pi$. Introduce the parameter: 

$$C = 2T\Omega/\pi = 2T2\Omega/(2\pi)$$

which is the area in phase space. 

$C$ corresponds to the “number of degrees of freedom of the space of signals with freq band $[-\Omega, \Omega]$ and time duration in $[-T, T]$.” Sharp decay at about $C = 2T\Omega/\pi$.

![Figure: Eigenvalues of T.](image-url)
“THE COMMUTING PROPERTY”

Natural Question: What are the other contexts where this miracle holds?

- Tomography, Radon transform, etc. Grünbaum’s work

EXAMPLE: (use $x$ instead of $t$)

\[
H = -\frac{d^2}{dx^2} + x^2.
\]

$L^2(\mathbb{R})$ can be decomposed in the eigenvector basis for $H$. First define the Hermite $\times \exp(-x^2/2)$.

\[
H_k(x) \overset{\text{def}}{=} (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \quad k = 0, 1, 2, \ldots ,
\]

and then

\[
h_k \overset{\text{def}}{=} H_k(x) e^{-x^2/2}.
\]
The function $h_k$ satisfies,

$$H h_k = (2k + 1) h_k.$$ 

Three term recursion relation:

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \quad n = 1, 2 \ldots \quad (2)$$

In this case Fourier corresponds to writing $g$ in the o.n. basis

$$\{ h_k \}_{k=0}^{\infty}.$$

$$g = \sum_{k=0}^{\infty} b_k h_k.$$ 

Analogue of the sinc kernel:

$$h(x, y) \overset{\text{def}}{=} \sum_{k=0}^{N} h_k(x) h_k(y).$$
\( \forall N \) there exists a diff. op. commuting with the integral operator with kernel \( h(x, y) \).
\( \forall T \) there exists a tridiagonal matrix (with simple spectrum) commuting with the matrix

\[
(G_{ij}(T))_{1 \leq i, j \leq N},
\]

where \( G_{ij}(T) = \int_{-\infty}^{T} h_i(\xi) h_j(\xi) d\xi \).

Main Point of the Proof: Existence of a 3-term recursion relation.
Generalizes to other orthogonal polynomials: 


The instances where the “commuting property” held were associated to the fact the family of eigenfunctions satisfied both an eigenvalue problem of the form

\[ L(x) \varphi = \lambda \varphi \quad \text{and} \quad B(\lambda) \varphi = \Theta(x) \varphi . \]
The Bispectral Problem

$L$ and $B$ are either **differential** operators or **difference** operators. Simplest case seems to be the continuous-continuous one. Find all diff ops. $L(x, \partial_x)$ s.t. there exists a family of eigenfunctions $\varphi(x, \lambda)$ satisfying simultaneously the equation

$$L(x, \partial_x) \varphi = \lambda \varphi .$$

and a differential equation in the spectral parameter of the form

$$B(\lambda, \partial_\lambda) \varphi = \Theta(x) \varphi .$$

**Note:** We can normalize the operator $L$. E.G. require that $L$ be constant and its second highest coefficient vanish.
Simplest Case

\[ L = -\partial_x^2 + u(x). \]

**Note:** We can set \( \lambda = \lambda(k) \). Say \( \lambda(k) = k^2 \).
Trivial Examples

Take $\varphi(x, k) = \exp(ikx)$

$$-\partial_x^2 \varphi = k^2 \varphi \quad & \quad - \partial_k^2 \varphi = x^2 \varphi$$

Take $f(z)$ a solution of Bessel’s eq.

$$f_{zz} + \frac{c}{z^2} f = f \quad & \quad \varphi(x, k) = f(xk)$$

Take $f(z)$ a solution of Airy’s eq.

$$f_{zz} = zf \quad & \quad \varphi(x, k) = f(x + k)$$
Non-Trivial Examples

Rational solutions of KdV: i.e.

\[ L = -\partial_x^2 + u, \]

with

\[ u(x) = -2\partial_x^2 \log \vartheta_n(x) \]

and \( \vartheta_n \) the \( n \)-th Adler-Moser polynomial.

or

\[ u(x) = \frac{l^2 - 1/4}{x^2} - 2\partial_x^2 \log W_n(x) \]

where \( W_n \) is a certain poly. (obtained by Darboux transformations)
The maximal dimension of the space of common eigenfunctions of equations $L\phi = \lambda\phi$ and $B\phi = \Theta\phi$ is called the \textit{rank} of the bispectral triple $(L, B, \Theta)$.

All the trivial examples lead to rank 2 triples.

Generic KdV potentials have rank 1.

The $W_n$ have fairly explicit formulas in terms of Wronskians.
Some References

Workshop on *The Bispectral Problem* CRM Proceedings & Lecture Notes, 14.
This question received contributions of a number of people: to cite a few:

- Duistermaat & Grünbaum (CMP ’86)
- Bakalov, Horozov, Yakimov
- L. Haine & P. Iliev
- A. Kasman
- J. Harnard
- J. Liberati
- F. Levstein
- M. Rothstein
- G. Wilson
- E. Horozov (CMP ’2002)
- Fastré
Darboux Method

Remark: Darboux-Moutard-Crum-Bäcklund (long story...)

Factor $L$ as

$$ L = PQ \, . $$

(3)

Consider

$$ \tilde{L} = QP \, . $$

(4)

$Q$ intertwines $L$ and $\tilde{L}$, i.e.,

$$ \tilde{L}Q = QL \, . $$

If $\psi$ an eigenfunction of $L$, in the kernel of $Q$, it follows that $\tilde{\psi} \overset{\text{def}}{=} Q\psi$ is also an eigenfunction of $\tilde{L}$. 
Darboux

Exercise

If we take \( L = -\partial_x^2 + u \) and impose that \( P \) and \( Q \) are first order operators

\[
P = -\partial_x - s \quad \& \quad Q = \partial_x - v ,
\]

Then, \( L = PQ \) implies that

\[
s = v
\]

& \( v \) satisfies the Riccati equation

\[
v_x + v^2 = u . \tag{5}
\]

The solutions of the Riccati equation are given by \( v = \partial_x \log(\phi) \), where \( \phi \) is in the kernel of \( L \).
New potential is given by

$$\tilde{u} = u - 2v_x .$$  \hspace{1cm} (6)

Hence

$$\tilde{\psi} = (\partial_x - \nu)\psi$$

is a solution of

$$\tilde{\mathcal{L}}\tilde{\psi} = \lambda\tilde{\psi} ,$$

whenever $\psi$ satisfies

$$\mathcal{L}\psi = \lambda\psi .$$
Rational Darboux Transformations

**DEFINITION**

We say that the process of going from $u$ to $\tilde{u}$ is rational Darboux transformation if both $u$ and $\tilde{u}$ are rational.

**Trivial Symmetries:**

1. translations (in $x$)
2. scalings (in $x$)
3. addition of a constant to the potential
THEOREM: (Duistermaat and Grünbaum) Modulo the trivial symmetries mentioned above, the bispectral Schrödinger potentials are

\[ u(x) = x \quad \text{or} \quad u(x) = \frac{c}{x^2} \]

or those obtained from \( u(x) = 0 \) or from

\[ u(x) = -\frac{1}{4x^2} \quad (7) \]

by means of finitely many rational Darboux transformations.

IMPORTANT REMARK: The rational solutions of the KdV hierarchy are obtained by iterating rational Darboux transformations staring from \( u = 0 \).

But starting from \( u = 0 \) accounts for only “half” of the solutions of the bispectral problem.
Results of Duistermaat & Grünbaum
(cont)

The other “half” does NOT remain rational by the flows of KdV. Those correspond to rational potentials that are obtained by applying Darboux to

$$u = \frac{-1/4}{x^2}$$

The so-called even family.

**QUESTION:** What hierarchy of nonlinear evolution equations preserves the bispectral potentials in the “even family”?

Answer in a joint work JPZ & F. Magri (CMP ’91) Further developments in JPZ & D. S. Valerio (CMP ’00)
Bi-Hamiltonian structure of KdV

The KdV equation

\[ u_t = -u_{xxx} + 6u_x u \overset{\text{def}}{=} X_1(u) . \]  

(8)

can be written as a Hamiltonian flow w.r.t. 2 distinct Poisson structures and compatible:

\[ X_1(u) = -\partial_x \frac{\delta H_1}{\delta u} , \]

where

\[ H_1 = \int \left( \frac{1}{2} u_x^2 + u^3 \right) dx , \]

and also

\[ X_1(u) = (\partial_x^3 - 4u\partial_x - 2u_x) \frac{\delta H_0}{\delta u} , \]

where

\[ H_0 = -\frac{1}{2} \int u^2 dx . \]
Crucial:

\[ K_u \overset{\text{def}}{=} \partial_x^3 - 4u \partial_x - 2u_x, \]

is a Poisson structure compatible with

\[ D_u \overset{\text{def}}{=} -\partial_x. \]

The KdV hierarchy could be written as:

\[ X_n = D_u \frac{\delta H_n}{\delta u}, \quad n = 1, 2, \ldots, \]

or

\[ X_{n+1} = K_u \frac{\delta H_n}{\delta u}, \quad n = 0, 1, 2, \ldots. \]

To generate the consecutive flows use the recursion operator

\[ N_u \overset{\text{def}}{=} K_u D_u^{-1} = -\partial_x^2 + 4u + 2u_x \partial_x^{-1}. \]
This gives

\[ X_{n+1} = N_u X_n \]  \hspace{1cm} (9)

It also gives:

**Master Symmetries:** Extensive subject (with contributions by many authors... To cite a few: Carillo, Fokas, Fuchssteiner, R. Oevel, W. Oevel, Ragnisco, Santini.

For our purposes define:

\[ \tau_0(u) \overset{\text{def}}{=} \frac{1}{2}xu_x + u \]

\[ \tau_n \overset{\text{def}}{=} N^n_u \tau_0, \ n = 1, 2, \ldots \]
THEOREM: The hierarchies \( \{X_k\}_{k=0}^\infty \) & \( \{\tau_j\}_{j=0}^\infty \) satisfy:

\[
[X_j, X_k] = 0 , \tag{10}
\]

\[
[\tau_j, \tau_l] = (l-j)\tau_{j+l} , \tag{11}
\]

&

\[
[X_j, \tau_l] = -(j + \frac{1}{2})X_{l+j} .
\]
A = \sum_{k} a_k z^{k+1} \frac{d}{dz} . \quad (12)

\{ V_k \overset{\text{def}}{=} z^{k+1} \frac{d}{dz} \}_{k=-\infty}^{\infty} . \quad (13)

[V_k, V_m] = (m - k) V_{k+m} .

**IMPORTANT:** The master symmetries mentioned above only comprise half of the Virasoro algebra.
Back to the main question: Which flows preserve bispectrality?

\[ \tau_j \overset{\text{def}}{=} N^j_u \tau_0, \quad (14) \]

where

\[ \tau_0(u) = \frac{1}{2} xu_x + u. \]

**THEOREM:** (Magri-JPZ) The flows \( \{\tau_j\}_{k=0}^{\infty} \) are tangential to the all manifolds of bispectral potentials decaying at infinity. (in particular to the even family)
As a consequence

The \textit{bispectral} potentials in the even family, when given the correct parametrization remain \textit{rational} by the flows of the mastersymmetry hierarchy.

\begin{align*}
X_1 &= -k \text{-th KdV flow} \\
\tau_j &= j \text{-th master-symmetry flow}
\end{align*}

\textbf{Figure:} Left: KdV rational potentials (rk 1) Right: Even family potentials (rk 2)
Remark

Many generalizations of the bispectral problem were solved by Bakalov, Yakimov and Horozov....
A nice result in connection with this for the KP context can be found in Bakalov, Yakimov, Horozov.
[Hor02, BHY98b, BHY98a, BHY97]

\( \tau \)-function belongs to a highest weight module over \( W_{1+\infty} \)
\( \tau \)-function belongs to a highest weight module over \( W_{1+\infty} \)
generated by a Bessel \( \tau \)-function iff it is the so-called monomial
Darboux transformation of this Bessel \( \tau \)-function.
Darboux transformation of this Bessel \( \tau \)-function.

2002 paper by Horozov on bispectral rings of diff. operators of prime order... [Hor02]
Rational Solutions of the KdV Master Symmetries

**QUESTION:**
Up to what extent does remaining rational by the master-symmetries of KdV characterize bispectrality?

**THEOREM:** (Valério-JPZ) If $u$ is a generic rational solution decaying at infinity of the master symmetry hierarchy $\{\tau_j\}_{j=0}^\infty$, then $u$ is bispectral.

**Main ideas behind the argument:**
If $u$ is a generic potential that remains rational by the flow of $\tau_1$, then

$$u(x,t) = \frac{c}{x^2} + \sum_{p \in \mathcal{P}} \frac{2}{(x - p(t))^2},$$

where $\mathcal{P} = \mathcal{P}(t)$ is the set of nonzero poles of $u(x,t)$ and $c$ is a constant independent of $t$.

Furthermore,

$$\dot{p} = -\frac{2c}{p} - 2 \sum_{q \in \mathcal{P} \setminus \{p\}} \frac{2p + q}{(p - q)^2}, \quad p \in \mathcal{P}, \quad (15)$$
One direction... since the rational solutions of KdV correspond to polynomial specific $\tau$ functions... of the KP hierarchy
It is natural to ask: Do general polynomial $\tau$ functions for KP lead to bispectral operators?
In a conveniently scaled set of variables the KP equation takes the form
\[ \frac{3}{4} u_{yy} = \left( u_t - \frac{1}{4} (u_{xxx} + 6 uu_x) \right)_x. \] (18)

(18) can be looked as the compatibility condition of
\[ \partial_y w = (\partial_x^2 + u) w \]
and
\[ \partial_t w = (\partial_x^3 + \frac{3}{2} u \partial_x + \nu) w, \]
where \( \nu \) can be expressed in terms of \( u \).
KP Hierarchy

Start: algebra of formal pseudodifferential operators:

\[
\sum_{-\infty}^{N} f_j(x) \partial_x^j ,
\]

Introduce the “space variable” \( x_1 \equiv x \), and the “time variables” \( x_j \), for \( j \geq 2 \). Set

\[
x \overset{\text{def}}{=} (x_1, x_2, \ldots) .
\]

Let \( Q(x, \partial_x) \) be the pseudodifferential operator

\[
Q(x, \partial_x) \overset{\text{def}}{=} \partial_x + u_1(x) \partial_x^{-1} + u_2(x) \partial_x^{-2} + \cdots .
\]
Def: The KP hierarchy is the infinite set of evolution equations for the coefficients of $Q$ given by

$$\partial_{x_n} Q = [Q_+^n, Q] ,$$

where $n = 1, 2, \ldots$ (where $Q_+^n$ denotes the differential part of the operator $Q^n$)
The KP Hierarchy is the compatibility condition of

\[ Qw = kw \] (19)

and for \( n \geq 2 \)

\[ \partial_{x_n} w = B_n(x, \partial_x) w, \] (20)

where

\[ B_n \overset{\text{def}}{=} Q^n_+ . \]

To obtain the Korteweg-de Vries hierarchy, one considers the constraint

\[ Q^2 = Q^2_+ . \]
The KP hierarchy can be rewritten as:

A system of infinitely many \textit{bilinear} differential equations in one single dependent variable. Those are the now celebrated \textit{Hirota bilinear equations}. (ref. Sato '81 and Jimbo - Miwa '83)

If \( \tau \) satisfies the Hirota bilinear equations, then the common solutions of (19) and (20) can be written in terms of vertex operators applied to the \( \tau \)-function.

\[
w(x; k) = \frac{X(k)\tau}{\tau} \equiv \exp(\xi(x; k)) \frac{\tau(x - \epsilon(k^{-1}))}{\tau(x)}, \quad (21)
\]

where \( \xi(x; k) \equiv \sum_{j \geq 1} x_j k^j \), and \( \epsilon(k^{-1}) \equiv (\frac{1}{k}, \frac{1}{2k^2}, \frac{1}{3k^3}, \ldots) \).
Polynomial $\tau$ Functions

*Elementary Schur functions* $q_0, q_1, q_2, \ldots$, are defined by the formula

$$\exp\left(\sum_{i=1}^{\infty} \lambda^i x_i\right) = \sum_{i=0}^{\infty} q_i(x_1, \ldots, x_i) \lambda^i.$$ 

We set $q_i \overset{\text{def}}{=} 0$ if $i \leq 0$. Let $f_1 \geq f_2 \geq \cdots f_m > 0$ be positive integers, and $Y = (f_1, \cdots, f_m)$

*Schur function*

$$\chi_Y \overset{\text{def}}{=} \det((q_{f_i-i+j}))_{1 \leq i,j \leq m}. \quad (22)$$

Found by M. Sato that $\chi_Y$ defined by equation (22) is a $\tau$-function for the KP hierarchy, i.e., it satisfies the Hirota bilinear equations for the KP hierarchy. In this case $w(x; k)$ given by equation (21) is a common solution of equations (19) and (20) for every positive integer $n$. 
**Lemma:** (see V. Kac's book on Inf. Dim. Lie Algebras) The polynomial solutions of the (Hirota form of the) KP hierarchy is exhausted by taking arbitrary finite matrices

\[ C = \begin{pmatrix} (c_{ij}) \end{pmatrix}, \]

and setting \( \tau \) as

\[ \tau = \text{Wr}_m \left( \sum c_{1,l} q_l, \ldots, \sum c_{m,l} q_l \right). \] (23)

**Theorem (JPZ):** If \( \tau \) is a polynomial solution of the KP hierarchy, then

\[ \varphi(x, k) \overset{\text{def}}{=} \exp(xk) \frac{\tau(x - \epsilon(k^{-1}))}{\tau(x)}, \]

is a bispectral eigenfunction.
G. Wilson’s Result

G. Wilson has characterized in a very elegant way the so called “rank one” commutative rings of bispectral operators. Here, the rank of a commutative algebra of differential operators is defined as the greatest common divisor of the order of the operators in the algebra.

**Theorem (Wilson):** Let $\mathcal{A}$ be a maximal rank-1 commutative algebra of ordinary differential operators. Then $\mathcal{A}$ is bispectral iff the curve $\text{Spec} \mathcal{A}$ is rational and unicursal.


[BW02, BW00, BW99]
Conclusions and Open Problems

- The analogues of the continuous case in the discrete case
- The characterization of high bispectral algebras of differential operators
- The characterization of the rational solutions of the $\mathcal{W}_\infty$ flows...
- Matrix case... some initial results by JPZ.
Figure: V WPDEs RIO DE JANEIRO 1997
Figure: V WPDEs RIO DE JANEIRO 1997
Figure: V WPDEs RIO DE JANEIRO 1997
A SPECIAL THANK YOU MESSAGE TO FRANCO ... 

Grazie!
Limited Bibliography

[DG86, Hor02, BHY98b, BHY98a, BHY97, Mag78, ZM91, Ili11, Ili99, Zub90]


Ideal classes of the Weyl algebra and noncommutative projective geometry.

With an appendix by Michel Van den Bergh.

JJ Duistermaat and FA Grunbaum.
Differential equations in the spectral parameter.

E. Horozov.
Bispectral operators of prime order.

P. Iliev.
*Discrete versions of the Kadomtsev-Petviashvili hierarchy and the bispectral problem.*
Bispectral commuting difference operators for multivariable askey-wilson polynomials.


F. Magri.

A simple model of the integrable hamiltonian equation.

*Journal of Mathematical Physics, 19:1156, 1978.*

J.P. Zubelli and F. Magri.

Differential equations in the spectral parameter, darboux transformations and a hierarchy of master symmetries for kdv.


J.P. Zubelli.

Differential equations in the spectral parameter for matrix differential operators.