A hyperbolic model for the laser cutting process

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ARTICLE INFO

Article history:
Received 3 April 2012
Received in revised form 21 January 2013
Accepted 21 February 2013
Available online 15 March 2013

Keywords:
Hyperbolic conservation laws
Non–local balance laws
Laser cutting

ABSTRACT

This paper presents a new hyperbolic model aimed at the description of an industrial continuous wave fusion laser cutting process. Analytically, it consists of a non standard $2 \times 2$ system of balance laws. Stationary solutions are studied analytically and have physically reasonable properties. They are also proved to be linearly strongly stable with respect to small perturbations. Numerical results illustrate the analytical properties.

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1. Introduction

We present a mathematical model aimed at the description of the cutting of steel through fusion by means of a high-energy laser in continuous wave operation mode. Similarly to [1,2], this 1D model describes the evolution of laser cutting along the direction of the laser movement. In industrial applications it is typically observed that the cutting induces small fluctuations in the thickness of the material along the sides of the cut [1,3–6]. The mechanism of striation is not yet completely understood: most experimental results point to the cutting speed as the main parameter affecting the formation of ripples. However, in pulsed laser cutting, which is not our main goal in this work, laser pulse frequency is also a relevant factor, see [7].

It is commonly believed that the profile of the steel being melted in front of the cut reflects the formation of ripples along the sides of the cut, see for instance [1,2,4]. A typical problem in laser cutting applications is to minimize these ripples by suitably tuning the various process parameters, like the cutting speed or the laser power, for instance. However, in order to investigate the effects of the many process parameters on the cutting dynamics, suitable models have to be developed. Recent research considered the treatment of the arising melt and its moving free boundaries, see for instance [1,2,5,8,9] and the references therein. Three–dimensional models including temperature effects as well as a very detailed description of the cutting gas are discussed in [10–12]. The role of the gas, in particular, is the subject of [13]. In [10], it is stated that the current computer hardware is insufficient to fully simulate this intricate process at a reasonable cost.

Therefore, we present below a simple model for the evolution of the melting front based on only two partial differential equations in one space dimension. To our knowledge, these balance laws do not fit in the present theoretical framework [14–21] on this subject. Nevertheless, we provide rigorous analytical information on stationary solutions and on the evolution of their first order variations. We also state below their relation with existing models in particular with the ones described in [2,22].
The literature on the various approaches to the laser cutting of steel is vast and the field of research very active, so that a full literature review is not provided. We refer to the classical work [6] and to the references therein for more details on laser cutting techniques. Also the more specific case of the phenomena related to fusion laser cutting has been widely considered, see [23]. From a more experimental point of view, we recall the recent work [24], focused on the instabilities of the thermal and hydrodynamics. Also [23] deals specifically with the dynamics of the melted material. Here glass was chosen as the material being cut, due to its absorbing the CO2 laser radiation and to its transparency to visible radiations.

Here we are interested in a more theoretical approach. Analytical models are presented for instance in [25], where the steel thickness is given a relevant role, see also the fundamental work [26].

The formation of ripples, a key issue from the industrial point of view, is at the heart of a fast developing research area. Theoretical approaches to striation are for example in [27] and in [28]. In the former work, striation is seen as due to a cyclic reaction of iron–oxygen oxidation, inducing a cyclic variation of the leading edge of the melting metal. The latter emphasizes the role of the strong nonlinearities that allow the description of a rather complex dynamics already within a model with a small number of degrees of freedom.

Other works entered the detail of specific physical phenomena, such as the laser beam polarization and its connections with the steel thickness in [29]. We refer to [30] for a relevant study of the role of heat conduction, leading to accurate estimates on the power absorbed by the steel at different temperatures. A steady state approach is followed in [31]. There, the laser beam intensity distribution, the hydrodynamics of the melted material and the geometry of the cutting front all enter in a model leading to interesting estimates on the fluid velocity, surface temperature and molten layer thickness. To the best of our knowledge the full mechanism of striation formation is still not completely understood and many parameters like for example cutting speed and laser pulse frequency influence this formation, see [7]. We want to contribute to this discussion by presenting a simple mathematical model to describe some of the observed properties.

The paper is organized as follows. Section 2 is devoted to the physical motivations and to the justification of the model. The available rigorous analytical information about this model are derived in Section 3. Numerical integrations of the model introduced below are displayed in Section 4, where paragraph 4.1 deals with the problem of estimating the relevant parameters appearing in the equations.

2. The model

2.1. Overview

We consider first the dynamics of the melted film and then the description of the remaining solid profile. Our starting point is the mass balance between the melted and non-melted material. The dynamics introduced below describes the movement of the melted material. Moreover, the source terms below model the material which is being melted. As a result, we obtain the following equations:

\[
\partial_t h + \partial_z \left( v_0 h + \frac{1}{2} \tau_r h^2 \right) = i(u + h) f(\partial_z (u + h)) \\
\partial_t u = -i(u + h) f(\partial_z (u + h)) + v_u \\
(h, u)(0, z) = (h_0, u_0)(z) \\
(h, u)(t, 0) = (0, 0),
\]

(2.1)

where \( h \) is the melted film thickness; \( u \) describes the profile of the remaining solid material; \( z \) is the space coordinate along the laser beam, see Fig. 1; \( i \) and \( f \) describe the laser intensity and its effect on the material; \( v_u \) is the speed of the laser beam relative to the material being cut. A key role is played by the parameters \( v_0 \) and \( \tau_r \). The flow of the melted material is here postulated to be the function Fig. 2

\[
h \rightarrow v_0 h + \frac{1}{2} \tau_r h^2.
\]

(2.2)

![Fig. 1. Position and orientation of the (x,y,z) axes. The model (2.13) describes the evolution of the material along the dotted line.](image)
This amounts to assume that the cutting gas is able to push the melted material at a uniformly positive speed also when its thickness is extremely small and the momentum transfer from the cutting gas to the melted material becomes negligible. Indeed, according to (2.2), the downwards speed of the melted film is \( v_0 + \tau_s h \), which does not vanish as \( h \to 0 \). Thus, the above assumption (2.2) allows to interpret \( v_0 \) and \( \tau_s \) as lumped parameters that also take into consideration the relevant role of the pressure exerted on the melted material, see [32]. In this simple setting, we have to neglect the effect of the recoil pressure exerted by the vaporized material that ejects the vaporized steel in the direction orthogonal to the cutting edge. Indeed, we recall that the latter effect may become relevant as soon as vaporization cannot be neglected, see [23,33]. On the other hand, although the model relies on conservation laws, the present choice of the moving reference frame hides the fact (which is here considered) that the molten material falls away through the lower boundary.

For the above assumption (2.2) to be reasonable, we need to require a condition stronger than merely \( v_0 > 0 \), see (2.14). If the gas speed is too low, then (2.2) is hardly acceptable and the movement of the melted material may not be described by (2.2). Note that this assumption also has relevant analytical consequences, see Section 3 below.

System (2.1) is related to various existing models as follows. In the classical paper [22], a single equation is used to model the melted film:

\[
\frac{\partial}{\partial t} h + \frac{1}{2} \tau_s h^2 - \frac{1}{3} p_o h^3 = v_p
\]  

for suitable constants \( p_o, \tau_s \) and a feeding velocity \( v_p \). In (2.3), the flow of the melted material \( h \to \frac{1}{2} \tau_s h^2 - \frac{1}{3} p_o h^3 \) is of second order in film thickness \( h \). Hence the speed of the melted material is \( \frac{1}{2} \tau_s h - \frac{1}{3} p_o h \) and vanishes as \( h \to 0 \), differently from what is prescribed in (2.2).

The two equations [2, Formulæ (33)–(34)], which read

\[
\begin{align*}
\frac{\partial}{\partial t} h + \frac{1}{2} \tau_s h^2 &= v_p + \text{Re}(s_r - s_l) \\
\frac{\partial}{\partial t} M &= v_p - 1.
\end{align*}
\]  

(2.4)

postulate a flow of the melted material again of the second order in \( h \), so that the speed vanishes as \( h \to 0 \). In (2.4), \( u \) describes the profile of the remaining material as above, \( M = \text{const} - u \), \( \text{Re} \) is the Reynolds number and \( s_r, s_l \) are suitable parameters, see [2, Formulæ (50)–(53)].

In the analytical treatment of (2.1) it is of use to introduce the total, i.e. solid and melted, material width \( U = u + h \) and differentiate both the resulting equations. The result is system (4.3), which is an equivalent system of balance laws with a non-local flow. Equations of this type recently appeared for various applications in applied mathematics literature. See for instance [16,19] where supply chains modeling and crowd dynamics are considered, or [34] which is devoted to a vehicular traffic model. However, the Eq. (4.3) obtained do not fit in these classes of non-local equations and, to our knowledge, no general well posedness result applies to them. In the present work, differently from [2], we are able to prove a strong linear stability of the stationary solutions to (2.1), in spite of the lack of general analytical results on balance laws of this type.

Concerning the key parameters in (2.2), we underline the difficulty of having their experimental values. In paragraph 4.1 we approach the problem of their estimation. Using various common parameters found in the literature we present some parameters in paragraph 2.2. In particular, analytical observations as well as numerical computations, show that the sign of \( \tau_s \) is related to the concavity of the stationary solutions to (2.1), which are reasonably related to the shape of the ripples.
2.2. Formalization

In this paragraph we rigorously present the notations for the model. The coordinates are \( t \geq 0 \) for time and \( (x,z) \) for space, the \( x \) axis is parallel to the laser beam axis movement, while \( z \) is parallel to the laser beam. The laser hits the material at \( z = 0 \), see Fig. 1. We assume that the material has thickness \( d \). The solid material is fed towards the laser at a constant velocity \( v_0 \) in the direction \( x \). Then, the laser heats the material while the cutting gas pushes the melted material downwards. The intensity of the laser beam is described by a function \( I = I(x) \), for instance

\[
I(x) = I_0 e^{2(2t-x)} Z_{p,2t}(x),
\]

but this particular shape of \( I \) is not relevant for the following analysis. We only require that \( I \) is non-negative, uniformly bounded from above and compactly supported. Denote by

\[ h \]

with \( h = h(t,z) \), the width of the melted part in the \( x \) direction; and

\[ u \]

with \( u = u(t,z) \), the width of the solid part in the \( x \) direction.

The cutting gas blows at constant vertical speed \( v_g > 0 \), is directed downwards in the \( z \) direction and pushes the melted part down. Because of momentum transfer and of the friction with the gas, we assume that the velocity of the melted part due to the gas stream is, at first order in \( h \),

\[
u(h) = v_\theta + \frac{1}{2} r_\theta h,
\]

where the parameters \( v_\theta > 0 \) and \( r_\theta \) are related to the effects of the cutting gas on the melted material (friction and pressure) as well as to the recoil pressure exerted by the vapor. Remark that the role of \( v_\theta \) is to ensure that some momentum is transferred from the gas to the melted material also when \( h \) is very small. Setting \( v_\theta = 0 \) would imply that the cutting gas has no effect on very thin layers of melted material. The term \( \frac{1}{2} r_\theta h \) is chosen as in \([2, \text{Formula (5.4)}]\) and in \([22]\), respectively, and we refer to the references for more details.

Let \( \theta \) be the angle under which the laser hits the melted surface. Clearly, for geometrical reasons, this angle varies along the cutting front. Moreover, it was shown to depend also on the cutting speed \( v_\theta \), see \([23,26]\), which is assumed approximately constant in the present setting. Call \( \mu = \sin \theta \) The rate of melting is then proportional to the product of the laser intensity \( I \) with \( \mu \) and with the Fresnel coefficient \( A(\mu) \), see \([1, \text{Formula (3)}]\), and \([35, \text{Chapter 9, Section 9.9}]\). We assume \( \theta \) so small that we can approximate \( \mu \sim \partial_z (u + h) \) (typical values for \( \theta \) are between 0° and 14°, see \([36]\)). The coefficient \( f(\mu) = \mu A(\mu) \) is given by

\[
f(\mu) = \frac{4 \mu^2 \epsilon}{2 \mu^2 + 2 \mu \epsilon + \epsilon^2},
\]

where \( \epsilon > 0 \) is a material dependent quantity. A sketch of the absorption coefficient \( f \) is given in Fig. 3. Observe for later that \( \max_{\mu > 0} f'(\mu) = 1 \) independently from \( \epsilon \). From a strictly rigorous point of view, \( (2.7) \) holds only for \( p \)-polarized light, whereas most lasers used nowadays in industries used circular polarized light.

With the quantities introduced above, the amount of energy absorbed by the solid material per unit time and per unit (vertical) length is \( \int f \). The energy per unit area (in the \((x,z)\) plane) required to melt the metal is \( \rho (L + C_p \Delta T) \). Here, \( L \) is the latent heat of fusion, \( C_p \) is the heat capacity of the solid, \( \rho \) is the density (mass per unit area) of the solid and \( \Delta T \) is
the difference between the fusion and room temperatures. Hence, the variation \( \Delta h \) in the thickness of the melted material, during the interval \( \Delta t \), satisfies
\[
\Delta h = \frac{i \Delta t}{\rho (L + C_p \Delta T)}.
\] (2.8)

Therefore, taking into consideration also the movement of the melted material, we lead to the following balance law:
\[
\partial_t h + \partial_x [v(h)h] = i(u + h)f(\partial_x (u + h)),
\] (2.9)
where
\[
i(x) = \frac{i(x)}{\rho (L + C_p \Delta T)}.
\] (2.10)
The source term in (2.9) is the rate of melting. The non-melted metal is fed into the laser at a velocity \( v_u \) and is melted at a rate \( i(u + h)f(\partial_x (u + h)) \), therefore the balance law for the non-melted mass is given by
\[
\partial_t u = -i(u + h)f(\partial_x (u + h)) + v_u.
\] (2.11)

We suppose that the melting begins at \( x = 0, z = 0 \), which leads to the boundary conditions: \( h(t, 0) = 0, u(t, 0) = 0 \). Hence, \( \partial_t h(t, 0) = 0 \) and \( \partial_t u(t, 0) = 0 \). Using (2.11), we get
\[
f(\partial_t h(t, 0) + \partial_x u(t, 0)) = v_u/i(0).
\]
Setting \( f_{\infty} = \lim_{\mu \to +\infty} f(\mu) \), so that \( f_{\infty} = 2c \) if \( f \) is as in (2.7), we then have the following condition on the point where melting begins:
\[
i(0)f_{\infty} \geq v_u.
\] (2.12)

This compatibility condition says that the higher the velocity under which the solid metal is fed into the laser beam, the higher is the laser intensity necessary to begin melting.

Summarizing, we obtain the following initial – boundary value problem for a system of balance laws:
\[
\begin{align*}
\partial_t h + \partial_x \left( v_x h + \frac{1}{2} \tau_x h^2 \right) &= i(u + h)f(\partial_x (u + h)) \\
\partial_t u &= -i(u + h)f(\partial_x (u + h)) + v_u \\
(h, u)(0, x) &= (h_0, u_0)(x) \\
(h, u)(t, 0) &= (0, 0).
\end{align*}
\] (2.13)

We introduce an additional condition among the physical parameters, namely
\[
v_u > 4i_{\text{max}},
\] (2.14)

since \( f'_{\text{max}} = 1 \) is the maximal value of the derivative of the Fresnel absorption (2.7) and \( i_{\text{max}} = \sup_i i(x) \). Since \( v_u \) is a function of \( v_x \), condition (2.14) essentially requires that the velocity of the gas needs to be sufficiently high. This condition is necessary to exclude physically unreasonable solutions and we show below that (2.14) also implies that any stationary solution \( u = u(z) \) is monotone increasing (see Lemma 3.1). Moreover, condition (2.14) ensures the hyperbolicity of both the non-linear system (4.3) and of the linearization of (2.13) around steady state solutions. The eigenvalues of the hyperbolic system are both positive (see (4.5)), therefore no boundary conditions are needed along \( z = d \). On the other hand, we need two boundary conditions along \( z = 0 \), coherently with (2.13) and with the physical motivations that lead to it.

Remark that (2.14) also requires that the speed of the melted material does not vanish as \( h \to 0 \), which is a distinctive feature of the present model (2.1), as noted in the introduction.

3. Analysis of the model

We consider the following system, which is equivalent to (2.13), where \( U = u + h \):
\[
\begin{align*}
\partial_t h + \partial_x \left( v_x h + \frac{1}{2} \tau_x h^2 \right) &= i(U)f(\partial_x U) \\
\partial_t U + \partial_x \left( v_x h + \frac{1}{2} \tau_x h^2 \right) &= v_u \\
(h, U)(t, 0) &= (0, 0).
\end{align*}
\] (3.1)

3.1. Stationary solutions

Imposing that the solution to (3.2) has the form \( U = U(z) \) and \( h = h(z) \), we obtain the decoupled Cauchy problems
\[
\begin{align*}
U' &= f^{-1} \left( \frac{\mu}{\rho} \right) \\
U(0) &= 0 \\
h' &= \frac{\mu}{\rho C_p i(0)} \\
h(0) &= 0.
\end{align*}
\] (3.2)
Note that \( f \), given by (2.7), is invertible for \( \mu > 0 \), so that the problem on the left in (3.2) is well defined. The explicit solution to the second problem in (3.2) is

\[
\begin{align*}
    h(z) &= \frac{\nu_0}{\nu_1} \left( \sqrt{1 + \frac{2\nu_0 u_0}{\nu_1^2}} - 1 \right) \quad \text{for} \quad \tau_g \neq 0 \quad \text{and} \quad \nu_0 + \tau_g h > 0; \\
    h(z) &= \frac{\nu_0}{\nu_1} z \quad \text{for} \quad \tau_g = 0.
\end{align*}
\]

(3.3)

This stationary solution is depicted in Fig. 4. Note that \( h \) is monotone increasing. Elementary computations show that the sign of \( \tau_g \) gives the concavity or convexity of \( h \). Furthermore, we prove below that under the compatibility condition (2.14), the width of the solid material \( u(z) \) is also monotone increasing.

**Lemma 3.1.** Let (2.14) hold. Assume \( \tau_g \geq 0 \) and that \( U = U(z) \) and \( h = h(z) \) is a stationary solution to (3.1). Then, setting \( u = U - h \), we have \( u'(z) > 0 \).

**Proof.** Due to (2.14) and since \( i(U) \) is strictly positive, we have

\[
    \frac{\nu_0}{i(U)} > \frac{4\lambda_{\max} f_{\max}}{i(U)}.
\]

Since \( f \) is monotone we have for all \( \xi \in [0, +\infty] \):

\[
    \frac{\nu_0}{i(U)} > f'(\xi).
\]

Multiplying both sides by \( \nu_0 / \nu_1 \) and since \( f(0) = 0 \), by the elementary Lagrange theorem we obtain \( \nu_0 / i(U) > f'(\xi) \nu_0 / \nu_1 = f(\nu_0 / \nu_1) \) for a suitable \( \xi \in [0, \nu_0 / \nu_1] \). Hence, \( f^{-1}(\nu_0 / i(U)) - \nu_0 / \nu_1 > 0 \). Consider now the explicit formula for \( h \) and note that, since \( u = U - h \):

\[
    u' = f^{-1} \left( \frac{\nu_0}{i(U)} \right) - \frac{\nu_0}{\nu_0 + \tau_g h} \geq f^{-1} \left( \frac{\nu_0}{i(U)} \right) - \frac{\nu_0}{\nu_0} > 0
\]

completing the proof. \( \square \)

**Lemma 3.2.** Let \( \tau_g \geq 0 \) and \( (h, U) \) be a smooth solution to (3.1) with \( h(0, z) > 0 \). Then,

\[
    h(t, z) \in \left[ 0, d - \frac{\lambda_{\max} f_{\max}}{\nu_0} \right] \quad \text{for all} \quad z \in [0, d] \quad \text{and} \quad t \geq 0.
\]

**Proof.** Let \( z = z(t) \) solve \( \dot{z} = \nu_0 + \tau_g h(t, z(t)) \) with \( z(t_0) = 0 \), where \( t_0 \geq 0 \). Then,

\[
    \frac{d}{dt} h(t, z(t)) = i(U(t, z(t))) f(\partial_z U(t, z(t))).
\]

![Fig. 4. Stationary solution to (3.1), i.e., solutions to the Cauchy problem (3.2). The layout of the geometry is as in Fig. 2. The red line is the width of the solid and melted material \( U_0(z) \) and the blue line is the solid part \( u(z) \). \( z \) is the depth of the material and the material is heated at \( z = 0 \). The parameters are given by Table 1, with \( \nu_0 = 1 \text{ m/s} \) and \( \tau_g = 1 \text{ s}^{-1} \). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](image-url)
Therefore
\[ \frac{d}{dt} h(t, z(t)) \geq 0 \quad \text{and} \quad \frac{d}{dt} h(t, z(t)) \leq i_{\max} f_{\infty}. \]

This implies that if \( h \geq 0 \) at \( t = 0 \), then \( h \) will be positive for all times. But this in turn implies \( \dot{z} \geq v_0 \). Next, compute
\[
\dot{h}(t, z(t)) = \dot{h}(t_0, 0) + \int_{t_1}^{t} \frac{d}{ds} h(s, z(s)) \, ds \leq (t - t_1) i_{\max} f_{\infty}
\]

since \( h(t_0, 0) = 0 \). Now \( \dot{d} \geq \dot{z}(t) = z(t) - z(t_0) \geq v_0(t - t_0) \) and we finally obtain \( h(t, z(t)) \leq d_{\max} f_{\infty} / v_0 \). □

3.2. Linearization along stationary solutions

We study small perturbations \( (\Delta h, \Delta U)(t, z) \) of stationary solutions \( (h_0, U_0) \) of (3.1). We consider sufficiently smooth functions \( h = h_0 + \Delta h \) and \( U = U_0 + \Delta U \). Then, the perturbations satisfy
\[
\begin{aligned}
\partial_t \Delta h + (v_0 + \tau_g h_0) \partial_z \Delta h - i(U_0 f'(U_0) \partial_z \Delta U) &= -\tau_g h_0 \Delta h \\
\partial_t \Delta U + (v_0 + \tau_g h_0) \partial_z \Delta h - i(U_0 f'(U_0) \partial_z \Delta h) &= -\tau_g h_0 \Delta h \\
(\Delta h, \Delta U)(t, 0) &= (0, 0) \\
(\Delta h, \Delta U)(0, z) &= (\Delta h_0, \Delta U_0)(z)
\end{aligned}
\]

Lemma 3.3. Assume \( \tau_g \geq 0 \) and that the compatibility condition (2.14) holds and let \( (h_0, U_0) \) solve (3.2). Then, system (3.4) is hyperbolic and the characteristic speeds are both positive.

Proof. System (3.4) is a linear system with the space dependent flux
\[
A(z) = \begin{bmatrix}
  v_0 + \tau_g h_0 & -i(U_0 f'(U_0)) \\
v_0 + \tau_g h_0 & 0
\end{bmatrix}
\]

The eigenvalues are
\[
\begin{aligned}
i_1(z) &= \frac{v_0 + \tau_g h_0}{2} \left( 1 - \sqrt{1 - \frac{4i(U_0 f'(U_0))}{v_0 + \tau_g h_0}} \right) \\
i_2(z) &= \frac{v_0 + \tau_g h_0}{2} \left( 1 + \sqrt{1 - \frac{4i(U_0 f'(U_0))}{v_0 + \tau_g h_0}} \right)
\end{aligned}
\]

Due to Lemma 3.2, we have \( h_0 \geq 0 \) and \( v_0 > 4i_{\max} \). Therefore, both eigenvalues are positive. □

We are now ready to prove the following strong stability of the linearized system (3.4).

Proposition 3.4. Assume \( \tau_g \geq 0 \) and that the compatibility condition (2.14) holds. Let \( (h_0, U_0) \) solve (3.2). Choose \( \Delta h \) and \( \Delta U \) in \( C^1([0, d]; \mathbb{R}^+) \) satisfying the boundary and compatibility conditions
\[
\begin{aligned}
\Delta h(0) &= 0 \\
\Delta U(0) &= 0 \\
\Delta h'(0) &= 0 \\
\Delta U'(0) &= 0.
\end{aligned}
\]

Then, there exists a unique solution \( (\Delta h, \Delta U) \in C^1([0, d]; \mathbb{R}^+ \times \mathbb{R}^+) \) to (3.4). Moreover, there exists a time \( T > 0 \), independent from the initial datum, such that \( (\Delta h, \Delta U)(t, z) = (0, 0) \) for all \( t \geq T \) and \( z \in [0, d] \).

Proof. Existence and uniqueness follow from [15, Theorem 3.6]. Call \( c \) the minimum of the eigenvalues: \( c = \inf_{z \in [0, d]} \lambda_1(z) \). By Lemma 3.3, \( c > 0 \). Note that \( h_0 \) is bounded away from zero by (3.2). Moreover, by Lemma 3.1, \( u_0^* \geq 0 \), hence also \( U_0^* \) and \( 4if' \) are bounded away from zero.

Consider now the characteristic line \( \tilde{z}_1 = \tilde{z}_1(t) \) which satisfies \( \tilde{z}_1 = \lambda_1(\tilde{z}_1) \), \( \tilde{z}_1(0) = 0 \). Call \( T \) the time at which this characteristic exits the interval \([0, d]\), that is \( \tilde{z}_1(T) = d \) with \( T \leq d/c \). The boundary conditions in (3.4) ensure that this classical solution vanishes for \( t \geq T \), thanks to the strictly positive velocity of the characteristics. □

4. Numerical computations

This section is devoted to the numerical simulations of the presented model. In practical applications, the initial stage in the laser cutting is the drilling of the solid material. For a model and a numerical method covering this stage see for instance to [36,37] and the references therein.
4.1. Parameter estimation

The parameters in Table 1 are standard values related to stainless steel, often found in the literature. Note that, with the parameters in Table 1, the compatibility condition (2.12) is clearly satisfied.

We are left to estimate \( \nu_s \) and \( \tau_s \). In the absence of available experimental data, we seek values that are coherent with the compatibility condition (2.14) and with the estimates based on the conservation of mass. According to the latter,

\[
\nu_s h(t, d) + \frac{1}{2} \tau_s h^2(t, d) = \nu_o d \quad \text{for all } t \geq 0.
\]  

(4.1)

Moreover, an upper bound of \( h \) at \( z = d \) comes from \( h \ll U = d \sin 14^\circ \approx 1.2 \times 10^{-1} \) m. With the data in Table 1, we get \( \nu_o d \approx 2 \times 10^{-4} \) m/s, while \( 4d_{\text{max}} \approx 8 \times 10^{-1} \) m/s. Therefore, a choice compatible with all the above conditions is

\[
\nu_s \approx 1 \text{m/s} \quad \text{and } h(t, d) \approx 10^{-4} \text{m}.
\]  

(4.2)

One may now assign to both terms in the left hand side of (4.1) the same magnitude, which leads to the following estimate for \( \tau_s \):

\[
\tau_s \approx 10^4 \text{ s}^{-1}.
\]

However, in this connection we have to note that the above value of \( \tau_s \) has little impact on the stationary solutions (3.2) to (3.1). Indeed, \( \tau_s \) does not enter the equation for \( U \). Moreover, straightforward computations show that the value \( h(d) \) of the melted thickness \( h \) at \( z = d \) as given by (3.3) is a decreasing function of \( \tau_s \). With the parameters in Table 1 and (4.2), the maximum value of \( h \) at \( z = d \) is \( h(d; \tau_s = 0) = (\nu_s/\nu_o) d \). Hence, the maximum displacement caused by the term \( \tau_s h^2 \) in the flow in (2.2) is bounded above by \( 1 \times 10^{-3} \) m.

Finally, we refer to the wide specialized literature in this subject for other sets of parameters. For instance, [38, Table 1] displays values analogous to those in Table 1 but referred to aluminum.

4.2. Numerical integrations

For the numerical integration we reformulate system (3.1) in the differentiated variables \( k = \partial_t h \) and \( W = \partial_t U \). Hence, \( U(t, z) = \int_0^z W(t, \xi) d\xi \) and \( h(t, z) = \int_0^z k(t, \xi) d\xi \). We differentiate the original system (3.1) to obtain the following system of conservation laws

\[
\begin{align*}
\partial_t k + \partial_z [\nu_s k + \tau_s k \int_0^z k(t, \xi) d\xi - i(\int_0^z W(t, \xi) d\xi) f(W)] &= 0, \\
\partial_t W + \partial_z [\nu_s k + \tau_s k \int_0^z k(t, \xi) d\xi] &= 0.
\end{align*}
\]  

(4.3)

The boundary conditions for \( k \) and \( W \) are obtained as follows. Since \( h(t, 0) = U(t, 0) = 0 \) we have \( \partial_t h(t, 0) = \partial_t U(t, 0) = 0 \). Using (3.1), we obtain \( W(t, 0) = f^{-1}(\nu_s/i(0)) \). Now, from the second equation in (3.1), we obtain the other boundary condition \( k(t, 0) = \nu_s/\nu_o \), giving finally

\[
\begin{align*}
\partial_t k + \partial_z [\nu_s k + \tau_s k \int_0^z k(t, \xi) d\xi - i(\int_0^z W(t, \xi) d\xi) f(W)] &= 0, \\
\partial_t W + \partial_z [\nu_s k + \tau_s k \int_0^z k(t, \xi) d\xi] &= 0, \\
k(t, 0) &= \nu_s/\nu_o, \\
W(t, 0) &= f^{-1}(\nu_s/i(0)).
\end{align*}
\]  

(4.4)

### Table 1

Reference values for the various parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon )</td>
<td>( 2.5 \times 10^{-1} )</td>
<td></td>
<td>[2, Formula (82)]</td>
</tr>
<tr>
<td>( d )</td>
<td>( 4.5 \times 10^{-3} )</td>
<td>m</td>
<td>typical value</td>
</tr>
<tr>
<td>Power</td>
<td>( 2 \times 10^{7} )</td>
<td>W</td>
<td>[36, fig. 3.5]</td>
</tr>
<tr>
<td>( l )</td>
<td>( 1.2 \times 10^{-3} )</td>
<td>m</td>
<td>[36, Section 3.5.1]</td>
</tr>
<tr>
<td>( \nu_s )</td>
<td>( 4 \times 10^{-2} )</td>
<td>m/s</td>
<td>[36, fig. 3.5]</td>
</tr>
<tr>
<td>( i (\text{mean}) )</td>
<td>( 2 \times 10^{-3} )</td>
<td>m/s</td>
<td>Formula (2.10)</td>
</tr>
<tr>
<td>( l (\text{mean}) )</td>
<td>( 1.7 \times 10^{6} )</td>
<td>W/m</td>
<td>Power; ( l )</td>
</tr>
<tr>
<td>( \rho (3D) )</td>
<td>( 8.03 \times 10^{3} )</td>
<td>Kg/m(^3)</td>
<td>[36, Table 3.3]</td>
</tr>
<tr>
<td>( \rho (2D) )</td>
<td>( 8.03 )</td>
<td>Kg/m(^2)</td>
<td>( l \cdot \rho (3D) )</td>
</tr>
<tr>
<td>( C_p )</td>
<td>( 5 \times 10^{2} )</td>
<td>J/(kg ( ^\circ \text{C} ))</td>
<td>[36, Table 3.3]</td>
</tr>
<tr>
<td>( L )</td>
<td>( 3 \times 10^{3} )</td>
<td>J/kg</td>
<td>[36, Table 3.3]</td>
</tr>
<tr>
<td>( \Delta T )</td>
<td>( 1.43 \times 10^{3} )</td>
<td>( ^\circ \text{C} )</td>
<td>[36, Table 3.3]</td>
</tr>
</tbody>
</table>
We verify the hyperbolicity of (4.3) writing these equations in their quasilinear form
\[
\begin{align*}
\partial_t k + (v_0 + \tau g h(t, z)) \partial_z k - i(U f' \langle W \rangle \partial_z W &= -\tau g k^2 + i(U f(W)(w + k) \\
\partial_t W + (v_0 + \tau g h(t, z)) \partial_z k &= -\tau g k^2.
\end{align*}
\]

The eigenvalues of the matrix of this system are
\[
\lambda_{1,2} = \frac{v_0 + \tau g h}{2} \left( 1 \pm \sqrt{1 - \frac{4(k + w)f'}{v_0 + \tau g h}} \right).
\] (4.5)

The compatibility condition (2.14), together with $h \geq 0$, $\tau g \geq 0$ and $f' \geq 1$, implies that $v_0 + \tau g h > 4f'(k + w)$, ensuring that both eigenvalues are real and, hence, that the system is hyperbolic.

Fig. 5. Time and spatial evolution of $h'(t, z)$ (left) and difference in log-scale of $|h'(t, z) - \bar{h}'(t, z)|$ for $t > 0.04$ (right). The layout of the geometry is as in Fig. 2. The time is $t$ and $z$ is the depth of the material and the material is heated at $z = 0$. The parameters are given by Table 1.
To numerically integrate the Eq. (4.4) we apply a standard finite-volume method. As a general reference for finite-volume methods we refer to [39]. We use an equidistant grid of \( N_z = 1000 \) points on the domain \( z \in [0, d] \). We apply a first-order central method [40] with local estimation of the characteristic speeds. This method is also known as Rusanov’s method [41] and is an extension of the Lax–Friedrichs method. The time-step is chosen according to the CFL-condition. A standard first-order discretization of the boundary conditions is used. The integrals \( (h, U) \) on the primitive variables \( (k, W) \) are computed using a second-order accurate Runge–Kutta method on the spatial grid for \( z \). The physical parameters of the simulation are given by Table 1. We use the following Gaussian function to model the laser intensity.

![Graph showing the evolution over time of \( h(t, d) \) (blue) and \( h'(t, d) \) (green) for a random perturbation of steady-state initial data. The parameters for the blue curve are given by Table 1. The parameters for the green curve are the same except for the velocity of the cutting gas with a 10% noise. Bottom: Clipping of the curves in the top part at different times. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)](image-url)
\[ i(x) = 2 \times 10^{-1} \cdot \exp \left( -\frac{x^2}{2\sigma^2} \right) = 2 \times 10^{-1} \cdot \exp \left( -\frac{(x - 0.5)^2}{2} \right) \quad \text{for} \quad x \in [0, l]. \]

Recall that the width of the melted part is \( h \), the width of the solid part \( u \) and \( U = u - h \). The stationary solution (3.2) corresponding to the parameters of Table 1 is depicted in Fig. 4. We perturb pointwise the melted thickness \( h_0(z) \) of the stationary solution by an equally distributed random noise of 10\% of \( h_0(z) \). We simulate (4.4) for \( t \in [0, T] \) and record the width \( h^*(t, d) \) of the melted part at \( z = d \). This quantity might be used to characterize ripples: it is a standard assumption that the width \( h^*(t, d) \) of the melted part is related to the ripple structures. A commonly accepted reason contributing to the rise of these irregularities is the convective transport of the melted material along the sides of the cut. As recalled in the introduction, the current literature considers several other factors, such as side burning, oscillations in the thin melt film, instabilities in the molten material flow, shocks in the impinging gas flow or fluctuations in the absorbed laser power. By induction, the current literature considers several other factors, such as side burning, oscillations in the thin melt film, instabilities in the molten material flow, shocks in the impinging gas flow or fluctuations in the absorbed laser power. By Proposition 3.4, we expect that the initial random perturbation decays over time and that the value of \( h(t, d) \) tends to the unperturbed width \( h_0(d) \), cf. Fig. 5, left. Therefore, in the presented model, the initial perturbations in the melting (or solid) part do not lead to the persistence of ripples. We present the time evolution of \( h^*(t, d) \) for the same initial data and for a simulation where we modify the speed of the cutting gas \( v_0 \). Differently to the previous simulation, we assume that the velocity of the cutting gas is not constant. Possibly due to the processing conditions there might be a small noise on the cutting gas velocity, which in turn leads to a variation in the parameter \( v_0 \). In the simulation we thus used a time-dependent value \( v_0 = v_0(t) \) with a uniformly distributed random noise of 10\%. Further observation on the role of the cutting gas are in [10].

The time evolution of \( h^* \) and \( h^* \) at maximal width of the material \( z = d \) is depicted in Fig. 6. Here, the blue curve corresponds to \( h^*(t, d) \) and the green one to \( h^*(t, d) \). We observe the decay of the initial perturbations for \( h^* \) and its persistence (at lower amplitude) for \( h^* \). The amplitude of the perturbations in \( h^* \) is of order \( 10^{-6} \) m. It is expected that the oscillations decay after time

\[ \frac{d}{\delta_1} \approx 4.5 \times 10^{-3} \text{ s}, \]

in the case of unperturbed cutting gas speed. This is also observed in the numerical simulations in Fig. 6. Finally, in Fig. 5 the time and spatial evolution of \( h^* \) and the evolution of the differences \( h^* - h^* \) is shown.

In Fig. 6 we observe the two competing effects of the stabilization due to Proposition 3.4 of the perturbed initial data and of the randomized velocity. Compared with the evolution of \( h^*(t, d) \) the oscillations in \( h^*(t, d) \) persist for a longer time period provided that there is a small noise on the speed of the cutting gas. This might lead to the conjecture that ripples persist and are due to slightly randomized processing conditions, in particular of the blowing gas.

5. Conclusion

We presented a model for the dynamics of the material being melted and expelled during a laser cutting process. Analytically, we obtained a system of non-local balance laws that does not fit in the currently available mathematical frameworks. Its linearization displays strong stability properties. From the engineering point of view, this stability suggests that imperfections in the laser cuts might be due to small variations in the various parameters governing the process, in particular to the speed of the cutting gas.

Acknowledgements

This work has been supported by DFG HE5386/8-1, DAAD 54365630, EXC128 Production Technology for High Wage Countries, and by the 2009 Vigoni project Transport Processes With Global Information: Applications, Modeling, Simulation and Numerics.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.apm.2013.02.031.

References


1 For interpretation of color in Fig. 6, the reader is referred to the web version of this article.