Marginally relevant polymer models
in the critical window

(joint work with R. Sun and N. Zygouras)

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I am going to talk about

- **Directed Polymer** in Random Environment in dim. $d = 2$

Our results also apply to other *marginally relevant* disordered systems

- Pinning Model with tail exponent $\alpha = 1/2$
- Directed Polymer with Cauchy tails in dim. $d = 1$
- Stochastic Heat Equation (SHE) with $d = 2$
Outline

1. Directed Polymer

2. Known Results

3. Critical Window

4. Techniques and proofs

5. Additional results
Directed Polymer in Random Environment

- **Reference Model:** simple random walk on $\mathbb{Z}^d$
  \[(S_n)_{n \geq 0} \quad P_{\text{rw}}(S_n - S_{n-1} = \pm e_i) = \frac{1}{2^d}\]

- **Disorder:** i.i.d. random variables $\omega(n, x)$
  zero mean, unit variance, expon. moments

  \[\lambda(\beta) := \log \mathbb{E}[e^{\beta \omega(n,x)}] < \infty\]

- **(-) Hamiltonian**
  \[H_N(S, \omega) := \sum_{n=1}^{N} \omega(n, S_n)\]

\[
\frac{dP_N^\omega(S_1, \ldots, S_N)}{dP_{\text{rw}}^\omega(S_1, \ldots, S_N)} \propto e^{\beta H_N(S, \omega)} = \frac{e^{\beta H_N(S, \omega)}}{Z_N^\omega}
\]

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Weak and strong disorder

- $(d \geq 3)$ There is a weak disorder phase: $\exists \beta_c > 0$ such that

  for $0 \leq \beta < \beta_c$  \( P^\omega_N \) is “similar” to \( P^{rw} \)

  \[
  \text{CLT} \quad P^\omega_N \left( \frac{S_N}{\sqrt{N}} \in \cdot \right) \xrightarrow{d} \mathcal{N}(0,1) \quad N \to \infty
  \]

  [Imbrie, Spencer 88] [Bolthausen 89] [Comets, Yoshida 06] [Chatterjee 16]

- $(d = 1, \ d = 2)$ There is always strong disorder:

  for any $\beta > 0$:  \( P^\omega_N \) “very different” from \( P^{rw} \)

  Conj. super-diffusivity  \( |S_N| \gg \sqrt{N} \) under \( P^\omega_N \)

  Macroscopic atoms  \( \max_{x \in \mathbb{Z}^d} P^\omega_N (S_N = x) \geq c > 0 \)

  [Carmona, Hu 02] [Comets, Shiga, Yoshida 03] [Vargas 07] [Lacoin 11] [Chatterjee 16]
Intermediate disorder

Henceforth we focus on the cases \( d = 1, \ d = 2 \)

Any fixed disorder strength \( \beta > 0 \) has dramatic effects as \( N \to \infty \)

Can we tune \( \beta = \beta_N \to 0 \) to see an interesting transition?

This is called intermediate disorder regime, because it interpolates between weak and strong disorder

(cf. near-critical percolation)

We do not focus on the probability \( P^\omega_N \) but rather on partition functions
Partition function

\[ Z^\omega_N = \mathbb{E}^{rw} \left[ e^{\beta H_N(S,\omega)} \right] = \mathbb{E}^{rw} \left[ e^{\beta \sum_{n=1}^{N} \omega(n,S_n)} \right] e^{-\lambda(\beta)N} \]

It amounts to redefine \( Z^\omega_N \rightsquigarrow Z^\omega_N / \mathbb{E}[Z^\omega_N] \)

- \( Z^\omega_N \) is a positive random variable with \( \mathbb{E}[Z^\omega_N] = 1 \) (martingale!)
- \((d = 1, \ d = 2)\) Strong disorder means

\[ \forall \beta > 0: \lim_{N \to \infty} Z^\omega_N = 0 \quad \mathbb{P}\text{-a.s.} \]
The random field of partition functions

\[ Z_\omega^N(z) := \text{partition function for RW starting at } z \in \mathbb{Z}^d \]

\[ = \mathbb{E}^{\text{rw}} \left[ e^{\beta H_N} \bigg| S_0 = z \right] e^{-\lambda(\beta)N} \]

Note that \( Z_\omega^N(z) \overset{d}{=} Z_\omega^N(0) = Z_\omega^N \xrightarrow{N \to \infty} 0 \) for every fixed \( \beta > 0 \)

Can we tune \( \beta = \beta_N \to 0 \) so that

\[ Z_\omega^N(\sqrt{N} \cdot x) \xrightarrow{N \to \infty} \mathcal{Z}(x) \] (random field on \( \mathbb{R}^d \))
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Case $d = 1$

For $d = 1$ the right scaling is $\beta_N = \frac{\hat{\beta}}{N^{1/4}}$

\textbf{Theorem} [Alberts, Khanin, Quastel (AOP '14)]

- Convergence in distribution
  \[ Z_{Nt}^\omega(\sqrt{N}x) \xrightarrow{d} \mathcal{Z}_t(x) \]
  \( N \to \infty \)

- $\mathcal{Z}_t(x)$ is solution of 1d Stochastic Heat Equation (SHE)
  \[
  \begin{cases}
  \partial_t \mathcal{Z} = \frac{1}{2} \Delta_x \mathcal{Z} + \hat{\beta} \cdot \dot{W} \mathcal{Z} \\
  \mathcal{Z}_0 = 1
  \end{cases}
  \]

\( W \) = Gaussian white noise on $[0, \infty) \times \mathbb{R}$
Case $d = 1$

Non-trivial limiting field: $\mathcal{Z}_t(x) > 0$ for every $\hat{\beta} \in (0, \infty)$

Corollary

Strong disorder emerges smoothly on the scale $\beta \propto \frac{1}{N^{1/4}}$

$$\mathcal{Z}_N^\omega \xrightarrow{d_{N \to \infty}} \begin{cases} 1 & \text{if } \beta \ll \frac{1}{N^{1/4}} \\ \mathcal{Z} > 0 & \text{if } \beta \sim \frac{\hat{\beta}}{N^{1/4}} \\ 0 & \text{if } \beta \gg \frac{1}{N^{1/4}} \end{cases}$$

$\mathcal{Z}_t(x) \rightsquigarrow$ Brownian Directed Polymer in Random Environment

[Alberts, Khanin, Quastel (JSP '14)]
Case $d = 2$: marginal relevance

Henceforth we focus on $d = 2$

The right scaling is $\beta_N \sim \sqrt{\frac{\pi}{\log N}} \hat{\beta}$

Logarithmic replica overlap

$$R_N := E^{rw} \left[ \sum_{n=1}^{N} 1 \{ S_n = S'_n \} \right] \sim \frac{1}{\pi} \log N$$

We look again for $Z^\omega_N(\sqrt{N}x) \xrightarrow{N \to \infty} Z(x)$

Unlike the case $d = 1$, there is a phase transition in $\hat{\beta}$
Phase transition

**Theorem [C., Sun, Zygouras (AAP to appear)]**

- For every fixed $x \in \mathbb{R}^2$
  \[
  Z_N^\omega(\sqrt{N}x) \xrightarrow{d} \tilde{Z}(x) \quad \begin{cases} 
  > 0 \text{ a.s. if } \hat{\beta} < 1 \\
  = 0 \text{ a.s. if } \hat{\beta} \geq 1
  \end{cases}
  \]

- $(\hat{\beta} < 1)$ Log-normal marginals with $\mathbb{E}[\tilde{Z}(x)] \equiv 1$
  \[
  \tilde{Z}(x) \overset{d}{=} \exp \left\{ N(0, \sigma^2) - \frac{1}{2} \sigma^2 \right\} \quad \text{with} \quad \sigma^2 = \log \frac{1}{1 - \hat{\beta}^2}
  \]

- $(\hat{\beta} < 1)$ Joint distributions: for any $x \neq x'$
  \[
  \tilde{Z}(x) \text{ and } \tilde{Z}(x') \text{ are independent (!)}
  \]

[ Dependence in $Z_N^\omega(z)$, $Z_N^\omega(z')$ at all scales $|z - z'| = o(\sqrt{N})$ ]
A different viewpoint

Recall that $\beta_N \sim \frac{\sqrt{\pi} \hat{\beta}}{\sqrt{\log N}}$

- $(\hat{\beta} < 1)$ Disorder has weak effects ($\tilde{Z}(x)$ indep. of $\tilde{Z}(x')$)
- $(\hat{\beta} \geq 1)$ Trivial limit $\tilde{Z}(x) \equiv 0$

Can we obtain an interesting limit $Z(x) \not\equiv 0$ for $\hat{\beta} \geq 1$?

$Z^\omega_N(\sqrt{N}x)$ is an irregular function of $x \in \mathbb{R}^2$

$\leadsto$ We should look for a limit in the space of (Schwartz) distributions!

(Instead of distributions we can focus on measures, because $Z^\omega_N \geq 0$)
Heuristic picture

\[ Z_\omega^N(\sqrt{N}x) \]

\[ x \in \mathbb{R}^2 \]
Averaged partition function

Henceforth we look at $Z_N^\omega(\sqrt{N}x)$ as a random measure on $\mathbb{R}^2$

For positive continuous $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ we define

$$\langle Z_N^\omega , \phi \rangle := \int_{\mathbb{R}^2} Z_N^\omega(\sqrt{N}x) \phi(x) \, dx$$

We can revisit our results for $\hat{\beta} < 1$

**Proposition**

For $\hat{\beta} < 1$ we have $Z_N^\omega(\sqrt{N}x) \xrightarrow{d} Z(x) \equiv 1$

$$\langle Z_N^\omega , \phi \rangle \xrightarrow{N \to \infty} \langle 1 , \phi \rangle = \int_{\mathbb{R}^2} \phi(x) \, dx$$
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What happens for $\hat{\beta} = 1$?

We now set $\hat{\beta} = 1$. More generally, we explore the critical window

$$\beta_N = \sqrt{\frac{\pi}{\log N} \left(1 + \frac{\vartheta}{\log N}\right)}$$

with $\vartheta \in \mathbb{R}$

For fixed $x \in \mathbb{R}^2$ we already know that $Z^\omega_N(\sqrt{N}x) \xrightarrow{d} 0$

We now look at $Z^\omega_N(\sqrt{N}x)$ as a random measure

**Conjecture**

$Z^\omega_N(\sqrt{N}x)$ converges to a generalized random field $Z(x)$ on $\mathbb{R}^2$

$$\langle Z^\omega_N, \phi \rangle \xrightarrow{d} \langle Z, \phi \rangle \quad \text{for every } \phi$$

$Z$ is a random measure on $\mathbb{R}^2$ (expected to be singular wrt Lebesgue)
Second moment in the critical window

What is known [Bertini, Cancrini ’95 (on 2d SHE)]

Tightness via second moment bounds

$$\mathbb{E}[\langle Z^\omega_N, \phi \rangle] \equiv \langle 1, \phi \rangle \quad \text{sup}_{N \in \mathbb{N}} \mathbb{E}[\langle Z^\omega_N, \phi \rangle^2] < \infty$$

More precisely

$$\text{Var}[\langle Z^\omega_N, \phi \rangle] \xrightarrow{N \to \infty} \langle \phi, K \phi \rangle < \infty$$

Explicit

$$K(x, x') \sim C \log \frac{1}{|x - x'|} \quad \text{as} \quad |x - x'| \to 0$$

Corollary

Existence of subsequential limits

$$\langle Z^\omega_N, \phi \rangle \xrightarrow{d, N \to \infty} \langle Z, \phi \rangle$$
New results: third moment

**Theorem** [C., Sun, Zygouras ’17+]

\[
\lim_{N \to \infty} \mathbb{E} \left[ \langle Z_N^\omega, \phi \rangle^3 \right] = C(\phi) < \infty
\]

**Corollary**

Any subsequential limit \( Z \) has the same covariance kernel \( K(x, x') \)

\( \sim Z \not\equiv 1 \) is non-degenerate!

- Explicit expression for \( C(\phi) \) as a series of multiple integrals
Work in progress

- Uniqueness of subsequential limit $\mathcal{Z}$ via coarse-graining arguments

$$\Rightarrow \quad \text{Existence of the limit} \quad Z^\omega_N \xrightarrow[d}{N \to \infty} Z$$

- Investigate properties of the limiting random measure $\mathcal{Z}$
  (it looks not so close to Gaussian Multiplicative Chaos)

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Partition function and polynomial chaos

\[ Z_{\omega}^{N} = E^{rw} \left[ e^{H_N(\omega, S)} \right] = E^{rw} \left[ e^{\sum_{1 \leq n \leq N} \sum_{x \in \mathbb{Z}^2} (\beta \omega(n, x) - \lambda(\beta)) \mathbb{1}_{\{S_{n}=x\}}} \right] \]

\[ = E^{rw} \left[ \prod_{1 \leq n \leq N} \prod_{x \in \mathbb{Z}^2} e^{(\beta \omega(n, x) - \lambda(\beta)) \mathbb{1}_{\{S_{n}=x\}}} \right] \]

\[ = 1 + \sum_{1 \leq n \leq N} P^{rw}(S_{n} = x) X_{n,x} \]

\[ + \sum_{1 \leq n < m \leq N} P^{rw}(S_{n} = x, S_{m} = y) X_{n,x} X_{m,y} + \ldots \]

\[ Z_{\omega}^{N} \text{ multi-linear polynomial of new RVs } X_{n,x} := e^{\beta \omega(n, x) - \lambda(\beta)} - 1 \]
Polynomial chaos

\[ \mathbb{E}[X_{n,x}] = 0 \quad \text{and} \quad \text{Var}[X_{n,x}] \sim \beta^2 \]

Let us pretend \( X_{n,x} = \beta Y_{n,x} \) with \( (Y_{n,x})_{n,x} \) i.i.d. \( \mathcal{N}(0, 1) \)

Then

\[
Z_N^{(1)} := \sum_{1 \leq n \leq N} \mathbf{P}^{\text{rw}}(S_n = x) Y_{n,x}
\]

\[
Z_N^{(2)} := \sum_{1 \leq n \leq m \leq N} \mathbf{P}^{\text{rw}}(S_n = x, S_m = y) Y_{n,x} Y_{m,y}
\]
The choice of $\beta$

$Z_N^{(1)}$ is Gaussian with variance given by the replica overlap $R_N$:

$$\text{Var}[Z_N^{(1)}] = \sum_{1 \leq n \leq N} \sum_{x \in \mathbb{Z}^2} P^{rw}(S_n = x)^2 = \sum_{1 \leq n \leq N} P^{rw}(S_n = S'_n)$$

$$\sim \frac{1}{\pi} \sum_{1 \leq n \leq N} \frac{1}{n} \sim \frac{\log N}{\pi}$$

To normalize $\beta Z_N^{(1)}$ we choose $\beta = \frac{\hat{\beta}}{\sqrt{\log N / \pi}}$

Similarly, $\text{Var}[Z_N^{(2)}] \sim \frac{1}{\pi^2} \sum_{1 \leq n < m \leq N} \frac{1}{n} \frac{1}{m-n} \ll \left( \frac{\log N}{\pi} \right)^2$
Variance bounds for $\hat{\beta} < 1$

More generally

$$\text{Var}[Z_N^{(k)}] \lesssim \left(\frac{\log N}{\pi}\right)^k$$ ($\ast$)

For $\hat{\beta} < 1$

$$\text{Var}[Z_N^\omega] \lesssim \sum_{k=1}^{\infty} \left(\frac{\hat{\beta}^2}{\log N}\right)^k \left(\frac{\log N}{\pi}\right)^k \lesssim \sum_{k=1}^{\infty} \hat{\beta}^{2k} < \infty$$

To deal with $\hat{\beta} = 1$ we need to refine ($\ast$)
Sharp asymptotics

Lemma

\[ \text{Var}[Z_N^{(k)}] \sim \sum_{0 < n_1 < \ldots < n_k \leq N} \frac{1}{n_1} \frac{1}{n_2 - n_1} \ldots \frac{1}{n_k - n_{k-1}} \]

\[ \sim \left( \frac{\log N}{\pi} \right)^k \mathbb{P} \left( T_{\frac{k}{\log N}} \leq 1 \right) \]

- \( (T_s)_{s \geq 0} \) increasing Lévy process (subordinator) with Lévy measure \( \nu(dt) = \frac{1}{t} 1_{(0,1)}(t) \)

- One can compute \( \mathbb{P} (T_s \leq 1) = \frac{e^{-\gamma s}}{\Gamma(1+s)} \)

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Variance

For $\hat{\beta} = 1$

$$\text{Var} \left[ Z_N^\omega \right] \sim \sum_{k=1}^{\infty} \mathbb{P} \left( T_{k \log N} \leq 1 \right) \sim C \log N$$

where

$$C := \int_0^\infty \mathbb{P}(T_s \leq 1) \, ds = \int_0^\infty \frac{e^{-\gamma s}}{\Gamma(1 + s)} \, ds$$

Covariances

$$\text{Cov} \left[ Z_N^\omega(x), Z_N^\omega(x') \right] \sim K(x, x')$$

$$= \int_0^1 \frac{e^{-\frac{|x' - x|^2}{2t}}}{2t} \left( \int_0^\infty \frac{e^{-\gamma s} (1 - t)^s}{\Gamma(1 + s)} \, ds \right) \, dt$$
Third moment in the critical window

\[ \langle Z_N^\omega, \phi \rangle \] is multilinear polynomial (sum of products) of i.i.d. RVs \( X_{n,x} \)

\[
\langle Z_N^\omega, \phi \rangle = \sum_{I \subseteq \{1, \ldots, N\} \times \mathbb{Z}^2} c(I) \prod_{(n,x) \in I} X_{n,x}
\]

for suitable \( c(I) = c(I, N, \phi) \)

- Expand \( \mathbb{E}[\langle Z_N^\omega, \phi \rangle^3] \) in 3 sums
- \( X \)'s from different sums match in pairs or triples (by \( \mathbb{E}[X_{n,x}] = 0 \))
- Triple matchings give negligible contribution

Pairwise matching of the \( X \)'s \( \rightsquigarrow \) highly non-trivial, yet manageable combinatorial structure \( \rightsquigarrow \) sharp asymptotics for \( \mathbb{E}[\langle Z_N^\omega, \phi \rangle^3] \)
Thanks
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Multi-scale correlations for $\hat{\beta} < 1$

**Theorem**

Fix $\hat{\beta} < 1$ and

$$|z - z'| \asymp N^\zeta \quad \zeta \in [0, \frac{1}{2}]$$

Then

$$\left( Z_N^\omega(z), Z_N^\omega(z') \right) \xrightarrow{d} \left( e^Y - \frac{1}{2} \text{Var}[Y], e^{Y'} - \frac{1}{2} \text{Var}[Y'] \right)$$

- $Y, Y'$ jointly Normal with variance $\sigma^2 = \log \frac{1}{1-\hat{\beta}^2}$

- $\text{Cov}[Y, Y'] = \log \frac{1-(2\zeta)\hat{\beta}^2}{1-\hat{\beta}^2}$
Path diffusivity for $\hat{\beta} < 1$

**Diffusivity**

- **Central Limit Theorem**

  \[ P_N^\omega \left( \frac{S_N}{\sqrt{N}} \in \cdot \right) \xrightarrow{N \to \infty} N(0, 1) \quad \text{in } \mathbb{P}(d\omega)-\text{probability} \]

- **Local Limit Theorem with random corrections**

  \[(\sqrt{N})^2 P_N^\omega \left( S_N = \lfloor x \sqrt{N} \rfloor \right) \xrightarrow{N \to \infty} e^{Y_x - \frac{1}{2} \text{Var}[Y_x]} \frac{e^{-|x|^2/2}}{2\pi}\]
Partition function fluctuations for $\hat{\beta} < 1$

For $\hat{\beta} < 1$  $\mathbb{Z}_N^{\omega}(\sqrt{N}x) \xrightarrow{P} 1$ (as a Schwartz distribution on $\mathbb{R}^2$)

This can be viewed as a LLN. Here is the corresponding CLT.

**Theorem [C., Sun, Zygouras (AAP to appear)]**

$$\mathbb{Z}_N^{\omega}(\sqrt{N}x) \overset{d}{\approx} 1 + \frac{1}{\sqrt{\log N}} G(x)$$

in $S'$

where $G(x)$ is a generalized Gaussian field on $\mathbb{R}^2$ with

$$\text{Cov} \left[ G(x), G(x') \right] \sim C \log \frac{1}{|x - x'|}$$

More precisely

$$\left\langle \sqrt{\log N} \left( \mathbb{Z}_N^{\omega}(\sqrt{N} \cdot) - 1 \right), \phi \right\rangle \xrightarrow{N \to \infty} \left\langle G, \phi \right\rangle \quad \forall \phi \in C_0(\mathbb{R}^2)$$
Second moment in the critical window

**Theorem (variance vs. covariances)**

\[ \nabla \text{Var}[Z_N^\omega(\sqrt{N}x)] \simeq \log N \to \infty \]

\[ \nabla \text{Cov}[Z_N^\omega(\sqrt{N}x), Z_N^\omega(\sqrt{N}x')] \xrightarrow{N \to \infty} K(x, x') < \infty \]

\[ K(x, x') \sim C \log \left| \frac{1}{|x - x'|} \right| \quad \text{as} \quad |x - x'| \to 0 \]

**Corollary**

\[ \nabla \text{Var}[\langle Z_N^\omega, \phi \rangle] \xrightarrow{N \to \infty} \langle \phi, K \phi \rangle < \infty \]

**Explicit kernel:**

\[ K(x, x') = \int_0^1 \frac{1}{2t} e^{-\frac{|x' - x|^2}{2t}} \left( \int_0^\infty \frac{e^{(\pi t^2 - \gamma)s} (1 - t)^s}{\Gamma(1 + s)} ds \right) dt \]

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The 2d Stochastic Heat Equation

\[
\begin{cases}
\partial_t u(t, x) = \frac{1}{2} \Delta_x u(t, x) + \beta \dot{W}(t, x) u(t, x) \\
u(0, x) \equiv 1
\end{cases}
\]

where \( \dot{W}(dt, dx) \) is Gaussian white noise on \([0, \infty) \times \mathbb{R}^2\)

Mollified noise:

\[ W_\delta(dt, dx) := \int_{y \in \mathbb{R}^2} \frac{1}{\delta} j\left(\frac{x-y}{\sqrt{\delta}}\right) W(dt, dy) \]

Mollified solution

\[ u_\delta(t, x) \overset{d}{=} Z_{\frac{\sqrt{N}}{\delta}}(\sqrt{N} x) \quad \text{for} \quad N = \frac{1}{\delta} \]

Generalized Feynman-Kac Formula

\[ u_\delta(t, x) \overset{d}{=} \mathbb{E}^{BM}\left[ \exp \left\{ \int_0^t (\beta W_1(ds, B_s) - \frac{1}{2}\beta^2 ds) \right\} \bigg| B_{t/\delta} = \frac{x}{\sqrt{\delta}} \right] \]

[Bertini, Cancrini ’95]