Scaling and Universality in Probability

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A more expressive (but less fancy) title would be

**Convergence of Discrete Probability Models to a Universal Continuum Limit**

This is a key topic of classical and modern probability theory

I will present a (limited) selection of representative results, in order to convey the main ideas and give the flavor of the subject
Outline

1. Weak Convergence of Probability Measures

2. Brownian Motion

3. A glimpse of SLE

4. Scaling Limits in presence of Disorder
Fix a set $\Omega$. A probability $P$ is a map from subsets of $\Omega$ to $[0,1]$ s.t.

$$P(\Omega) = 1, \quad P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i) \quad \text{for disjoint } A_i$$

[ $P$ is only defined on a subclass ($\sigma$-algebra) $\mathcal{A}$ of “measurable” subsets of $\Omega$ ]

$(\Omega, \mathcal{A}, P)$ is an abstract probability space. We will be “concrete”:

( Metric space $E$, “Borel $\sigma$-algebra”, Probability $\mu$ )

- Integral $\int_E \varphi \, d\mu$ for bounded and continuous $\varphi : E \to \mathbb{R}$
- Discrete probability $\mu = \sum_i p_i \delta_{x_i}$ with $x_i \in E$, $p_i \in [0,1]$

$$\int_E \varphi \, d\mu := \sum_i p_i \varphi(x_i)$$
Riemann sums and integral on $[0, 1]$

- **Partition** $t = (t_0, t_1, \ldots, t_k)$ of $[0, 1]$
  
  $0 = t_0 < t_1 < \ldots < t_k = 1 \quad (k \in \mathbb{N})$

- **Riemann sum** of a function $\varphi : [0, 1] \to \mathbb{R}$ relative to $t$
  
  $R(\varphi, t) := \sum_{i=1}^{k} \varphi(t_i) (t_i - t_{i-1})$

**Theorem**

Let $t^{(n)}$ be partitions with

$$\text{mesh}(t^{(n)}) := \max_{1 \leq i \leq k_n} (t_i^{(n)} - t_{i-1}^{(n)}) \xrightarrow{n \to \infty} 0$$

If $\varphi : [0, 1] \to \mathbb{R}$ is **continuous**, then

$$R(\varphi, t^{(n)}) \xrightarrow{n \to \infty} \int_{0}^{1} \varphi(x) \, dx$$
A probabilistic reformulation

Partition $\mathbf{t} = (t_0, t_1, \ldots, t_k) \leadsto$ discrete probability $\mu_\mathbf{t}$ on $[0, 1]$

$$\mu_\mathbf{t}(\cdot) := \sum_{i=1}^{k} p_i \delta_{t_i}(\cdot) \quad \text{where} \quad p_i := t_i - t_{i-1}$$

Uniform partition

$\mathbf{t} = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1) \leadsto \mu_\mathbf{t} = \text{uniform probability on } \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$
A probabilistic reformulation

Key observation: Riemann sum is ... integral w.r.t. \( \mu_t \)

\[
R(\varphi, t) = \sum_{i=1}^{k} \varphi(t_i) p_i = \int_{[0,1]} \varphi \, d\mu_t
\]

Theorem

If \( \text{mesh}(t^{(n)}) \to 0 \) and \( \varphi : [0,1] \to \mathbb{R} \) is continuous, then

\[
\int_{[0,1]} \varphi \, d\mu_{t^{(n)}} \quad \xrightarrow{n \to \infty} \quad \int_{[0,1]} \varphi \, d\lambda
\]

with \( \lambda := \text{Lebesgue measure (probability) on [0,1]} \)

- **Scaling Limit:** convergence of \( \mu_{t^{(n)}} \) toward \( \lambda \)
- **Universality:** the limit \( \lambda \) is the same, for any choice of \( t^{(n)} \)
Weak convergence

- $E$ is a **Polish space** (complete separable metric space), e.g.
  
  \[ [0, 1], \quad C([0, 1]) := \{ \text{continuous } f : [0, 1] \to \mathbb{R} \} , \quad \ldots \]

- $(\mu_n)_{n \in \mathbb{N}}, \mu$ are probabilities on $E$

### Definition (weak convergence of probabilities)

We say that $\mu_n$ converges weakly to $\mu$ (notation $\mu_n \Rightarrow \mu$) if

\[
\int_E \varphi \, d\mu_n \xrightarrow{n \to \infty} \int_E \varphi \, d\mu
\]

for every $\varphi \in C_b(E) := \{ \text{continuous and bounded } \varphi : E \to \mathbb{R} \}$

[Analysts call this **weak-* convergence**; note that $\mu_n, \mu \in C_b(E)^*$]
A useful reformulation

- $\mu_n \Rightarrow \mu$ does not imply $\mu_n(A) \rightarrow \mu(A)$ for all meas. $A \subseteq E$?

**Example**

$$\mu_n = \text{uniform probability on } \left\{ \frac{1}{n}, \frac{2}{n}, \ldots, 1 \right\} \quad A := \mathbb{Q} \cap [0, 1]$$

$$\mu_n \Rightarrow \lambda \ (\text{Lebesgue}) \quad \text{but} \quad 1 = \mu_n(A) \nrightarrow \lambda(A) = 0$$

- Weak convergence means $\mu_n(A) \rightarrow \mu(A)$ for “nice” $A \subseteq E$

**Theorem**

$$\mu_n \Rightarrow \mu \quad \text{iff} \quad \mu_n(A) \rightarrow \mu(A) \quad \forall \text{ meas. } A \subseteq E \quad \text{with} \quad \mu(\partial A) = 0$$

- Weak convergence links measurable and topological structures
Three interesting examples of weak convergence, leading to

- Brownian motion
- Schramm-Löwner Evolution (SLE)
- Continuum disordered pinning models

Common mathematical structure

- A Polish space $E$
- A sequence of discrete probabilities $\mu_n$ (easy) on $E$
- A “continuum” probability $\mu$ (difficult!) such that $\mu_n \Rightarrow \mu$
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From random walk to Brownian motion

- $E := C([0, 1]) = \{ \text{continuous } f : [0, 1] \to \mathbb{R} \}$ (with $\| \cdot \|_\infty$)

- $E_n := \left\{ \text{piecewise linear } f : [0, 1] \to \mathbb{R} \text{ with } f(0) = 0 \text{ and } f\left(\frac{i+1}{n}\right) = f\left(\frac{i}{n}\right) \pm \sqrt{\frac{1}{n}} \right\} \subseteq C([0, 1])$

$|E_n| = 2^n$

$\Delta f = \pm \sqrt{\Delta t} \leadsto \text{slope}(f) = \pm \sqrt{n}$

Case $n = 40$
Let $\mu_n$ be the probability on $C([0, 1])$ which is uniform on $E_n$:

$$\mu_n(\cdot) = \sum_{f \in E_n} \frac{1}{2^n} \delta_f(\cdot)$$

**Theorem (Donsker)**

The sequence $(\mu_n)_{n \in \mathbb{N}}$ converges weakly on $C([0, 1])$: $\mu_n \Rightarrow \mu$

The limiting probability $\mu$ on $C([0, 1])$ is called **Wiener measure**

- Deep result!
- Wiener measure is the law of Brownian motion
- Wiener measure is a “natural” probability on $C([0, 1])$ (like Lebesgue for $[0, 1]$)
Reminders (II). Random variables and their laws

A random variable (r.v.) is a measurable function $X : \Omega \to E$ [where $(\Omega, \mathcal{A}, P)$ is some abstract probability space]

The law (or distribution) $\mu_X$ of $X$ is a probability on $E$

$$\mu_X(A) = P(X^{-1}(A)) = P(X \in A) \quad \text{for } A \subseteq E$$

- $X$ describes a random element of $E$
- $\mu_X$ describes the values taken by $X$ and the resp. probabilities

Instead of a probability $\mu$ on $E$, it is often convenient to work with a random variable $X$ with law $\mu$

When $E = C([0, 1])$, a r.v. $X = (X_t)_{t \in [0, 1]}$ is a stochastic process
Simple random walk

Let us build a stochastic process $X^{(n)}$ with law $\mu_n$

Fair coin tossing: independent random variables $Y_1, Y_2, \ldots$ with

$$P(Y_i = +1) = P(Y_i = -1) = \frac{1}{2}$$

Simple random walk: $S_0 := 0$ $S_n := Y_1 + Y_2 + \ldots + Y_n$

Diffusive rescaling: space $\propto \sqrt{\text{time}}$

$$X^{(n)}(t) := \text{linear interpol. of } \frac{S_{nt}}{\sqrt{n}} \quad t \in [0, 1]$$

The law of $X^{(n)}$ (r.v. in $C([0,1])$) is $\mu_n$ uniform probab. on $E_n$

Donsker: The law of simple random walk, diffusively rescaled, converges weakly to the law of Brownian motion
General random walks

Instead of coin tossing, take independent random variables $Y_i$ with a generic law, with zero mean and finite variance (say 1)

Define random walk $S_n$ and its diffusive rescaling $X^{(n)}(t)$ as before

E.g. $P(Y_i = +2) = \frac{1}{3}$, $P(Y_i = -1) = \frac{2}{3}$

The law $\mu_n$ of $X^{(n)}$ is a (non uniform!) probability on $C([0,1])$
Universality of Brownian motion

Theorem (Donsker)

\[ \mu_n \Rightarrow \mu := \text{Wiener measure} \]

The law of any RW (zero mean, finite variance) diffusively rescaled converges weakly to the law of Brownian motion (Wiener measure)

Universality:

\[ \mu_n(A) \rightarrow \mu(A) \quad \forall A \subseteq C([0,1]) \text{ with } \mu(\partial A) = 0 \]

Example (Feller I, Chapter III)

- \( U_+(f) := \text{Leb}\{t \in [0,1] : f(t) > 0\} \)
  \[ = \{\text{amount of time in which } f > 0\} \]

- \( A := \{f: U_+(f) \geq 0.95 \text{ or } U_+(f) \leq 0.05\} \subseteq C([0,1]) \)

Then \( \mu_n(A) \rightarrow \mu(A) \approx 0.290.13 \). Random walk has a chance of 29%13% of spending 95%99% or more of its time on the same side!
Some sample paths of the SRW

\[ U_{N/N} = 97\% \]
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A glimpse of SLE

Even the simplest randomness (coin tossing) can lead to interesting models, such as random walks and Brownian motion

Brownian motion is at the heart of Schramm-Löwner Evolution (SLE), one of the greatest achievements of modern probability

[Fields Medal awarded to W. Werner (2006) and S. Smirnov (2010)]

We present an instance of SLE, which emerges as the scaling limit of percolation (spatial version of coin tossing)

Fix a simply connected Jordan domain $D \subseteq \mathbb{R}^2$ and $A, B \in \partial D$

$E := \{ \text{continuous } f : [0, 1] \to \overline{D} \text{ with } f(0) = A, f(1) = B \}$

$= \{ \text{curves in } \overline{D} \text{ joining } A \text{ to } B \}$ \hspace{1cm} [\| \cdot \|_\infty \text{ norm, up to reparam.}]

We now introduce discrete probabilities $\mu_n$ on $E$
1. The rescaled hexagonal lattice

- Fix $n \in \mathbb{N}$ and consider the hexagonal lattice of side $\frac{1}{n}$
- Approximate $\partial D$ with a closed loop in the lattice
2. Percolation

- Boundary hexagons colored yellow (A to B) and blue (B to A)
- Inner hexagons colored by coin tossing (critical percolation)
3. The exploration path

- Exploration path: start from $A$ and follow the boundary between yellow and blue hexagons, eventually leading to $B$
4. The law $\mu_n$

- Forgetting the colors, the exploration path is an element of $E$ (continuous curve $A \to B$)
- It is a random element of $E$ (determined by coin tossing)
- Its law $\mu_n$ is a discrete probability on $E$ ($\frac{1}{n}$ = lattice mesh)
Scaling limit of the exploration path

Fix a (simply connected) Jordan domain $D$ and points $A, B \in \partial D$

$$E := \{\text{curves in } \overline{D} \text{ joining } A \text{ to } B\}$$

**Theorem (Schramm; Smirnov; Camia & Newman)**

The sequence $(\mu_n)_{n \in \mathbb{N}}$ converges weakly on $E$: $\mu_n \Rightarrow \mu$

The limiting probability $\mu$ is the law of (the trace of) SLE(6)

- Extremely challenging!
- **Universality**? Independence of lattice (loop soup - conj.)
- **Conformal Invariance.** For another Jordan domain $D'$

$$\mu_{D';A',B'} = \phi \# (\mu_{D;A,B})$$

where $\phi : D \to D'$ is conformal with $\phi(A) = A'$, $\phi(B) = B'$
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From simple to Bessel random walk

The simple random walk is $S_n := Y_1 + \ldots + Y_n$ [Y_i coin tossing]

Fix $\alpha \in (0, 1)$ and define the $\alpha$-Bessel random walk as follows:

- $\alpha = \frac{1}{2}$: no drift ($c_\alpha = 0$) $\sim\sim$ simple random walk
- $\alpha < \frac{1}{2}$: drift away from the origin ($c_\alpha > 0$)
- $\alpha > \frac{1}{2}$: drift toward the origin ($c_\alpha < 0$)

$c_\alpha := \frac{1}{2} - \alpha$
Diffusively rescaled $\alpha$-Bessel RW

**Definition**

$$\mu_{n,\alpha} := \text{law of diffusively rescaled } \alpha\text{-Bessel RW}$$

Discrete probability on $E_n \subseteq C([0, 1])$

Not uniform for $\alpha \neq \frac{1}{2}$

**Theorem** (Extension of Donsker)

$$\forall \alpha \in (0, 1), \quad \mu_{n,\alpha} \text{ converges weakly on } C([0, 1]): \quad \mu_{n,\alpha} \Rightarrow \mu_{\alpha}$$

$$[\mu_{\alpha} := \text{law of } \alpha\text{-Bessel process} \quad \text{(Brownian motion for } \alpha = \frac{1}{2} \text{)}]$$
The disordered pinning model

Idea: reward/penalize $\alpha$-Bessel RW $\mu_{n,\alpha}$ each time it visits zero

- Fix a real sequence $\omega = (\omega_i)_{i \in \mathbb{N}}$ (charges attached to $t = \frac{i}{n}$)
- Total charge (energy) of a path $H_n^\omega(f) := \sum_{i=1}^{n} \omega_i \mathbb{1}_{\{f(\frac{i}{n})=0\}}$

Disordered pinning model $\mu_{n,\alpha}^\omega$ (Gibbs measure)

$$\mu_{n,\alpha}^\omega(f) := \frac{1}{\text{(normaliz.)}} e^{H_n^\omega(f)} \mu_{n,\alpha}(f), \quad \forall f \in E_n$$
The disordered pinning model

\( \mu_{n,\alpha}^\omega \) is a probability on \( C([0,1]) \) that depends on the sequence \( \omega \).

How to choose the charges \( \omega \)? **In a random way!**

\( (\omega_i)_{i \in \mathbb{N}} \) independent \( \mathcal{N}(h,\beta^2) \) \( [\text{mean } h \in \mathbb{R}, \text{ variance } \beta^2 > 0] \)

**Disordered systems: two sources of randomness!**

- First we sample a typical \( \omega \), called (quenched) **disorder**
- Then we have a probability \( \mu_{n,\alpha}^\omega \) on the space \( E_n \) of RW paths

The **disordered pinning model** \( \mu_{n,\alpha}^\omega \) is a **random** probability on \( E_n \)

\[ \text{[i.e. a random variable } \omega \mapsto \mu_{n,\alpha}^\omega \text{ taking values in } \mathcal{M}_1(E_n) \]}

Weak convergence of \( \mu_{n,\alpha}^\omega \) [of its law] to some random probab. \( \mu_\alpha \)?
Scaling limits of disordered pinning model

Inspired by [Alberts, Khanin, Quastel 2014]

**Theorem (F. Caravenna, R. Sun, N. Zygouras)**

Rescale suitably $\beta$, $h$ (disorder mean and variance) and let $n \to \infty$

- $(\alpha < \frac{1}{2})$ Disorder disappears in the scaling limit!
  
  $$\mu_{n,\alpha}^{\omega} \Rightarrow \mu_{\alpha}$$
  
  law of $\alpha$-Bessel process (as if $\omega \equiv 0$)

- $(\alpha > \frac{1}{2})$ Disorder survives in the scaling limit!
  
  $$\mu_{n,\alpha}^{\omega} \Rightarrow \mu_{\alpha}^{\omega}$$
  
  truly random probability on $C([0, 1])$

Recall that $\mu_{n,\alpha}^{\omega} \ll \mu_{n,\alpha}$ for every $n \in \mathbb{N}$ (Gibbs measure)

However $\mu_{\alpha}^{\omega} \ll \mu_{\alpha}$ for a.e. $\omega$! (no continuum Gibbs measure)

- $(\alpha = \frac{1}{2})$ Work in progress...
Thanks
Weak convergence in presence of disorder

- $E$ is a Polish space (complete separable metric space)
- $\mathcal{M}_1(E) :=$ probability measures on $E$
- Notion of convergence $\mu_n \Rightarrow \mu$ (weak convergence) in $\mathcal{M}_1(E)$

What if $\mu_n^\omega, \mu^\omega$ are random probabilities on $E$?

- $[\omega \in \Omega$ probability space$]$

- The space $\tilde{E} := \mathcal{M}_1(E)$ is also Polish
- Random probabilities $\mu_n^\omega, \mu^\omega$ are $\tilde{E}$-valued random variables
- Their laws are probabilities on $\tilde{E}$: weak convergence applies!

We still write $\mu_n^\omega \Rightarrow \mu^\omega$ for this convergence
(heuristics/intuition analogous to the non-disordered case)