Restriction of characters to Sylow $p$-subgroups

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Topics on Groups and their Representations

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Introduction
Let $G$ be a finite group, $p$ prime, $P \in \text{Syl}_p(G)$.

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Let $G$ be a finite group, $p$ prime, $P \in \text{Syl}_p(G)$.

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Notice that $\text{Irr}_{p'}(P) = \{ \lambda \in \text{Irr}(P) \mid \lambda(1) = 1 \} =: \text{Lin}(P)$.

**Conjecture (McKay; 1972)**

*Let $G$ be a finite group, $p$ prime. Then $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(\text{N}_G(P))|$.***
Theorem (Malle, Spaeth; 2015)

Let $G$ be a finite group, and $p = 2$. Then $|\text{Irr}_2'(G)| = |\text{Irr}_2'(\text{N}_G(P))|$. 

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Let $S_n$ be the symmetric group and let $P_n \in \text{Syl}_2(S_n)$.
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Find a canonical bijection $\Phi : \text{Irr}_2(S_n) \rightarrow \text{Irr}_2(N_{S_n}(P_n))$.
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Fact: $N_{S_n}(P_n) = P_n$. Hence $\text{Irr}_2'(N_{S_n}(P_n)) = \text{Lin}(P_n)$. 
Since $P_n = P_{2^{n_1}} \times P_{2^{n_2}} \times \cdots \times P_{2^{n_t}}$, the case $n = 2^k$ is key.
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**Theorem A (G, 2016)**

Let $\chi \in \text{Irr}_{2'}(S_{2^k})$ then:

(i) There exists a unique $\chi^* \in \text{Lin}(P_{2^k})$ such that $\chi \downarrow_{P_{2^k}} = \chi^* + \Delta$. 
(Here $\Delta$ is a sum of irreducible characters of even degree).

(ii) Moreover, $\star : \text{Irr}_{2'}(S_{2^k}) \longrightarrow \text{Irr}_{2'}(N_{S_{2^k}}(P_{2^k}))$ is a bijection.
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**Theorem B (G, 2016)**

Let $n \in \mathbb{N}$ and $\chi \in \text{Irr}(S_n)$, then:

(i) There always exists a $\lambda \in \text{Lin}(P_n)$ such that $\lambda \mid \chi \downarrow_{P_n}$.

(ii) $\lambda$ is unique if and only if $n = 2^k$ and $\chi \in \text{Irr}_2(S_{2^k})$. 
Theorem C (G, Kleshchev, Navarro, Tiep 2016)

There exists a combinatorially defined canonical bijection

$$\Phi : \text{Irr}_{2'}(S_n) \longrightarrow \text{Irr}_{2'}(N_{S_n}(P_n)).$$

Moreover $$\Phi(\chi) \mid \chi \downarrow_{P_n},$$ for all $$\chi \in \text{Irr}(S_n).$$
Restriction to Sylow $p$-subgroups
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**Facts**

- If \( \chi \in \text{Irr}_{p'}(G) \) then \( |L_\chi| \neq 0 \).
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- If $p \mid \chi(1)$ then $|L_\chi|$ could in principle take any value $\{0, 1, 2, \ldots\}$.
- If $G = S_n$ and $p = 2$ then $|L_\chi| \neq 0$ for all $\chi$. 
Theorem A

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- $P_{p^k} \cong C_p \lhd \cdots \lhd C_p \lhd C_p = P_{p^{k-1}} \lhd C_p = B \rtimes C_p$,
- where $B = P_{p^{k-1}} \times P_{p^{k-1}} \times \cdots \times P_{p^{k-1}}$ is the base group above.
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Lemma

Let $\lambda \in \text{Irr}(P_{p^k})$. Then $\lambda(1) = 1$ if and only if there exists $\varphi \in \text{Lin}(P_{p^{k-1}})$ such that $\varphi \times \varphi \times \cdots \times \varphi \mid \lambda \downarrow_B$.

...whiteboard...
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**Theorem B (q-sections of characters/partitions)**

Let $\chi \in \text{Irr}(S_n)$. Then, there exists $\Delta(\chi) \in \text{Irr}(S_m)$ such that

$$\Delta(\chi) \times \Delta(\chi) \times \cdots \times \Delta(\chi) \mid \chi \downarrow_D.$$
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What about arbitrary groups?
Conjecture C

Let $\chi \in \operatorname{Irr}(G)$ be such that $p | \chi(1)$. If $|L_\chi| \neq 0$ then $|L_\chi| \geq p$. 
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Theorem D

Conjecture C holds for the following classes of groups:

- Solvable groups.
- Groups with abelian Sylow $p$-subgroup. (Strong form).
- Symmetric and Alternating groups. (Strong form).
- All the sporadic simple groups.
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**Groups with abelian Sylow $p$-subgroups**
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Groups with abelian Sylow $p$-subgroups

Roughly speaking, the same as above holds. Moreover we can choose $D$ to be a defect group of the $p$-block of $\chi$. 
Future work: Prove Conjecture C, for all finite groups.....

Suspect

Let $\chi \in \text{Irr}(G)$ be such that $p \mid \chi(1)$. If $|L_\chi| \neq 0$ then there exists a subgroup $D \leq P$ and $\lambda \in \text{Lin}(D)$ such that $(\lambda) \uparrow^P$ is a constituent of $\chi \downarrow_P$. 
On a question of Alex Zalesski
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**Equivalent Question**

*Given $\lambda \vdash n$, is $1_{P_n}$ an irreducible constituent of $(\chi^\lambda) \downarrow_{P_n}$?*
Theorem (G, Law; 2017)

Let $p$ be an odd prime and let $n > 10$ be a natural number. Then the trivial character $1_{P_n}$ is a constituent of $(\chi^\lambda)_{|P_n}$ for all $\lambda \vdash n$, unless $n = p^k$ and $\lambda \in \{(p^k - 1, 1), (2, 1^{p^k - 2})\}$.
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Some remarks:

- Almost all irreducible characters of $S_n$ are irreducible constituents of $1_{P_n} \uparrow^{S_n}$, when $p$ is odd.
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Some remarks:

- Almost all irreducible characters of $S_n$ are irreducible constituents of $1_{P_n} \uparrow^{S_n}$, when $p$ is odd.
- The situation is completely different, and more chaotic when $p = 2$.  

Thank you very much!!