Abstract. Invariants of topological spaces of dimension three play a major role in many areas, in particular …

Introduction by the Organisers

The workshop *Invariants of topological spaces of dimension three*, organised by Max Muster (München) and Bill E. Xample (New York) was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds …
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Workshop: Hyperbolic Techniques in Modelling, Analysis and Numerics

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A Traffic Model with Phase Transitions at a Junction

Francesca Marcellini

(joint work with Mauro Garavello)

We consider the Phase Transition traffic model in [6], based on a non-smooth $2 \times 2$ system of conservation laws,

\begin{align}
\partial_t \rho + \partial_x (\rho v(\rho, w)) &= 0 \\
\partial_t (\rho w) + \partial_x (\rho w v(\rho, w)) &= 0
\end{align}

with $v = \min \{V_{\text{max}}, w \psi(\rho)\}$,

where $\rho$ is the traffic density, $w = w(t, x)$ is the maximal speed of each driver, $\psi$ is a $C^2$ function and $V_{\text{max}}$ is a uniform bound on the speed. This is a macroscopic description displaying 2 phases, the Free phase $F$ and Congested phase $C$, described by the sets

$F = \{(\rho, w) \in [0, R] \times [\bar{w}, \hat{w}]: v(\rho, \rho w) = V_{\text{max}}\}$,

$C = \{(\rho, w) \in [0, R] \times [\bar{w}, \hat{w}]: v(\rho, \rho w) = w \psi(\rho)\}$,

where $R$ is the maximal traffic density. This model is an extension of the classical Lighthill-Whitham [12] and Richards [14] model and it falls into the class of second order traffic models introduced by Aw and Rascle in [1] and independently by Zhang in [15]. In 2002, Colombo proposed the first second order model with two different phases in [4, 5]. See also the phase transition models in [2, 10, 13].

1. The Riemann Problem at a Junction

We propose a Riemann solver at a junction for the model in (1) which conserves the number of cars and also the maximal speed $w$ of each vehicle, see [8]. Note that $w$ is a peculiar characteristic of (1), being a specific feature of every single driver.

We consider a junction with $n$ incoming arcs $I_1, ..., I_n$ and $m$ outgoing arcs $I_{n+1}, ..., I_{n+m}$, where each incoming arc is given by $I_i = [-\infty, 0]$ and each outgoing arc is $I_j = [0, +\infty]$, see [3, 7, 9, 11]. On each arc we consider the phase transition model in (1) with the change of variable $\eta = \rho w$; we get a system where the conserved variables are $\rho$ and $\eta$ and the speed is $v(\rho, \eta) = \min \{V_{\text{max}}, \frac{\eta}{\rho} \psi(\rho)\}$.

We consider the following Riemann problem

\begin{align}
\begin{cases}
\partial_t \rho + \partial_x (\rho v(\rho, \eta)) = 0 & (\rho, \eta) \in I_i \\
\partial_t \rho + \partial_x (\rho v(\rho, \eta)) = 0 & (\rho, \eta) \in I_j \\
(\rho_i, \eta_i)(0, x) = (\bar{\rho}_i, \bar{\eta}_i) \\
(\rho_j, \eta_j)(0, x) = (\bar{\rho}_j, \bar{\eta}_j),
\end{cases}
\end{align}

where $(\bar{\rho}_i, \bar{\eta}_i) \in F \cup C$ are the initial data in each incoming arc $I_i$, $i = 1, ..., n$, and $(\bar{\rho}_j, \bar{\eta}_j) \in F \cup C$ are the initial data in each outgoing arc $I_j$, $j = 1, ..., m$. 
We define the concept of Riemann solver at a generic junction.

**Definition 1.1.** A Riemann solver at a junction is a function

\[ \mathcal{R}_J^{n+m} : \prod_{i=1}^{n+m} (F \cup C) \rightarrow \prod_{i=1}^{n+m} (F \cup C) \]

\[ ((\rho_1, \eta_1), \ldots, (\rho_{n+m}, \eta_{n+m})) \rightarrow ((\rho_1^*, \eta_1^*), \ldots, (\rho_{n+m}^*, \eta_{n+m}^*)) \]

satisfying the following properties.

1. The consistency condition holds, i.e.: \( \mathcal{R}_J^{(\rho_1^*, \eta_1^*), \ldots, (\rho_{n+m}^*, \eta_{n+m}^*)} = ((\rho_1^*, \eta_1^*), \ldots, (\rho_{n+m}^*, \eta_{n+m}^*)) \).
2. For every \( i \in \{1, \ldots, n\} \), the Riemann problem in (2) with initial data \( (\rho, \eta)(0, x) = (\rho_i, \eta_i) \), with \( x < 0 \), is solved with waves with negative speed.
3. For every \( i \in \{n+1, \ldots, n+m\} \), the Riemann problem in (2) with initial data \( (\rho, \eta)(0, x) = (\rho_i, \eta_i) \), with \( x > 0 \), is solved with waves with positive speed.
4. The traffic distribution

\[ A \left[ \begin{array}{c} \rho_1^* v(\rho_1^*, \eta_1^*) \\ \vdots \\ \rho_m^* v(\rho_m^*, \eta_m^*) \end{array} \right] = \left[ \begin{array}{c} \rho_{n+1}^* v(\rho_{n+1}^*, \eta_{n+1}^*) \\ \vdots \\ \rho_{n+m}^* v(\rho_{n+m}^*, \eta_{n+m}^*) \end{array} \right] \]

holds, where \( A = (\alpha_{i,j})_{i=1,\ldots,n; j=n+1,\ldots,n+m} \), whose coefficients indicate the percentage of traffic that passes from \( I_i \) to \( I_j \), with \( \sum_{j=n+1}^{n+m} \alpha_{i,j} = 1 \).
5. The mass conservation holds, i.e.: \( \sum_{i=1}^{n} \rho_i^* v(\rho_i^*, \eta_i^*) = \sum_{i=n+1}^{n+m} \rho_i^* v(\rho_i^*, \eta_i^*) \).
6. The distribution of the maximal speed holds, i.e.: \( w_{n+1}^* = \frac{1}{\sum_{i=1}^{n+1} \alpha_{i,n+1} \gamma_i^*} \left[ \alpha_{1,n+1} \gamma_1^* w_1^* + \ldots + \alpha_{n,n+1} \gamma_n^* w_n^* \right] \),

\[ \vdots \]

\( w_{n+m}^* = \frac{1}{\sum_{i=1}^{n+m} \alpha_{i,n+m} \gamma_i^*} \left[ \alpha_{1,n+m} \gamma_1^* w_1^* + \ldots + \alpha_{n,n+m} \gamma_n^* w_n^* \right] \),

where \( w_i^* = \frac{\eta_i^*}{\rho_i^*} \) and \( \gamma_i^* = \rho_i^* v(\rho_i^*, \eta_i^*) \) for every \( i \in \{1, \ldots, n+m\} \).

For special junctions, the cases of \( 1 \times m \) and \( 2 \times 1 \) junctions, we prove that the Riemann solver is well defined. The following result holds (see [8] for the proof).

**Theorem 1.2.** Under the assumptions

- (H-1): \( R, \bar{v}, \bar{w}, V_{\text{max}} \) are positive constants, with \( \bar{w} < \bar{v} \); \( \bar{w} \) and \( \bar{v} \) are the minimum, respectively, maximum, of the maximal speeds of each vehicle;
- (H-2): \( \psi \in C^2([0, R]; [0, 1]) \) with \( \psi(0) = 1 \), \( \psi(R) = 0 \), \( \psi'(\rho) \leq 0 \) and \( \frac{d}{d\rho} (\rho \psi(\rho)) \leq 0 \), for all \( \rho \in [0, R] \);
- (H-3): \( \bar{w} > V_{\text{max}} \);
- (H-4): the waves of the first family in \( C \) have negative speed,
the Riemann solver $\mathcal{RS}_J$ for the cases of $1 \times m$ and $2 \times 1$ junctions, constructed as in [8, Section 4, Section 5], satisfies all the conditions of Definition 1.1 and produces a solution to the Riemann problem (2).

**Remark 1.3.** We note that the distribution of the maximal speed in (6) of Definition 1.1 is given by

$$w^\ast_2 = \cdots = w^\ast_{1+m} = \bar{w}_1,$$

in the case of $1 \times m$ junction and is given by

$$w_3 = \frac{\gamma_1}{\gamma_1 + \gamma_2} \bar{w}_1 + \frac{\gamma_2}{\gamma_1 + \gamma_2} \bar{w}_2,$$

where $\gamma_1 = \rho_1 v(\rho_1, \eta_1)$ and $\gamma_2 = \rho_2 v(\rho_2, \eta_2)$, in the case of $2 \times 1$ junction, see [8].

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**References**


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