Polynomial Chaos and Scaling Limits of Disordered Systems

3. Marginally relevant models

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We present some results on the the continuum partition function

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All these different models share a crucial feature: logarithmic overlap

$$\mathbf{R}_N = \begin{cases} \displaystyle\sum_{1 \le n \le N} \mathbf{P}^{\mathrm{ref}}(n \in \tau)^2 \\ \displaystyle\sum_{1 \le n \le N} \sum_{x \in \mathbb{Z}^d} \mathbf{P}^{\mathrm{ref}}(S_n = x)^2 \end{cases} \sim C \log N$$

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For simplicity, we will perform our computations on the pinning model

The 2d Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x) + \beta\, W(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2$$

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Mollification in space: fix $j \in C_0^\infty(\mathbb{R}^d)$ with $\|j\|_{L^2} = 1$

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Then $$u_\delta(t,x) \overset{d}{=} \mathrm{E}_{\frac{x}{\sqrt{\delta}}}\left[\exp\left\{\int_0^{\frac{t}{\delta}} \left(\beta\, W_1(s,B_s) \;-\; \tfrac{1}{2}\beta^2\right) \mathrm{d}s\right\}\right]$$

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By soft arguments $u_\delta(1,x) \stackrel{d}{\approx} \mathbf{Z}_N^\omega$ (partition function of 2d DPRE)

Pinning in the relevant regime $\alpha > \frac{1}{2}$

Recall what we did for $\alpha > \frac{1}{2}$ (for simplicity $h = 0$)

$$\mathbf{Z}_N^{\omega} = \mathbf{E}^{\mathrm{ref}}\left[e^{H_N^{\omega}}\right] = \mathbf{E}^{\mathrm{ref}}\left[e^{\sum_{n=1}^{N}(\beta\omega_n - \lambda(\beta))\mathbb{1}_{\{n\in\tau\}}}\right]$$

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- $\mathbf{P}^{\mathrm{ref}}(n\in\tau) \sim \dfrac{c}{n^{1-\alpha}}$

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$$\mathbf{Z}_N^{\omega} = 1 + \beta \sum_{0<n\leq N} \frac{Y_n}{n^{1-\alpha}} + \beta^2 \sum_{0<n<m\leq N} \frac{Y_n \, Y_m}{n^{1-\alpha}\,(m-n)^{1-\alpha}} + \ldots$$

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Lattice $\frac{\mathbb{Z}}{N}$ has cells with volume $\frac{1}{N}$, hence if $\frac{\beta}{N^{1-\alpha}} \approx \sqrt{\frac{1}{N}}$ that is

$$\beta = \frac{\hat{\beta}}{N^{\alpha-\frac{1}{2}}}$$

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What happens for $\alpha = \frac{1}{2}$? Stochastic integrals ill-defined: $\frac{1}{\sqrt{t}} \notin L^2_{\mathrm{loc}}$ ...

The marginal regime $\alpha = \frac{1}{2}$

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$$\sigma^2 = \beta^2 \sum_{0<n\le N} \frac{1}{n} \sim \beta^2 \log N$$

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We then rescale $\boxed{\beta = \beta_N \sim \frac{\hat\beta}{\sqrt{\log N}}}$ Other terms converge?

Interestingly, every sum gives contribution 1 to the variance!

$$\mathbb{V}\mathrm{ar}\left[\mathbf{Z}_N^\omega\right] = 1 + \hat\beta^2 + \hat\beta^4 + \ldots = \frac{1}{1-\hat\beta^2}$$ blows up at $\hat\beta = 1!$

Scaling limit of marginal partition function

Theorem 1. [C., Sun, Zygouras '15b]

Consider DPRE $d = 2$ or Pinning $\alpha = \frac{1}{2}$ or 2d SHE

(or long-range DPRE with $d = 1$ and Cauchy tails)

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$$\mathbf{Z}_N^\omega \xrightarrow[N\to\infty]{d} \mathcal{Z}^W = \begin{cases} \text{log-normal} & \text{if } \hat{\beta} < 1 \\ 0 & \text{if } \hat{\beta} \geq 1 \end{cases}$$

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$$\mathcal{Z}^W \stackrel{d}{=} \exp\left\{ \sigma_{\hat\beta} W_1 - \frac{1}{2}\sigma_{\hat\beta}^2 \right\} \qquad \text{with} \quad \sigma_{\hat\beta} = \log \frac{1}{1-\hat\beta^2}$$

Multi-scale correlations for $\hat{\beta} < 1$

Define $\mathbf{Z}_N^{\omega}(t, x)$ as partition function for rescaled RW starting at (t, x)

$$\mathbf{Z}_N^{\omega}(t, x) = \mathbf{E}^{\mathrm{ref}}\left[e^{H^{\omega}(S)} \,\middle|\, S_t^{\delta} = x\right]$$

where $\{S_t^{\delta} = x\} = \{S_{Nt} = \sqrt{N} x\}$ $\qquad [\delta = \frac{1}{N}]$

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More generally, if $\mathrm{X} = (t_N, x_N)$ and $\mathrm{X}' = (t'_N, x'_N)$ are such that

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Y, Y' joint $\mathcal{N}(0, \sigma_{\hat\beta}^2)$ with $\mathbb{C}\mathrm{ov}\left[Y, Y'\right] = \log \dfrac{1 - \zeta\hat\beta^2}{1 - \hat\beta^2}$

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We can integrate $\mathbf{Z}_N^\omega$ against a test function $\phi \in C_0([0,1] \times \mathbb{R}^2)$

$$\langle \mathbf{Z}_N^\omega , \phi \rangle := \int_{[0,1]\times\mathbb{R}^2} \phi(t,x)\, \mathbf{Z}_N^\omega(t,x)\, \mathrm{d}t\, \mathrm{d}x$$

$$\simeq \frac{1}{N^2} \sum_{t \in [0,1]\cap \frac{\mathbb{Z}}{N},\; x \in (\frac{\mathbb{Z}}{\sqrt{N}})^2} \phi(t,x)\, \mathbf{Z}_N^\omega(t,x)$$

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Corollary

$$\langle \mathbf{Z}_N^\omega \,,\, \phi \rangle \to \langle 1 \,,\, \phi \rangle \quad \text{in probability} \quad \text{as } N \to \infty$$

Fluctuations for $\hat{\beta} < 1$

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$$\mathbf{Z}_N^{\omega}(t,x) \approx 1 + \frac{1}{\sqrt{\log N}} G(t,x) \qquad (\text{in } \mathcal{S}')$$

Fluctuations for $\hat{\beta} < 1$

Theorem 3. [C., Sun, Zygouras '15b]

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where G(t,x) is a generalized Gaussian field on $[0,1] \times \mathbb{R}^2$ with

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Conjecture

For $\hat{\beta} = 1$ the partition function $\mathbf{Z}_N^\omega(t,x)$ has a non-trivial limit in law, viewed as a random Schwartz distribution in (t,x)

Proof of Theorem 1. for pinning

$$\mathbf{Z}_N^{\omega} = \sum_{k=0}^{N} \beta^k \sum_{0<n_1<\ldots<n_k\leq N} \frac{Y_{n_1} Y_{n_2} \cdots Y_{n_k}}{\sqrt{n_1}\sqrt{n_2 - n_1}\cdots\sqrt{n_k - n_{k-1}}}$$

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Goal: find the joint limit in distribution of all these sums

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$\rightsquigarrow$ blackboard!

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4th Moment Theorem [de Jong 90] [Nualart, Peccati, Reinert 10]

Consider homogeneous (deg. ℓ) polynomial chaos $Y_N = \sum_{|I|=\ell} \psi_N(I) \prod_{i \in I} Y_i$

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Then
$$Y_N \xrightarrow[N\to\infty]{d} \mathcal{N}(0, \sigma^2)$$

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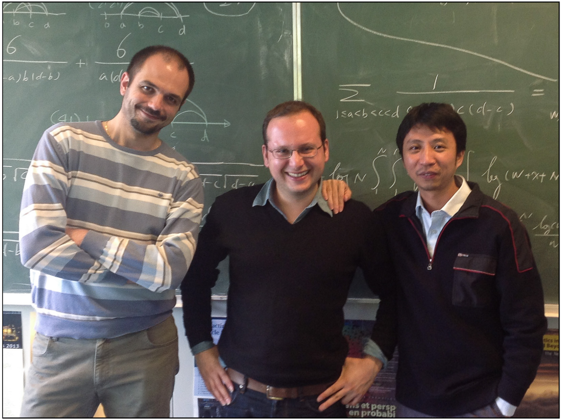
Collaborators



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)