INTENSE AUTOMORPHISMS OF FINITE GROUPS

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Topics on Groups and their Representations
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Joint work with
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Intense automorphisms of groups
Let $G$ be a finite group. An automorphism $\alpha$ of $G$ is intense if for all $H \leq G$ there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Write $\alpha \in \text{Int}(G)$.

Motivation: Intense automorphisms appear naturally as solutions to a certain cohomological problem. They (surprisingly!) give rise to a very rich theory.

Example:
- Every automorphism of a cyclic group is intense.
- Inner automorphisms are intense.
- Power automorphisms are intense.
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$$\text{Int}(G) \cong P \rtimes C$$

where

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- $C$ is a subgroup of $\mathbb{F}_p^*$. 
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The intensity of $G$ is $\text{int}(G) = |C|$. 
The problem

Can we classify all $p$-groups $G$ satisfying $\text{int}(G) > 1$?
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YES!
Abelian groups

Let $p$ be a prime number and let $\mathbb{Z}_p$ denote the ring of $p$-adic integers. Let $G$ be a finite abelian $p$-group.
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$$\mathbb{Z}_p^* \rightarrow Aut(G) \rightarrow \mathbb{F}_p^* \rightarrow Int(G)$$

**Theorem**

*Let $p$ be a prime number and let $G \neq 1$ be a finite abelian $p$-group. Then $\text{int}(G) = p - 1$.***
Strategy

Let $p$ be a prime number and let $G$ be a finite $p$-group. Let $N$ be a normal subgroup.
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and with a little extra work

3. if $N \neq G$, then $\text{int}(G)$ divides $\text{int}(G/N)$. 

Since we want $G$ to have $\text{Int}(G) > 1$, we can forget about $p = 2$. 

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Since we want \( G \) to have \( \text{int}(G) > 1 \), we can forget about \( p = 2 \!! \).
Let $p$ be an odd prime and let $G$ be an extraspecial group of exponent $p$. 

Theorem

Let $p$ be a prime number and let $G$ be a finite $p$-group of class 2. Then $\int(G) > 1$ if and only if $G$ is extraspecial of exponent $p$ (in which case $\int(G) = p - 1$).
Let \( p \) be an odd prime and let \( G \) be an extraspecial group of exponent \( p \). Then, for \( \lambda \in \mathbb{Z}_p^* \), we have

\[
\frac{G}{\gamma_2(G)} \times \frac{G}{\gamma_2(G)} \twoheadrightarrow \gamma_2(G)
\]

\[
\begin{array}{ccc}
\lambda & \lambda & \lambda^2 \\
\downarrow & \downarrow & \downarrow \\
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Class 3

Theorem
Let $p$ be an odd prime and let $G$ be a finite $p$-group of class 3. Then the following are equivalent.

1. One has $\text{int}(G) > 1$.
2. One has $|G : \gamma_2(G)| = p^2$.

Corollary
Let $p$ be a prime number and let $c \in \mathbb{Z} \geq 3$. Then there exist, up to isomorphism, only finitely many finite $p$-groups of class $c$ and intensity greater than 1.
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Necessary conditions

Let $p$ be a prime number and let $G$ be a finite $p$-group of class $c \geq 3$. Define

$$w_i = \log_p |\gamma_i(G) : \gamma_{i+1}(G)|.$$
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- \(|\alpha| = 2 \) and \( \text{int}(G) = 2 \).
- \( \gamma_i(G)/\gamma_{i+1}(G) \) is elementary abelian and \( \alpha \equiv (-1)^i \) on it.
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- $|\alpha| = 2$ and $\text{int}(G) = 2$.
- $\gamma_i(G)/\gamma_{i+1}(G)$ is elementary abelian and $\alpha \equiv (-1)^i$ on it.
- $(w_i)_{i \geq 1} = (2, 1, 2, 1, \ldots, 2, 1, w, 0, 0, 0, \ldots)$ with $w \in \{0, 1, 2\}$. 
Normal subgroups structure

\[ f = 0 \] \hspace{2cm} \[ f = 1 \] \hspace{2cm} \[ f = 2 \]
Pro-$p$-help?
Let $p > 3$ be a prime number and let $t \in \mathbb{Z}_p$ satisfy $(\frac{t}{p}) = -1$. Set

$$A_p = \mathbb{Z}_p + \mathbb{Z}_pi + \mathbb{Z}_pj + \mathbb{Z}_p ij$$

with defining relations $i^2 = t$, $j^2 = p$, and $ji = -ij$. Then $A_p$ is a non-commutative local ring such that $A_p/jA_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : A_p \to A_p$ is defined by

$$a = s + ti + uj + vij \mapsto \overline{a} = s - ti - uj - vij.$$

Let $G = \{a \in A^*_p \mid a\overline{a} = 1 \text{ and } a \equiv 1 \mod jA_p\}$ and, for all $a \in G$, define $\alpha(a) = iai^{-1}$. 

Theorem $G$ is a non-nilpotent pro-$p$-group and $\alpha$ induces an intense automorphism of order 2 on every non-trivial discrete quotient of $G$. Moreover, $G$ is unique with this property.
Let $p > 3$ be a prime number and let $t \in \mathbb{Z}_p$ satisfy $(\frac{t}{p}) = -1$. Set

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$G$ is a non-nilpotent pro-$p$-group and $\alpha$ induces an intense automorphism of order 2 on every non-trivial discrete quotient of $G$. Moreover, $G$ is “unique with this property”.
Are 3-groups really special?

Lemma

Let $p$ be a prime number and let $G$ be a finite $p$-group of class at least 4. If $|\text{int}(G)| > 1$, then $p$-th powering induces a bijection $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

We define a $\kappa$-group to be a finite 3-group $G$ with $|G:\gamma_2(G)| = 9$ such that cubing induces a bijection $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

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There is, up to isomorphism, a unique $\kappa$-group of class 3.
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There is, up to isomorphism, a unique \( \kappa \)-group of class 3.
3-groups are really special

Let $R = \mathbb{F}_3[\epsilon]$ be of cardinality 9, with $\epsilon^2 = 0$. Set

$$\Delta = R + Ri + Rj + Rij$$

with defining relations $i^2 = j^2 = \epsilon$ and $ji = -ij$. The standard involution is

$$a = s + ti + uj + vij \mapsto \bar{a} = s - ti - uj - vij.$$  

Write $m = \Delta i + \Delta j$ and define $MC(3) = \{x \in 1 + m : \bar{x} = x^{-1}\}$. 
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The group \( \text{MC}(3) \) has order 729, class 4, and it is a \( \kappa \)-group.
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**Theorem**

*Let $G$ be a finite 3-group of class at least 4. Then $\text{int}(G) > 1$ if and only if $G \cong MC(3)$.***
The actual classification

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<th>3</th>
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<td>$p - 1$ if $G$ extraspecial of exponent $p$; 1 otherwise</td>
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</tr>
<tr>
<td>3</td>
<td></td>
<td>2 if $</td>
<td>G : G_2</td>
<td>= p^2$; 1 otherwise</td>
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<tr>
<td>4</td>
<td>1</td>
<td>2 if $G \cong MC(3)$; 1 otherwise</td>
<td>2 if $G$ is a $p$-obelisk with a concrete automorphism; 1 otherwise</td>
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<td>G_5</td>
<td>= p$, $\Phi(C_G(G_4)) = G_3$, and $G$ has a concrete automorphism; 1 in all other cases</td>
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THANK YOU