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Chapter 8

ON THE MATHEMATICAL DESCRIPTION OF THE EFFECTIVE BEHAVIOUR OF ONE-DIMENSIONAL BOSE-EINSTEIN CONDENSATES WITH DEFECTS

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Abstract

Bose-Einstein condensation and the related topic of Gross-Pitaevskiĭ equation have become an important source of models and problems in mathematical physics and analysis. In particular, in the last decade, the interest in low-dimensional systems that evolve through the nonlinear Schrödinger equation has undergone an impressive growth. The reason is twofold: on the one hand, effectively one-dimensional Bose-Einstein condensates are currently realized, and the investigation on their dynamics is nowadays a well-developed field for experimentalists. On the other hand, in contrast to its higher-dimensional analogous, the one-dimensional nonlinear Schrödinger equation allows explicit solutions, that simplify remarkably the analysis. The recent literature reveals an increasing interest for the dynamics of nonlinear systems in the presence of so-called defects, namely microscopic scatterers, which model the presence of impurities. We review here some recent achievements on such systems, with particular attention to the cases of the Dirac's "delta" and "delta prime" defects. We give rigorous definitions, recall and comment on known results for the delta case, and introduce new results for the delta prime case. The latter system turns out to be richer and interesting since it produces a bifurcation with symmetry breaking in the ground state. Our purpose lies mainly on collecting and conveying results, so proofs are not included.

1. Physical Motivation

In this chapter we review and comment on a recent research line that focuses on the properties of the dynamics of the time-dependent one-dimensional Gross-Pitaevskiĭ equation with singular scatterers, often called, in this context, "defects". Such equation has become very popular in the last twenty years because of its relevance in the description of the Bose gases, and of its link with the Bose-Einstein condensates (BEC). Let us just remind that a Bose gas is simply a system of a large number of identical Bosons, possibly interacting, while a BEC is a particular regime that a Bose gas can experience: such a regime is attained at very low temperatures, for very dilute and weakly interacting gases. The transition from a non-condensed to a condensed Bose gas can be described as a phase transition. It was first theoretically foreseen by Bose and Einstein ([16, 23, 24]), and realized seventy years later by Cornell and Wieman at JILA and by Ketterle at MIT. For these experimental achievements the three of them were awarded Nobel Prize in 2001.

In the last decade the interest for the dynamics of Bose gases in low dimension has undergone an impressive growth, due to the experimental realization of the so-called *cigar-shaped* and *disc-shaped* condensates, namely, BEC confined in regions that are approximately one- or two-dimensional. From the point of view of the theory the one-dimensional case is qualitatively different from the three-dimensional one as, for instance, in the former no condensation occurs unless a sufficiently strong confinement of the gas is provided. Nonetheless, already from the first realizations of Bose-Einstein condensation, experimentalists made use of asymmetric trapping fields and obtained asymmetric condensates. Such structures can be accomplished by using suitably anisotropic magneto-optical traps. In particular, the occurrence of condensation has been observed in low dimension too.

An explanation for this disagreement of theory and experiments was provided in 1996 by Ketterle and Van Druten ([41]), who investigated the occurrence of the condensation in a gas with a finite number of particles confined in a one-dimensional trap. Obviously, in this approach the condensation does not manifest itself as a phase transition, that, strictly speaking, involves a thermodynamic limit and then an infinite number of particles. In spite of that, in [41] the signature of BEC is provided by the fraction of particles in the ground state. Ketterle and van Druten argued that even at a theoretical level the condensation in a gas of one-dimensional bosons actually occurs, and the famous negative prediction "is only an artifact of the usual thermodynamic limit which does not apply in the situation realized in atom traps where a finite number of atoms is given".

In order to perform a rigorous mathematical analysis of a Bose gas, one can use the framework of the first quantization. From the physical point of view this choice is justified by the fact that the physics of the BEC does not involve creation nor annihilation of particles, but it has to be stressed that this is not always the favorite choice in theoretical physics (see e.g. [15, 21]). However, in the first-quantized framework the state of a system of N three-dimensional bosons at time t is described by a function $\Psi_N(t)$, called the *wave function of the system at time* t, that belongs to the functional space $L^2(\mathbb{R}^{3N})$ and is invariant under permutation of variables, namely

$$\Psi_N(t, \mathbf{x}_1, \ldots, \mathbf{x}_N) \;=\; \Psi_N(t, \mathbf{x}_{\sigma_1}, \ldots, \mathbf{x}_{\sigma_N})$$

where $\mathbf{x}_i \in \mathbb{R}^3$ and σ is a permutation of the set $\{1, \ldots, N\}$.

As usual, the interpretation of the wave function is supplied by the Born's rule, that states that the probability density of finding the N particles in the positions $(\mathbf{x}_1, \dots, \mathbf{x}_N)$ at time t is given by $|\Psi_N(t, \mathbf{x}_1, \dots, \mathbf{x}_N)|^2$.

The dynamics of the system is then described by a linear N-body Schrödinger equation, namely

$$i\partial_t \Psi_N(t) \;=\; H_N \Psi_N(t)$$

where the Hamiltonian operator H_N acts on wave function as follows:

$$H_N \Psi_N(t) \;=\; -\sum_{j=1}^{N} \Delta_j \Psi_N(t) + \sum_{1 \leq i < j \leq N} v(\mathbf{x}_i - \mathbf{x}_j)\Psi_N(t) + \sum_{j=1}^{N} V(\mathbf{x}_i)\Psi_N(t). \quad (1.1)$$

Here the function v represents the two-body interaction potential between the particles of the gas (notice that we neglected three-body interactions or more), while V describes the effect of the magnetic trap, and is usually assumed to be confining to a region whose effective linear size is called L. Both $V(x_i)$ and $v(x_i - x_j)$ act by multiplication on the wave function $\Psi_N(t)$. In fact, in many models the confining potential V is replaced by a box of finite length L, with various boundary conditions (periodic, Neumann, Dirichelet, and so on). We don't report here on the technical hypotheses commonly assumed on v and V. We warn the reader that in (1.1) and in what follows we use units such that $\hbar = 2m = 1$.

To give a mathematically rigorous definition of Bose-Einstein condensation, we restrict to the important case of *condensation in the ground state*. Let Φ_N be the ground state of the operator H_N. We define the *one-particle density matrix* as the linear integral operator $\widehat{\gamma}_{N,1}$ on $L^2(\mathbb{R}^3)$ whose integral kernel reads

$$\gamma_{N,1}(\mathbf{x}, \mathbf{x}') \;:=\; \int_{\mathbb{R}^{3(N-1)}} \Phi_N(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_{N-1}) \overline{\Phi_N(\mathbf{x}', \mathbf{y}_1, \dots, \mathbf{y}_{N-1})}\, d\mathbf{y}_1 \dots d\mathbf{y}_{N-1}.$$

The general theory guarantees that $\widehat{\gamma}_{N,1}$ is a positive, trace class operator whose trace equals one.

According to the definition given by Penrose and Onsager ([48]), and following Lieb, Seiringer, Solovej and Yngvason (see [45] and references therein), we say that Bose Einstein condensation occurs if the largest eigenvalue of $\widehat{\gamma}_{N,1}$ does not vanish in the thermodynamic limit, i.e. in the limit $N \to \infty$, $L \to \infty$, N/L constant. Notice that, from one side, this definition is consistent with the idea of "macroscopic occupation of the ground state" used by Ketterle and van Druten, on the other hand it cannot avoid the thermodynamic limit which, according to [41], prevents one-dimensional gases from condensing.

The breakthrough result in the proof of the condensation for a gas of (three-dimensional) bosons is due to Lieb and Seiringer ([44]). It is to be stressed that, strictly speaking, they did not prove condensation for the ground state of the Hamiltonian (1.1), but rather for a rescaled version of this Hamiltonian. More specifically, they proved that in the limit $N \to \infty$ and choosing a rescaling for the interaction potential v such that the scattering length a scales as N^{-1}, namely

$$a = \frac{a_0}{N}, \quad a_0 > 0, \quad (1.2)$$

the one-body density matrix converges to the projection on a particular one-body wave function ϕ^{GP}. In the Dirac notation, such result can be expressed as

$$\lim \widehat{\gamma}_{N,1} \ = \ |\phi^{GP}\rangle\langle\phi^{GP}|,$$

where the symbol "lim" summarizes together the infinite-particle and the thermodynamic limit. The wavefunction ϕ^{GP} can be characterized as the unique minimizer of the Gross-Pitaevskiĭ functional

$$\mathcal{E}^{GP}(\phi) \ = \ \|\nabla\phi\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V(\mathbf{x})|\phi(\mathbf{x})|^2 \, d\mathbf{x} + 4\pi a_0 \int_{\mathbb{R}^3} |\phi(\mathbf{x})|^4 \, d\mathbf{x}, \tag{1.3}$$

where a_0 is the scattering length of the unscaled potential v, under the constraint $\|\phi\|_{L^2(\mathbb{R}^3)} = 1$. Obviously, to find explicitly the function ϕ^{GP} one has to solve the Euler-Lagrange equation for the constrained functional $\mathcal{E}^{GP}$, namely

$$-\Delta\phi(\mathbf{x}) + V(\mathbf{x})\phi(\mathbf{x}) + 8\pi a_0|\phi(\mathbf{x})|^2\phi(\mathbf{x}) \ = \ \omega\phi(\mathbf{x}). \tag{1.4}$$

and then look, among the solutions, for the one of minimal energy. Equation (1.4) is called the *stationary Gross-Pitaevskiĭ equation*.

In fact, the Gross-Pitaevskiĭ theory is more general, and covers the time-dependent case too. The picture one should have in mind is the following: once the condensate has been produced inside a trap, the trap is removed. Then the condensate starts diffusing outside the boundary of the trap. Since the state of the condensate is represented by a unique wave function, the diffusion process could be expected to evolve following the ordinary linear Schrödinger equation of quantum mechanics. In fact, this is not the case: the diffusion of the condensate is influenced by the fact that the condensate is made of many particles that share the same wave function, and interact one another. This interaction emerges in the effective evolution as a nonlinear term.

To be more specific, consider the evolution of a gas of N no longer confined bosons, described by the equation

$$i\partial_t\Psi_N(t) \ = \ -\sum_{j=1}^{N}\Delta_j\Psi_N(t) + \sum_{1\le i<j\le N} N^2 v(N(\mathbf{x}_i - \mathbf{x}_j))\Psi_N(t). \tag{1.5}$$

where $v \geq 0$. Notice that, thanks to the rescaling of the interaction potential, the condition (1.2) is fulfilled. Choose a completely symmetric initial datum

$$\Psi^{1D}_{N,0}(\mathbf{x}_1, \ldots \mathbf{x}_N) \ = \ \varphi(\mathbf{x}_1)\ldots\varphi(\mathbf{x}_N).$$

Then, the N-body reduced density matrix $\widehat{\gamma}_{N,n}$ having the function

$$\begin{aligned} &\gamma_{N,n}(\mathbf{x}_1, \ldots \mathbf{x}_n, \mathbf{x}'_1, \ldots \mathbf{x}'_n) \\ &= \int_{\mathbb{R}^{3(N-n)}} \Psi_N(\mathbf{x}_1, \ldots \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_{N-n})\overline{\Psi_N(\mathbf{x}'_1, \ldots \mathbf{x}'_n, \mathbf{y}_1, \ldots, \mathbf{y}_{N-n})}\, d\mathbf{y}_1 \ldots d\mathbf{y}_{N-n} \end{aligned}$$

as integral kernel, satisfies

$$\lim_{N\to\infty} \mathrm{Trace}|\gamma_{N,n}(t) - |\psi(t)\rangle\langle\psi(t)|^{\otimes n}| \ = \ 0, \tag{1.6}$$

where $\psi(t)$ solves the *time-dependent Gross-Pitaevskiĭ equation*

$$i\partial_t\psi(t,\mathbf{x}) \;=\; -\partial_z^2\psi(t,\mathbf{x})+\alpha|\psi(t,\mathbf{x})|^2\psi(t,\mathbf{x}). \tag{1.7}$$

The operator convergence given in (1.6) is called *trace-class convergence*. It is stronger than the ordinary operator convergence, and is the only physically meaningful convergence for density matrices, since it implies the convergence of all observable quantities.
The rigorous derivation of the three-dimensional Gross-Pitaevskiĭ equation was accomplished by Erdős, Schlein, and Yau in a series of papers ([26, 27, 28]), while the trace-class convergence was first recognized by Michelangeli ([47]).

Let us now consider the case of an elongated trap. The key result is the one given by Lieb, Seiringer, and Yngvason ([46]), which establishes that in a suitable "squeezing limit" for the trap, the longitudinal energy of the ground state of the three-dimensional gas equals the ground-state energy of the Lieb-Liniger gas (i.e., a gas of one-dimensional bosons interacting through a delta potential, see [43]) with a properly chosen coupling among the particles. Besides, the one-dimensional probability amplitude associated to the ground state of the three-dimensional gas shows the profile of the minimizer of a one-dimensional Gross-Pitaevskiĭ functional. This result has been recently extended by Seiringer and Yin to the case of excited states ([52]).

Let us describe in some more details this result. First, define the z axis in correspondence with the longitudinal side of the trap, so the x and y axes lie in the transversal directions. The position vector $\mathbf{x}$ can be correspondingly represented as $\mathbf{x} \;=\; (\mathbf{x}^\perp, z)$. Second, write the confining potential V as the sum of a longitudinal plus a transversal term, namely

$$V(\mathbf{x}) \;=\; V^\perp(\mathbf{x}^\perp)+V^\parallel(z). \tag{1.8}$$

Now, let us introduce some notation:
- $e^\perp$ is the ground state energy of the two-dimensional operator $H^\perp = -\Delta^\perp + V^\perp$;
- $E(N,L,r,a)$ denotes the ground state energy of the Hamiltonian operator

$$\begin{aligned} H_{N,L,r,a} \quad = \quad & -\sum_{j=1}^N \Delta_j + \sum_{1\le i<j\le N} a^{-2}v(a^{-1}(\mathbf{x}_i-\mathbf{x}_j)) + \\ & +\sum_{j=1}^N r^{-2}V^\perp(r^{-1}\mathbf{x}_i^\perp)+\sum_{j=1}^N L^{-2}V^\parallel(L^{-1}z_i), \end{aligned} \tag{1.9}$$

where we decomposed the coordinates of the i-th particle in the gas as $\mathbf{x}_i \;=\; (\mathbf{x}_i^\perp, z_i)$.
- $E^{1D}(N,L,g)$ is the ground state energy of the Hamiltonian

$$H_{1D} \;=\; -\sum_{i=1}^N \partial_z^2 + g\sum_{1\le i<j\le N}\delta(z_i-z_j), \tag{1.10}$$

where, denoted by b the L^2-normalized ground state of the "transversal" non-interacting Hamiltonian $-\Delta^\perp + V^\perp(\mathbf{x}^\perp)$, we defined

$$g := 8\pi a r^{-2}\int_{\mathbb{R}^2}|b(\mathbf{x}^\perp)|^4\,d\mathbf{x}^\perp.$$

We stress that in (1.10) the Dirac's distribution defines a potential that acts on wave functions by multiplication.

Then,

$$\lim \frac{E(N,L,r,a) - Nr^{-2}e^{\perp}}{E^{1D}(N,L,g)} = 1 \tag{1.11}$$

where by the symbol "$\lim$" is meant a limit procedure in which:
- the particle number N tends to infinity;
- the ratio r/L between the transversal and the longitudinal dimension tends to zero;
- the quantity a/r tends to zero;
- denoted by ρ a particular measure of the one-dimensional density of the gas (for a precise definition see [45]), the quantity $r^2 \rho \min(\rho, g)$ tends to zero.

In the same papers it was proved that there is a natural notion of "one-particle one-dimensional wave function" emerging in the limit. To show this, denote by $\Phi_{N,L,r,a}$ the ground state of the three-dimensional Hamiltonian (1.1), define the one-dimensional density amplitude

$$\psi_{N,L,r,a}(z) \;=\; N \left[\int_{\mathbb{R}^{3N-1}} |\Phi_{N,L,r,a}(\mathbf{x}^{\perp}, z, \mathbf{y}_1, \ldots, \mathbf{y}_{N-1})|^2 \, d\mathbf{x}^{\perp} d\mathbf{y}_1 \ldots d\mathbf{y}_{N-1} \right]^{\frac{1}{2}}.$$

Furthermore, let us recall that the variational problem

$$\min_{\|\phi\|_{L^2(\mathbb{R})} = N} \left[\int_{\mathbb{R}} \left[|\phi'(z)|^2 + V_L^{\|}(z) |\phi(z)|^2 + g |\phi(z)|^4 \right] dz \right]$$

has a unique minimizer, and let us call it $\varphi_{N,L,g}$. Then, in a suitable sense, one has

$$\lim \frac{\psi_{N,L,r,a}(z)}{\varphi_{N,L,g}(z)} = 1.$$

As stated before, a subsequent paper by Seiringer and Yin ([52]) considerably improves the result, extending it to excited states and therefore giving a full derivation of the Lieb-Liniger model as a shrinking limit of a three-dimensional Bose gas.

Once realized that a three-dimensional Bose gas in an elongated trap is effectively one-dimensional, one is led to the problem of describing the effective one-dimensional dynamics. The cited theorems suggest that such dynamics should look like

$$\begin{aligned} i\partial_t \Psi_N^{1D}(t, Z_N) \;&=\; -\sum_{i=1}^{N} \partial_{z_i}^2 \Psi_N^{1D}(t, Z_N) + \\ &+ g \sum_{1 \le i < j \le N} \delta(x_i - x_j) \Psi_N^{1D}(t, Z_N) + \sum_{i=1}^{N} V^{\|}(x_i) \Psi_N^{1D}(t, Z_N), \end{aligned} \tag{1.12}$$

where we denoted the string of the longitudinal coordinates of the particles by $Z_N := (z_1, \ldots, z_N)$.

Even though for the dynamical problem a rigorous proof of the dimensional reduction is still lacking, it has been proved that, in a suitable scaling limit and for some special initial data, the evolution driven by equation (1.12) can be reduced to a one-body dynamics, even though a nonlinear one.

The main theorem is a one-dimensional version of the analogous result in dimension three with singular potentials. It is due to Ammari and Breteaux ([12]), with previous similar results by A., Bardos, Golse, and Teta ([2, 4]). One ends up with the rigorous derivation of the one-dimensional time-dependent Gross-Pitaevskiĭ equation

$$i\partial_t\psi(t,z) \;=\; -\partial_z^2\psi(t,z)+\alpha|\psi(t,z)|^2\psi(t,z). \tag{1.13}$$

In what follows we focus on equation (1.13). The core of this chapter is the report on some recent results on the dynamics described by (1.13) and its generalizations to arbitrary power non linearities with additional point interactions, i.e., potentials that are concentrated in a point. Such interactions were introduced, in the linear case, in 1936 by Fermi in order to describe the scattering of nuclei by slow neutrons ([30]). In the context of the Bose-Einstein condensates they describe the interaction with an "impurity" or "defect" that is localized with respect to the characteristic length of the condensate.

The most celebrated among these interactions is the Dirac's delta potential, but it is far from being the unique possibility. Indeed, fixed a point on the line, there exists a 4-parameter family of interactions localized on that point, and any of them can be conveniently expressed through a boundary condition. Point interactions have been widely studied as associated to the linear Schrödinger equation, where they proved useful as non trivial yet exactly solvable models. Nonetheless, their use in the context of the nonlinear Schrödinger equation is just at the beginning, therefore such models offer several mathematical open problems: once the preliminary question about their well-posedness has been solved ([6]), very little is known about the dynamics (see e.g. [37, 38], see also [51] for numerical investigations), or the problem of bound states. Concerning the last one it is to be stressed that, in spite of their explicit character, point interactions provide a detailed description for complicated phenomena: for instance, the mechanism of symmetry breaking in case of strongly oscillating potentials. As explained in the next sections, this is a feature of a peculiar point interaction, called δ'. On the other hand, the popular δ interaction does not exhibit a bifurcation phenomenon, and this confirms the fact that properties of point interactions are far from being exhausted by the Dirac's delta case.

This chapter is organized as follows: in section 2 we describe and classify general point interactions in the linear context, giving domain and action of the relevant hamiltonian operators and of the related quadratic forms; in section 3 we restrict to two peculiar point interactions (called "Dirac's delta" and "delta prime"), introduce the nonlinearity, and compute the stationary states; in section 4 we discuss the problem of the stability of the stationary states: to this aim we cannot avoid some technicalities related to the variational approach to this kind of problem. Unfortunately, any general theory of stability of stationary states is based on variational techniques. Finally, in section 5 we give a very brief description of a new and very concrete approach, due to Holmer, Zworski and coworkers, to the problem of the dynamics of solitons in the presence of point interactions.

We end this introduction by collecting some notation used throughout the paper (and sometimes appeared already in the introduction).

1. The set $L^p(\mathbb{R})$ contains any complex-valued function f of a real variable such that

$\int_{\mathbb{R}} |f(x)|^p \, dx$ is finite. One defines the L^p-norm of a function f by

$$\|f\| := \left[\int_{\mathbb{R}} |f(x)|^p \, dx \right]^{\frac{1}{p}}.$$

Notice that the symbol of the norm does not carry any index when referred to $L^2(\mathbb{R})$, i.e. $p = 2$.
2. The set $H^1(\mathbb{R})$ contains any function in $L^2(\mathbb{R})$ whose derivative is in $L^2(\mathbb{R})$ too. Such functions are continuous.
3. The set $H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-)$ consists of any function that can be decomposed as follows:

$$\psi := \chi_+ \psi_+ + \chi_- \psi_-,$$

where $\chi_\pm$ is the characteristic function of $\mathbb{R}^\pm$, while the functions $\psi_\pm$ are even and belong to $H^1(\mathbb{R})$. Let us stress that such functions can bear a jump discontinuity at the origin. For such functions, with an abuse of notation, we will denote

$$\|\psi'\|^2 \; = \; \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)}.$$

4. Given a functional A, its functional derivative evaluated at the function ψ is denoted by $A'(\psi)$. Such a functional derivative is a linear operator, so its action on a function ϕ is written as $A'(\psi)\phi$.

2. Review of Point Interactions

In this section we describe all interactions in dimension one, that are localized at a single point. This is done for the sake of completeness and to give an idea of how many possibly interesting models are still to be explored. However, we warn that from the next section we will make use of two such interactions only, the δ and δ' potential.

For a rigorous mathematical treatment of nonlinear Schrödinger equation with defects, one must at first model the defect. The main feature characterizing it is the tiny length-scale, so it appears reasonable to model a defect via point interactions. Point interactions in dimension one are by now well understood and classified.

By definition, the family of hamiltonian operators describing the dynamics of a particle in dimension one under the influence of a scattering center located at the origin, is obtained as the set of self-adjoint extensions (s.a.e.) of the symmetric operator

$$\widehat{H}_0 = -\partial_x^2 \tag{2.1}$$

defined on the domain

$$D(\widehat{H}_0) = C_0^\infty(\mathbb{R} \backslash \{0\}) \tag{2.2}$$

of the infinitely regular functions with compact support out of the origin.
By the Krein's theory of s.a.e. for symmetric operators on Hilbert spaces [8] one easily proves that there is a 4-parameter family of s.a.e. of (2.1). Besides, such a family can be translated into a 4-parameter family of boundary conditions at the point 0. The explicit action and domain of the so constructed operators, following [10, 11, 9, 14] and reference therein, can be given in one of the of the two following ways:

- Given $\omega \in \mathbb{C}, a, b, c, d \in \mathbb{R}$ such that $|\omega| = 1,\ ad - bc = 1$, we define the s.a.e. H_U as follows:

$$
\begin{aligned}
U = &\ \omega \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
D_U\ := D(H_U) = &\ \left\{ \psi \in H^2(\mathbb{R}\backslash\{0\}),\ \begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = U \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix} \right\}, \\
(H_U\psi)(x) = &\ -\psi''(x), \qquad x \neq 0, \qquad \forall\, \psi \in\ D(H_U).
\end{aligned}
\tag{2.3}
$$

  We stress that the dynamics generated by any Hamiltonian H_U couples the negative real halfline with the positive one. In other words, a wave packet initially confined in the negative half line instantaneously diffuses in the positive one, and vice versa. The class of point interactions defined by these self-adjoint extension of $\widehat{H}_0$ is therefore called *coupling*.

- Given $p, q \in \mathbb{R} \cup \{\infty\}$ we define the s.a.e. $H_{p,q}$ as follows:

$$
\begin{aligned}
D_{p,q}\ := D(H_{p,q})\ = &\ \left\{ \psi \in H^2(\mathbb{R}\backslash\{0\}),\ \psi(0+) = p\psi'(0+),\ \ \psi(0-) = q\psi'(0-) \right\}, \\
(H_{p,q}\psi)(x)\ = &\ -\psi''(x), \qquad x \neq 0 \qquad \forall\, \psi \in\ D(H_{p,q}).
\end{aligned}
\tag{2.4}
$$

This second class of point interactions has domains in which the right halfline is decoupled from the left halfline.
We recall that with the notation $H^s(\Omega)$ we indicate the Sobolev spaces of the (classes of equivalence) of functions which are in $L^2(\Omega)$ and which have distributional derivatives up to the order s in $L^2(\Omega)$. In the following we will need just the cases $s = 1, 2$ and $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R} \setminus \{0\}$.

Notice that, using the description provided by the matrix U, and choosing $\omega = a = d = 1$, $b = c = 0$, one reconstructs the free-particle Hamiltonian $H_0 = -\partial_x^2$ on its standard domain $H^2(\mathbb{R})$.
Furthermore, the choice $\omega = a = d = 1$, $b = 0$, $c \neq 0$ corresponds to the well-known case of a pure Dirac's δ interaction of strength c. Explicitly

$$
\begin{aligned}
\psi(0+) = &\ \psi(0-) \\
\psi'(0+) - \psi'(0-) = &\ c\psi(0-)
\end{aligned}
\tag{2.5}
$$

On the other hand, the case $\omega = a = d = 1$, $c = 0$, $b \in \mathbb{R}$ corresponds to the case of the so-called δ' interaction of strength b, namely, to the boundary condition

$$
\begin{aligned}
\psi'(0+) = &\ \psi'(0-) \\
\psi(0+) - \psi(0-) = &\ b\psi'(0-)
\end{aligned}
\tag{2.6}
$$

Note that in the δ interaction the functions in the domain are continuous and their derivatives have a jump at the origin, while in the δ' case the functions have a jump at the origin, and their left and right derivatives coincide.

The second form (2.4) allows to include the Neumann or Dirichlet boundary conditions: in $0+$ they are realized by choosing $p = \infty$ or $p = 0$, respectively, and in $0-$ by choosing $q = \infty$ or $q = 0$, respectively.

In [6] it has been proved that the integral form of the Gross-Pitaevskiĭ equation described in the introduction,

$$\psi(t, x) \ = \ e^{-itH}\psi_0 - i\lambda \int_0^t e^{-i(t-s)H} |\psi(s)|^2 \psi(s) \, ds, \tag{2.7}$$

where H is any self-adjoint extension of the operator $\widehat{H}_0$, admits a unique solution global in time for any initial datum $\psi_0 \in L^2(\mathbb{R})$. We recall that the use of the integral form in the place of the differential one is motivated by two main reasons. In the first place the integral form is better adapted to show existence and uniqueness results by means of usual fixed point theorems. Moreover, it allows the study of the evolution of finite energy initial data (the so called weak or mild solutions), which is physically a more sensible choice than the operator domain data (strong solutions).
In the quoted paper it is furthermore proved that

- if ψ_0 belongs to the operator domain $D(H)$, then the solution $\psi(t)$ is, at any t, an element of the operator domain too.
- if ψ_0 belongs to the form domain $Q(H)$, then the solution $\psi(t)$ is, at any t, an element of the operator domain too.

Now we discuss the quadratic form associated to the point interactions described by the operators H.
We recall (see for details for example [49]) that the quadratic form B_A associated to a self-adjoint operator A is the closure (ever existing) of the quadratic form given by $b_A(\phi, \psi) = (\phi, A\psi)$, for $\phi, \psi \in D(A)$ and $(,)$ is the inner product of the underlying Hilbert space. The form domain $Q(A)$ of the closure turns out to be an extension of the operator domain $D(A)$. Due to the above definition, the form B_A assumes often the meaning of energy, and the form domain $Q(A)$ that of domain of the finite energy states.
In any case, the quoted results are particularly relevant from a physical point of view, since they allow to choose any finite energy state as initial datum.

The quadratic forms associated to the self-adjoint extensions of $\widehat{H}_0$ are defined as follows:
1. For the Hamiltonian $H_{0,0}$ the energy space is

$$Q_0 := \{\psi \in H^1(\mathbb{R}), \ \psi(0) = 0\} \tag{2.8}$$

and the form reads

$$B_0(\psi) \ = \ \|\psi'\|^2. \tag{2.9}$$

2. For the Hamiltonian $H_{0,q}$, $q \neq 0$,

$$Q_{0+} := \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-), \ \psi(0+) = 0\} \tag{2.10}$$

and

$$B_{0,q}(\psi) \ = \ \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} - |q|^{-1}|\psi(0-)|^2. \tag{2.11}$$

Analogously,

$$Q_{0-} := \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-),\ \psi(0-) = 0\} \tag{2.12}$$

and the form reads

$$B_{p,0}(\psi) \;=\; \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + |p|^{-1}|\psi(0+)|^2. \tag{2.13}$$

3. For the Hamiltonian H_U, defined in (2.3), with $b = 0$ the energy space is

$$Q_{\omega a} := \{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-),\ \psi(0+) = \omega a \psi(0-)\} \tag{2.14}$$

and the form reads

$$B_{\omega a}(\psi) \;=\; \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + ac|\psi(0-)|^2\ . \tag{2.15}$$

4. For any other s.a.e. of $\widehat{H}_0$ the energy space is given by

$$Q := H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-) \tag{2.16}$$

To describe the action of the form we have to consider two cases:
4.a. if the Hamiltonian is of the type H_U described in (2.3), with $b \neq 0$, then

$$B_U(\psi) \;:=\; \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + b^{-1}[d|\psi(0+)|^2 + a|\psi(0-)|^2 - 2\mathrm{Re}(\omega\overline{\psi(0+)}\psi(0-))] \tag{2.17}$$

4.b. if the Hamiltonian is of the type $H_{p,q}$ described in (2.4), with p, q both different from zero, then

$$B_{p,q}(\psi) \;:=\; \|\psi'\|^2_{L^2(\mathbb{R}^+)} + \|\psi'\|^2_{L^2(\mathbb{R}^-)} + p^{-1}|\psi(0+)|^2 - q^{-1}|\psi(0-)|^2 \tag{2.18}$$

All energy spaces can be endowed with the structure of Hilbert space by introducing the scalar product

$$(\psi,\phi)_X \;=\; (\psi,\phi) + \lim_{\varepsilon\to 0+}\int_{\varepsilon}^{+\infty} \overline{\psi'(x)}\phi'(x)\,dx + \lim_{\varepsilon\to 0+}\int_{-\infty}^{-\varepsilon} \overline{\psi'(x)}\phi'(x)\,dx. \tag{2.19}$$

In the following sections we will be mainly interested to the cases of δ and δ' interactions. For a detailed discussion of the mathematical properties of the δ' interaction see [10], [11]. For reference, we recall here the precise action and domain of these operators and their forms, and we name them with distinctive symbols, respectively H_α and H_β. We add moreover some further remarks on their properties.

- δ with strength α: $H_\alpha\psi = -\psi''$ $\forall x \neq 0$ and
$$D(H_\alpha) = \{\psi \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}),\ \psi'(0+) - \psi'(0-) = \alpha\psi(0)\} \tag{2.20}$$

- δ' with strength β: $H_\beta\psi = -\psi''$ $\forall x \neq 0$ and

$$\begin{aligned} D(H_\beta) \;=\; & \{\psi \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}\backslash\{0\}); \\ & \psi(0+) - \psi(0-) = \beta\psi'(0+),\ \psi'(0^+) = \psi'(0^-)\} \end{aligned} \tag{2.21}$$

Note that for $\alpha = 0$ and $\beta = 0$ we get in both cases the free 1-d laplacian. However, the limit $\beta \to 0$ is singular, in the sense that the energy space is not stable in the limit.
Concerning quadratic forms and form domains we have the following

- δ interaction with $\alpha \neq 0$: formula (2.15) applies.

$$D(B_\alpha) = H^1(\mathbb{R}), \qquad B_\alpha(\psi) \; = \; \|\psi'\|^2 + \alpha|\psi(0)|^2 \tag{2.22}$$

- δ' interaction with $\beta \neq 0$

$$D(B_\beta) = H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-), \quad B_\beta(\psi) \; := \; \|\phi'\|^2 + \beta^{-1}|\psi(0^+) - \psi(0^-)|^2 \tag{2.23}$$

In both cases $\alpha = 0$ and $\beta = 0$ the δ and δ' respectively reduce to the free laplacian form.

Besides, if ψ belongs to the operator domain of a δ'-interaction with strength β, then one has

$$B_\beta(\psi) \; := \; \|\phi'\|^2 + \beta|\psi'(0)|^2 \tag{2.24}$$

which is the reason to attribute the name of δ' to H_β, with an often questioned abuse of interpretation. Finally, we recall the main spectral properties of H_α and H_β.
Both operators have their essential spectrum which is purely absolutely continuous and precisely: $\sigma_{ess}(H_\alpha) = \sigma_{ess}(H_\beta) = \sigma_{ac}(H_\alpha) = \sigma_{ac}(H_\beta) = [0, +\infty)$.
Concerning the discrete spectrum, if nonempty it is purely point, and precisely one has the following.
If $\alpha \geq 0$, then $\sigma_p(H_\alpha) = \emptyset$; if $\alpha < 0$ there exists a unique eigenvalue, given by $\sigma_p(H_\alpha) = \{-\frac{\alpha^2}{4}\}$
If $\beta \geq 0$, then $\sigma_p(H_\beta) = \emptyset$; if $\beta < 0$ there exists a unique eigenvalue, given by $\sigma_p(H_\beta) = \{-\frac{4}{\beta^2}\}$
The corresponding normalized eigenfunctions are given, $\forall\, \alpha,\ \beta \in (-\infty, 0)$, by

$$\psi_\alpha(x) = \left(-\frac{\alpha}{2}\right)^{\frac{1}{2}} \, e^{\frac{\alpha}{2}|x|}, \qquad \psi_\beta(x) = \left(-\frac{2}{\beta}\right)^{\frac{1}{2}} \, \epsilon(x) e^{\frac{2}{\beta}|x|} \,, \quad \epsilon(x) \equiv \frac{x}{|x|} \,.$$

The singular continuous spectrum is empty: $\sigma_{sc}(H_\alpha) = \sigma_{sc}(H_\beta) = \emptyset$.

3. Stationary NLS with Point Interactions

We recall some known facts about the stationary one-dimensional unperturbed nonlinear Schrödinger equation (i.e. without point perturbation)

$$-\psi'' + \lambda|\psi|^{2\mu}\psi = -\omega\psi, \qquad \|\psi\| < \infty, \qquad \omega \in \mathbb{R} \tag{3.1}$$

The power μ of nonlinearity is positive; the nonlinearity is named *attractive* if $\lambda < 0$ and *repulsive* if $\lambda > 0$.
We are interested in real solutions; through the rescaling $\varphi = |\lambda|^{\frac{1}{2\mu}}\psi$ we get the equivalent problem

$$-\varphi'' + q|\varphi|^{2\mu}\varphi = \omega\varphi, \qquad \|\varphi\| < \infty, \qquad \omega \in \mathbb{R}, \quad q = sign(\lambda)$$

Taking the product of both members with φ' yields the equation

$$\frac{1}{2}(\varphi')^2 - \frac{q}{2(\mu+1)}\varphi^{2\mu+2} - \frac{\omega}{2}\varphi^2 = \mathcal{E}'$$

where $\mathcal{E}'$ is an unprejudiced integration constant.
Finally, the nonlinear change of wavefunction $\varphi(x) = \left[\frac{\mu+1}{2}\right]^{\frac{1}{2\mu}} \phi^{\frac{1}{\mu}}(\mu x)$ yields the standard problem for a nonlinear quartic oscillator

$$\frac{1}{2}(\phi')^2 - \frac{q}{4}\phi^4 - \frac{\omega}{2}\phi^2 \equiv \frac{1}{2}(\phi')^2 + V(\phi) = \mathcal{E}$$

The previous equation can be interpreted as the energy equation for a one-dimensional conservative system with a quartic potential $V(\phi)$. Bound states of the NLS equation correspond to solutions lying on the level curves of the energy equation and vanishing as $|x| \to \infty$, the remaining ones being not in L^2. These solutions vanishing at spatial infinity necessarily lie on the homoclinic curves of the energy equation connecting equilibrium points of the potential V with $\phi = 0$, if they exist, while heteroclinic curves correspond to solutions with different limits as $x \to \pm\infty$. Solutions in this second class, while however interesting theoretically and experimentally, are not in L^2; we do not treat them and we refer to [60] for results on their existence and stability under various hypotheses.

3.1. Repulsive Nonlinearity

The almost totality of rigorous results on NLS equation with point interaction is devoted to the case of attractive nonlinearity and the present review makes no exception. Nevertheless we give in the following lines a few remarks on the repulsive quartic NLS.
Let us note preliminarily that for a repulsive nonlinearity ($q = 1$) with $\gamma = 0$ (absence of point interaction) there are no L^2 solutions to the stationary NLS, apart from the trivial ones. In the case $\omega < 0, \quad \mathcal{E} = \frac{\omega^2}{4}$ there exist two families of *finite energy* solutions, (i.e. $\phi(x) \to \phi_\pm, \quad \phi'(x) \to 0, \quad ||\phi'||_{L^2} < \infty$), corresponding to the heteroclinic curve in the phase space of the energy equation. These are the so called *dark solitons* or *topological solitons* of the one-dimensional NLS equation, with the explicit expression

$$\phi_{x_0}(x) = \pm\sqrt{|\omega|}\tanh\left(\sqrt{\frac{|\omega|}{2}}(x - x_0)\right) ,$$

and correspondingly for the original equation with arbitrary power nonlinearity. We remark here that dark solitons have actually observed in BEC (see for example [25]).
In the presence of a δ interaction a boundary condition has to be satisfied at the singularity position, matching a couple of solutions as above, $\phi(x) = \phi_{x_\pm}(x)$ with $x_\pm$ corresponding to $\pm x > 0$. An easy check shows that for both attractive and repulsive δ interaction, one has that $x_\pm = 0$ and the only nonlinear stationary state is the dark soliton without modifications.
Let us consider now the case of $\omega > 0$. In this case there are no separatrices, but the (non trivial) solutions corresponding to $\mathcal{E} = 0$ are at least one-side L^2. Their explicit form, after an easy integration of the differential equation, is given by

$$\phi_{x_0}(x) = \sqrt{2\omega}\ \mathrm{cosech}(\sqrt{\omega}(x - x_0)) .$$

The interesting fact is that in the presence of attractive δ interaction they could be matched at the origin to form a nonlinear bound state $\phi(x) = \phi_{x_\pm}(x)$, where x_+ has to be used for $x > 0$, x_- has to be used for $x < 0$, and $x_- = -x_+ > 0$.
The correct matching turns out to give

$$\phi(x) = \sqrt{2\omega}\, \text{cosech}(\sqrt{\omega}(|x| + |x_+|)) \ ,$$

where, due to the boundary condition,

$$\tanh(\sqrt{\omega}|x_+|) = -2\frac{\sqrt{\omega}}{\alpha}$$

This relation shows that, as expected, a solution could exist only for attractive ($\alpha < 0$) delta interactions.
The last condition to impose is normalization of the stationary state, which is given by

$$1 = 4\omega \int_0^\infty \text{cosech}^2(\sqrt{\omega}(|x| + |x_+|))dx = 4\sqrt{\omega}(\coth(\sqrt{\omega}|x_+|) - 1) \ .$$

The two constraints on $\alpha, \omega, |x_+|$ and the condition $\omega > 0$ yield the relations

$$|x_+| = \frac{1}{\sqrt{\omega}} \tanh(1 + \frac{1}{4\sqrt{\omega}}), \qquad \alpha = -\frac{1 + 4\sqrt{\omega}}{2}, \qquad \alpha < -\frac{1}{2}$$

In particular, there is a threshold in the strength of the attractive δ interaction which allows the existence of stationary states in the repulsive NLS. Beyond the threshold, a stationary state exists and it is unique for every value of the frequency ω.
In the very recent paper [40], to which we refer for details, results are given on stability of these stationary states.

3.2. Attractive Nonlinearity

From now on we focus on the attractive case, i.e. $q = -1$.
It is immediate that if $\omega < 0$ then the stationary state equation (3.1) does not admit L^2 solutions, just because there are no separatrices, and the solutions do not vanish as $|x| \to \infty$.
For $\omega > 0$ the separatrix curve corresponds to the homoclinic orbit with $\mathcal{E} = 0$ and there exist solutions rapidly vanishing at infinity. They read

$$\phi(x) = \pm\sqrt{2\omega}\, \text{sech}(\sqrt{\omega}x)$$

Correspondingly, for the stationary state of the original equation with generic power μ we find the family of waveforms centered at x_0

$$\psi_{x_0}(x) = \pm|\lambda|^{-\frac{1}{2\mu}} [(\mu + 1)\omega]^{\frac{1}{2\mu}} \text{sech}^{\frac{1}{\mu}} [\mu\sqrt{\omega}(x - x_0)]$$

The properties of these solutions are generic for the stationary focusing Schrödinger equation, even in the presence of more general nonlinearities not of power type: every solution $\phi \in H^1(\mathbb{R})$ of the stationary equation (up to a translation of the origin) is: i) even or radial); ii) positive iii) decreasing for $x > 0$; iv) $\phi \in C^2(\mathbb{R})$; v) exponentially decaying for $|x| \to \infty$. For details see [13].

To construct stationary states for the point perturbation of the NLS equation, as before, the strategy consists in matching the two branches of a solution for $x < 0$ and $x > 0$ so to satisfy the boundary conditions related to the particular point interaction. This is equivalent to the request that a nonlinear stationary state is an element of the operator domain of the considered point interaction. The only case widely treated in the literature is the δ interaction H_α (see [17, 22, 32, 33, 34, 42, 59] and references therein) and more recently some results have been obtained for the attractive δ' interaction H_β (see [7]).
In any case we search for a solution of the form

$$\psi_{x_1,x_2}(x) = \begin{cases} A_1 \mathrm{sech}^{\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_1)] & (x<0) \\ A_2 \mathrm{sech}^{\frac{1}{\mu}}[\mu\sqrt{\omega}(x-x_2)] & (x>0) \end{cases} \tag{3.2}$$

where

$$A_1 = \pm\left[\frac{(\mu+1)\omega}{|\lambda|}\right]^{\frac{1}{2\mu}} \qquad \& \qquad A_1 = \pm A_2$$

3.2.1. δ interaction

Let us begin with the δ interaction. For the convenience of the reader we write here again the stationary Schrödinger equation (3.1) and the δ-like boundary conditions (2.20):

$$\begin{cases} -\psi'' + \lambda|\psi|^{2\mu}\psi + \omega\psi \;=\; 0 \\ \psi'(0+) - \psi'(0-) \;=\; \alpha\psi(0) \end{cases} \tag{3.3}$$

Notice that, with respect to (3.1), we drop the normalization condition. The reason for this choice is that, from a mathematical point of view, this condition turns out to be somewhat artificial, and hides the structure of the family of stationary states.

From the second equation in (3.3), or, equivalently, from (2.20), one has $D(H_\alpha) \subset H^1(\mathbb{R})$, and this forces $A_1 = A_2$ in (3.2) (and we can choose without loss of generality $A_1 > 0$) and $x_1 = \pm x_2$. It turns out after a simple calculation that the only function of the form (3.2) satisfying these properties is given by ($x_2 = \mathrm{artanh}(\frac{\alpha}{2\sqrt{\omega}}) = -x_1$, and $\epsilon(x)$ is sign of x)

$$\begin{aligned} \psi_{x_1,x_2}(x) &= [\frac{(\mu+1)\omega}{|\lambda|}]^{\frac{1}{2\mu}} \mathrm{sech}^{\frac{1}{\mu}}[\mu\sqrt{\omega}x - \epsilon(x)\mathrm{artanh}(\frac{\alpha}{2\sqrt{\omega}})] \\ &= [\frac{(\mu+1)\omega}{|\lambda|}]^{\frac{1}{2\mu}} \mathrm{sech}^{\frac{1}{\mu}}[\mu\sqrt{\omega}|x| - \mathrm{artanh}(\frac{\alpha}{2\sqrt{\omega}})] \equiv \psi_{\omega,\alpha}(x). \end{aligned} \tag{3.4}$$

This solution exists only if $\omega > \frac{\alpha^2}{4}$, the value corresponding to the absolute value of the energy of the linear bound state generated by a delta interaction with strength α. See [34] for a discussion of this general phenomenon of bifurcation of the nonlinear eigenfrequency from the linear one, both from the mathematical and physical points of view.
The given solution has a simple structure: it is symmetric for every α, and has a single maximum in $x = 0$ for an attractive δ and two maxima in $x = \pm\mathrm{artanh}(\frac{\alpha}{2\sqrt{\omega}})$ and a local minimum in 0 for a repulsive δ.
Moreover, the unperturbed stationary state centered at $x = 0$ is a solution of the delta perturbed equation, because of the fact that its derivative at the origin is vanishing, so the required boundary condition is satisfied in a trivial way.

No other stationary states exist for the δ perturbed NLS equation. In particular note the threshold in the energy E (or frequency ω, which in this context and these units is equivalent), or, stated other way, the fact that at a fixed energy (or frequency) a sufficiently strong δ interaction is needed to sustain a nonlinear bound state, and this both for the attractive or negative δ interaction.

3.2.2. δ' interaction

Now we come to δ' interaction (see [7]).
The domain of the δ' interaction is given by (2.21). From now on we consider the case of attractive δ' interaction, i.e. $\beta < 0$. To normalize signs, we define $\gamma = -\beta > 0$, and correspondingly

$$D(H_\gamma) = \{\psi \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}\backslash\{0\}),\ \psi(0+)-\psi(0-) = -\gamma\psi'(0+) = -\gamma\psi'(0-)\}, \tag{3.5}$$

therefore the search for stationary states translates into the problem

$$-\psi''(x) + \lambda|\psi(x)|^{2\mu}\psi(x) + \omega\psi(x) = 0, \quad x \neq 0, \qquad \psi \in D(H_\gamma). \tag{3.6}$$

In this case the structure of the family of stationary states is richer than in the δ-case. Notice that for any element of the operator domain a discontinuity at $x = 0$ is allowed, while right and left derivatives are prescribed to coincide at the origin. The jump of the function at zero is proportional to the common value of the right and left derivatives at the same point.
As a consequence, one immediately sees that the regular bright soliton centered at $x = 0$ satisfies the boundary condition, since the continuity is restored by vanishing derivative at the origin. But in general stationary states are expected to be discontinuous at $x = 0$, and indeed they are.
The novel feature of the generic solutions is that boundary conditions allow stationary states that are not everywhere of constant sign, in contrast to the case of δ interaction (and of the ground states of the ordinary regular Schrödinger operators).
More precisely, it results that there are two families of stationary states, defined as follows:
1. A family $\mathcal{F}_1$ of “non-changing sign” stationary states, namely

$$\mathcal{F}_1 = \left\{\psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-),\ \text{s.t.}\ \psi\ \text{fulfils (3.6) and}\ \epsilon\left(\frac{\psi(0+)}{\psi(0-)}\right) = 1\right\} \tag{3.7}$$

The boundary conditions that define $D(H_\gamma)$ on the elements of $\mathcal{F}_1$ give the system

$$\begin{cases} t_1^{2\mu} - t_1^{2\mu+2} = t_2^{2\mu} - t_2^{2\mu+2} \\ t_1^{-1} - t_2^{-1} = \gamma\sqrt{\omega} \end{cases} \tag{3.8}$$

where $t_{1,2} = \tanh(\mu\sqrt{\omega}x_{1,2})$.
First, the bright soliton centered at $x = 0$ belongs to this family. Second, it is easily seen, for instance by graphical methods, that there is a unique solution with $x_2 > x_1 > 0$, that provides a pair of stationary states: ψ_{x_1,x_2} and $\psi_{-x_2,-x_1}$.

2. A family $\mathcal{F}_2$ of "changing sign" stationary states, namely

$$\mathcal{F}_2 \;=\; \left\{ \psi \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-),\ \text{s.t.}\ \psi\ \text{ fulfils (3.6) and } \epsilon \left(\frac{\psi(0+)}{\psi(0-)} \right) \;=\; -1 \right\} \qquad (3.9)$$

The boundary conditions provide the system

$$\begin{cases} t_1^{2\mu} - t_1^{2\mu+2} &= \;\; t_2^{2\mu} - t_2^{2\mu+2} \\ t_1^{-1} + t_2^{-1} &= \;\; \gamma\sqrt{\omega} \end{cases} \qquad (3.10)$$

where $t_i = |\tanh(\mu\sqrt{\omega}x_i)|$.
Then, an analytical and graphical study of the system above gives the following structure for the solutions:

- $\omega \le \frac{4}{\gamma^2}$: there are no solutions. We stress that, in analogy with the case of the δ defect, the threshold corresponds to (the absolute value of) the energy of the linear bound state sustained by a δ' defect.

- $\frac{4}{\gamma^2} < \omega \le \frac{4(\mu+1)}{\gamma^2\mu}$: the system (3.8) has solutions with $t_1 = t_2$ only and the unique corresponding stationary state can be easily and explicitly computed. It is given by $\psi_{x_1,-x_1}$, with
$$x_1 = \frac{1}{2\mu\sqrt{\omega}} \log \frac{\gamma\sqrt{\omega}+2}{\gamma\sqrt{\omega}-2}.$$
Note that this solution fulfills the symmetry condition $\psi(-x) = -\psi(x)$.

- $\omega > \frac{4(\mu+1)}{\gamma^2\mu}$: besides of the previous symmetric solution, two more solutions appear, corresponding to roots of (3.10) with $t_1 \neq t_2$. These solutions exhibit no parity symmetry, because $x_1 \neq \pm x_2$.

Summarizing, the family $\mathcal{F}_1$ is composed of a single branch of stationary states of constant sign, which exists for every positive value of the frequency ω. The family $\mathcal{F}_2$ of changing sign solutions is composed of two subfamilies. A first subfamily $\mathcal{F}_2^s$ of symmetrical stationary states which exist for every value of the frequency $\omega > \frac{4}{\gamma^2}$. In correspondence to the frequency value $\omega = \frac{4(\mu+1)}{\gamma^2\mu}$ and for every $\omega > \frac{4(\mu+1)}{\gamma^2\mu}$ appears a second subfamily $\mathcal{F}_2^{ns}$ of *couples* of non symmetrical stationary states. As we will see in the following section, the ground states of the system belong to the family $\mathcal{F}_2$, while $\mathcal{F}_1$ is entirely composed of excited states.

Let us remark that in the case of a cubic nonlinearity ($\mu = 1$) a more explicit description of the family of stationary state is feasible, due to exact solubility of the above algebraic systems.

4. Stability and Instability of Stationary States

The issue of stability of stationary states of NLS is a well developed subject, with several classical and well-known results (see for example [18, 35, 36, 54, 55, 57, 58] and references therein). Here we are interested in the stability or instability of stationary states, described in the previous section, of the 1-dim focusing NLS with δ or δ' point perturbation. The subject is new, and the relevant rigorous literature is not very extended (see [5, 7, 32, 33, 34, 39, 42]). We give an account of the known results referring to the quoted papers for proofs and details. As in the case of the free NLS equation, several conserved quantities exist in the presence of point perturbations also, and they are related to the presence of symmetries in the associated dynamical system. Among these there are i) energy (or Hamiltonian)

$$\mathcal{H}(\psi) = \frac{1}{2}B(\psi) + \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2}$$

where B is the quadratic form associated to the particular point interaction and according to the usual notations in L^p spaces,

$$\|\psi\|_{2\mu+2}^{2\mu+2} = \int_{\mathbb{R}} |\psi(x)|^{2\mu+2}\, dx \; ;$$

ii) and *charge* (sometimes called *mass*), proportional to the L^2 norm of the wave function:

$$\mathcal{Q}(\psi) = \frac{1}{2}\|\psi\|^2 .$$

Let us note that for the free case there is one more conserved quantity, namely the linear momentum, associated to the translational invariance; in fact, in the perturbed case, this additional conservation law ceases to hold due to the translational symmetry breaking caused by the defect.
For the sake of completeness we recall the well known fact that in the free nonlinear case and $\mu = 1$ the NLS equation is an integrable system and there is an infinite family of constants of motion.

We are interested in the existence and stability of *ground states*, possibly not unique, of the NLS equation with defect, which in an informal way can be defined as the stationary states which minimize the energy.
Preliminarily, let us note the following fact about the free case.
The free stationary equation (3.1) is the Euler-Lagrange equation associated to the action

$$S_\omega(\psi) = \frac{1}{2}\|\psi'\|_2^2 + \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2} + \frac{\omega}{2}\|\psi\|^2 = \mathcal{H}(\psi) + \omega\mathcal{Q}(\psi) \qquad (4.1)$$

By considering ω as a Lagrange multiplier, a solution of the stationary equation (3.1) is also, formally, a stationary point of $\mathcal{H}$ at constant $\mathcal{Q}$.
It is a well-known property of standard NLS equation (not only in one dimension) that the stationary points of $\mathcal{H}$ at constant $\mathcal{Q}$ are the minimizers of the action 4.1 among the non trivial solutions of (3.1). In other words, given $\lambda < 0$, for every $\omega > 0$ there is a unique positive solution of (3.1); moreover, this solution is the unique positive minimizer of the variational problem

$$d(\omega) = \inf\{S_\omega(\psi),\ \psi \in H^1(\mathbb{R}),\ \psi \neq 0,\ S'_\omega(\psi)\psi = 0\} \qquad (4.2)$$

The constraint in the previous variational problem is equivalent (for the minimizers) to the validity of (3.1). Note that it is explicitly given by

$$S'_\omega(\psi)\psi = \|\psi'\|^2 + \lambda\|\psi\|_{2\mu+2}^{2\mu+2} + \omega\|\psi\|^2 \equiv I_\omega(\psi) = 0$$

We will define as *ground states* of the NLS equation the minimizers of $\mathcal{H}$ at constant $\mathcal{Q}$, if they exist, i.e. solutions of the variational problem

$$g(\omega) = \inf\{\mathcal{H}(\psi),\ \psi \in H^1(\mathbb{R}),\ \psi \neq 0,\ \mathcal{Q}(\psi) = \mathrm{cost}\}\ . \tag{4.3}$$

We give here the definition of neighborhood and stability used in this context. Due to the $U(1)$ symmetry exhibited by the problem, the notion of stability we are referring to is necessarily the one of *orbital stability*. In the free case the definition of stability should be different (see [18] chapter 8 for details and discussion). Correspondingly, to define neighborhoods we refer to $U(1)$-orbits instead of single elements:

Definition 4.1. *Let Q be the domain form. The set*

$$U_\eta(\phi) := \{\psi \in Q,\ \text{s.t.} \inf_{\theta\in[0,2\pi)} \|\psi - e^{i\theta}\phi\|_Q \leq \eta\}$$

is called the spherical neighborhood with radius η of the function ϕ.

According to the given definition, we define as stable any stationary solution $e^{i\omega t}\,\phi$ such that, starting suitably close to it, a solutions remains arbitrarily close to its $U(1)$-orbit in the future. More precisely,

Definition 4.2. *A stationary state ϕ is* stable *if for any $\varepsilon > 0$ there exists $\eta > 0$ such that*

$$\inf_{\theta\in[0,2\pi)} \|\psi - e^{i\theta}\phi\|_Q \leq \eta \Rightarrow \sup_{t\geq 0} \inf_{\theta\in[0,2\pi)} \|\psi(t) - e^{i\theta}\phi\|_Q \leq \varepsilon,$$

where $\psi(t)$ is the solution to the problem (2.7) *with ψ as initial datum. A stationary state which is not stable, is called* unstable.

Let us stress again that, if a stationary solution $e^{i\omega t}\phi$ is stable, then another solution remains close time by time not to $e^{i\omega t}\phi$, but to some element $e^{i\theta(t)}\phi$, if it starts close enough to ϕ. The stability in the presence of symmetry has been widely studied for several decades. In the analysis of the examples of the following sections the main technical tool is the so-called Q method, known in the physical literature from the seventies and often referred to as Vakhitov-Kolokolov method [56], and rigorously studied by Weinstein and Shatah-Strauss in the eighties ([57, 58, 53, 35, 36]). According to this method, the stability of the stationary states which are minima of the action S_ω is controlled by the convexity of a real function of one real variable (at least in the case of a one dimensional and commutative symmetry group; for more general situations see [36]). This function is so constructed: preliminarily, by variational methods or other methods a family of stationary states ψ_ω is found. The dependence in the frequency ω is continuous and differentiable. Then define

$$d : (\omega_1, \omega_2) \to \mathbb{R}\ , \qquad d(\omega) = S(\psi_\omega), \tag{4.4}$$

where (ω_1, ω_2) is a suitable open interval in $\mathbb{R}$; so $d(\omega)$ coincides with the action evaluated on the stationary state ψ_ω.
The stability criterion, roughly speaking, is the following: if $d''(\omega) > 0$ the stationary state corresponding to ω is stable, and if $d''(\omega) < 0$ it is unstable.

More precisely, for the validity of the criterion, some precise spectral conditions have to hold for the operator describing the linearization around the stationary state, i.e $S''(\psi_\omega)$. These condition are discussed, for example, in the original papers ([36] [GSS2]).
The name Q-method is due to the fact that $d'(\omega) = \mathcal{Q}(\phi_\omega)$ when $S = \mathcal{H} + \omega\mathcal{Q}$. The utility of the method mainly relies to the degree of explicit knowledge of the family of stationary states, and its dependence on ω. In the case of the free cubic NLS or δ and δ' perturbed cubic NLS this knowledge at our disposal is actually explicit, so the method can be directly applied. For a more general power nonlinearity the solution cannot be expressed in terms of elementary functions, however the method works as well. In the case of a general potential V the method is of more difficult applicability.
Let us stress the fact that other methods are known for the study of orbital stability of families of stationary states of the NLS. In particular for the stability of ground states, an alternative to the Q method is given by the variational method developed by Cazenave and Lions (see [19, 18]).

4.1. Stability of Solitary Waves in the Presence of δ Interactions

The focusing NLS with δ interaction is the main topic of the recent papers [33, 32, 42]. However, a first analysis of this equation appeared previously in [34, 39]; in particular, in the second paper an analysis based on dynamical systems methods appears for the case of two δ interactions. This case is not a trivial extension of the case with a single delta, since it displays a relevant phenomenon of symmetry breaking for the set of ground states.
Let us start by the case of a single δ interaction. Here, as one can expect, the dynamics perturbed by an attractive defect, which cooperates with the focusing nonlinearity, is remarkably different from the dynamics perturbed by a repulsive defect, with is competing with the nonlinearity. Moreover, there must be a threshold in the stability, depending on the nonlinearity power, on account of the onset of blowup for supercritical nonlinearities.
Let us begin to analyze attractive δ interactions.
In the first place one provides a variational characterizations of stationary solutions of the problem. The stationary equation for NLS with a δ interaction is given by

$$H_\alpha\psi + \lambda|\psi|^{2\mu}\psi = -\omega\psi, \qquad \|\psi\| = \text{cost}, \qquad \omega \in \mathbb{R}, \quad \psi \in D(H_\alpha)\,. \tag{4.5}$$

The following theorem is proved in [33].

Theorem 4.3. *Let $\omega > \frac{\alpha^2}{4}$ and $\alpha < 0$. Then, there exists a unique real positive solution $\psi_{\omega,\alpha}$ of equation* (4.5)*. This solution is the unique positive minimizer of the constrained variational problem*

$$d(\omega) = \inf\{S_{\omega,\alpha}(\psi),\ \psi \in H^1(\mathbb{R}),\ \psi \neq 0,\ S'_{\omega,\alpha}(\psi)\psi = 0\}$$

where

$$S_{\omega,\alpha}(\psi) = \frac{1}{2}\|\psi'\|^2 + \frac{\alpha}{2}|\psi(0)|^2 + \frac{\lambda}{2\mu+2}\|\psi\|_{2\mu+2}^{2\mu+2} + \frac{\omega}{2}\|\psi\|^2$$

Explicitly, $\psi_{\omega,\alpha}$ is given by (3.4)*, and every other complex solution of* (4.5) *is of the form $e^{i\theta}\psi_{\omega,\alpha}$ for some $\theta \in \mathbb{R}$. Finally, there not exist solutions of* (4.5) *for $\omega \leq \frac{\alpha^2}{4}$.*

Note that

$$S'_{\omega,\alpha}(\psi)\psi = \|\psi'\|^2 + \alpha|\psi(0)|^2 + \lambda\|\psi\|_{2\mu+2}^{2\mu+2} + \omega\|\psi\|^2 \equiv I_{\omega,\alpha}(\psi) \tag{4.6}$$

The natural constraint $I_{\omega,\alpha}(\psi) = 0$ defines the so-called ”Nehari manifold” associated to the action of the variational problem. It is a standard tool to control a variational problem where the action is not bounded from below or from above, as it is here the case and in analogy with the free case. According to [33, 7], to prove that the minimum problem has a solution one exploits a) the boundedness from below of the action on its Nehari manifold (i.e. the natural constraint given by the set of solution of Euler Lagrange equations); b) the fact that the Nehari manifold is away from zero; c) some inequalities due to Brézis-Lieb to show that the minimizing sequence is in fact an element of the minimization domain; d) finally, minimizers turn out to be elements of $D(H_\alpha)$ and satisfy the stationary equation in (4.5).
Concerning stability, the situation for an attractive δ interaction is completely described by the following theorem, proved in [33, 32].

Theorem 4.4. *Let $\omega > \frac{\alpha^2}{4}$ and $\alpha < 0$.*

1. *Let $0 < \mu \leq 2$; then the stationary state $\psi_{\omega,\alpha}$ is stable in $Q = H^1(\mathbb{R})$ for any $\omega \in (\frac{\alpha^2}{4}, +\infty)$.*

2. *For $\mu > 2$ there exists $\omega_1 = \omega_1(\alpha,\mu)$ such that $\psi_{\omega,\alpha}$ is stable in $Q = H^1(\mathbb{R})$ for $\omega \in (\frac{\alpha^2}{4}, \omega_1)$ and unstable for $\omega \in (\omega_1, +\infty)$.*

In [33] a equation for the stability frequency threshold $\omega_1(\alpha,\mu)$ is given. It is not known the stability character of $e^{i\omega t}\psi_{\omega,\alpha}$ at the threshold ω_1 when $\mu > 2$.
Let us note that the attractive δ interaction stabilizes the critical case $\mu = 2$, which is unstable in the absence of the defect. This result follows from the general Grillakis-Shatah-Strauss theory ([35]).

Now we come to the repulsive δ case. Here the situation is more complicated, due to the fact that a variational characterization of stationary states is missing. Indeed, the action $S_{\omega,\alpha}(\psi)$ constrained on the Nehari manifold does not attain its minimum in $Q = H^1(\mathbb{R})$. Nevertheless, it has a minimum in $H^1_{rad}(\mathbb{R}) = \{H^1(\mathbb{R})|\psi(x) = \psi(-x)\}$, the subspace of even (or ”radial”) functions ([32]). Moreover, there is stability in $H^1_{rad}(\mathbb{R})$ for $0 < \mu \leq 1$ for $\omega \in (\frac{\alpha^2}{4}, +\infty)$. and for $1 < \mu < 2$ in $(\omega_2 + \infty)$ for a certain $\omega_2 = \omega_2(\alpha,\mu)$. A more complete analysis ([42]) reveals the following situation.

Theorem 4.5. *Let $\omega > \frac{\alpha^2}{4}$ and $\alpha > 0$.*

1. *Let $0 < \mu \leq 1$; then the stationary state $\psi_{\omega,\alpha}$ is unstable in $Q = H^1(\mathbb{R})$ for any $\omega \in (\frac{\alpha^2}{4}, +\infty)$.*

2. *For $1 < \mu < 2$ there exists $\omega_2 = \omega_2(\alpha,\mu)$ such that $\psi_{\omega,\alpha}$ is unstable in $Q = H^1(\mathbb{R})$ for $\omega \in (\frac{\alpha^2}{4}, \omega_2) \cup (\omega_2, +\infty)$*

3. *For $\mu \geq 2$ the stationary state $\psi_{\omega,\alpha}$ is unstable in $Q = H^1(\mathbb{R})$.*

Again, it is possible to give an equation for the stability frequency threshold $\omega_2(\alpha,\mu)$ but it is not known the stability character of $\psi_{\omega,\alpha}$ at the threshold ω_2 when $\mu > 2$. Note that in case of stability, the threshold would be an isolate (and in fact the unique) stability point in $H^1(\mathbb{R})$ of the family of the stationary states in the repulsive δ case. For the repulsive δ defect too, one makes use of the Grillakis-Shatah-Strauss theory to prove instability, and in particular a non trivial analysis of the eigenvalues of linearization $S''_\omega(\psi_{\omega,\alpha})$ of the action around the stationary state is needed. See [42] for details.

4.2. Stability in the Presence of δ' Interaction

We give here a brief description of the recent results contained in the paper [7].
We recall that in the δ' interaction the form domain is $Q = H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^-)$, and we treat only the attractive δ' interaction, so we pose $-\beta = \gamma > 0$ and $H_\gamma = H_{-\beta}$.
The stationary equation for NLS with a δ' interaction is given by

$$H_\gamma \psi + \lambda |\psi|^{2\mu}\psi = -\omega\psi, \qquad \|\psi\| = \text{cost}, \qquad \omega \in \mathbb{R}, \quad \psi \in D(H_\gamma)\,. \tag{4.7}$$

The form of stationary states has been discussed in the previous sections. As in the case of the δ interaction we have

Theorem 4.6. *Let $\omega > \frac{4}{\gamma^2}$ and $\gamma > 0$. Let us consider the constrained variational problem*

$$d(\omega) = \inf\{S_{\omega,\gamma}(\psi),\ \psi \in Q,\ \psi \neq 0,\ S'_{\omega,\gamma}(\psi)\psi = 0\} \tag{4.8}$$

where

$$S_{\omega,\gamma}(\psi) = \frac{1}{2}\|\psi'\|^2 + \frac{1}{2\gamma}|\psi(0+) - \psi(0-)|^2 + \frac{\lambda}{2\mu+2}\|\psi\|^{2\mu+2}_{2\mu+2} + \frac{\omega}{2}\|\psi\|^2.$$

Then:

if $\omega \in (\frac{4}{\gamma^2}, \frac{4(\mu+1)}{\gamma^2\mu})$ there exists a unique minimizer $\psi_{\omega,\gamma}$ of the variational problem (4.8) *which is the unique changing sign solution of the equation* (4.7)*, and the minimizer $\psi_{\omega,\gamma} \in \mathcal{F}_2^s$;*

if $\omega > \frac{4(\mu+1)}{\gamma^2\mu}$ there exist exactly two minimizers of the constrained variational problem (4.8)*, they solve* (4.7) *and belong to $\mathcal{F}_2^{ns}$.*

Finally, there are no solutions of (4.7) *for $\omega \leq \frac{4}{\gamma^2}$.*

The states of the family $\mathcal{F}_1$ do not minimize the problem (4.8), because a minimizer cannot have constant sign. To clarify this point, let us consider a non-negative function

$$\psi = \chi_+\psi_+ + \chi_-\psi_-,$$

lying in the energy domain, and define

$$\widetilde{\psi} = \chi_+\psi_+ - \chi_-\psi_-.$$

Then one easily sees

$$S_\omega(\widetilde{\psi}) \leq S_\omega(\psi).$$

So, the ground states are members of the family $\mathcal{F}_2$, while the elements of $\mathcal{F}_1$ turn out to be excited states of the system.
We concentrate on the analysis of stability of the ground states. In the case of a δ' interaction thanks to an argument which combines (see [31]) variational method and the estimation (elementary but lengthy) of the sign of the function $d''(\omega)$, it turns out that for lower frequency, precisely in the interval $\omega \in (\frac{4}{\gamma^2}, \omega^*)$ for a certain finite ω^*, the (anti)symmetrical states are stable; crossing the critical frequency ω^* the couple of newly non symmetrical states which bifurcate from $\psi_{\omega*}$ are stable. It is an open question the stability character of the symmetrical states passing the critical frequency ω^*, but there are indications of their instability. So the global emerging picture is that of a pitchfork bifurcation with an exchange of stability.
Taking in account the theorem (4.8), one concludes that

Theorem 4.7. *Let $1 \leq \mu \leq 2$. Then:*

i) any ground state is stable;

ii) the point $\omega^* := \frac{4(\mu+1)}{\gamma^2 \mu}$ is a bifurcation point;

iii) the set of ground states

$$\mathcal{G} \supseteq \left\{ \psi_\omega \in \mathcal{F}_2^s\,, \quad \omega \in \left(\frac{4}{\gamma^2}, \frac{4(\mu+1)}{\gamma^2 \mu} \right) \right\} \cup \left\{ \psi_\omega \in \mathcal{F}_2^{ns}\,, \quad \omega \in \left(\frac{4(\mu+1)}{\gamma^2 \mu}, \infty \right) \right\}$$

iv) if $\psi_{\omega*}$ is unstable, the above inclusion is an equality; if it is stable, one has

$$\mathcal{G} = \left\{ \psi_\omega \in \mathcal{F}_2^s\,, \quad \omega \in \left(\frac{4}{\gamma^2}, \frac{4(\mu+1)}{\gamma^2 \mu} \right] \right\} \cup \left\{ \psi_\omega \in \mathcal{F}_2^{ns}\,, \quad \omega \in \left(\frac{4(\mu+1)}{\gamma^2 \mu}, \infty \right) \right\}$$

Concerning the last point, the stability of the bifurcation state $\psi_{\omega*}$ is still an open problem. Summarizing, there is a spontaneous symmetry breaking which results in the birth of asymmetrical states out of the family of symmetrical ones.
We remark that no such a phenomenon shows in the case of a single δ perturbation of NLS. Nevertheless an analogous situation appears in the case of NLS with *two* attractive δ interactions (see [39] where an analysis of this model is given by means of dynamical systems methods). This is a model (although singular) of a double well potential, and a bifurcation with symmetry breaking should be considered as expected. Concerning NLS with double well potentials we should remark that the associated bifurcation phenomenon has been recently discussed for any nonlinearity power μ and any spatial dimension in the semiclassical limit [51]. With more details, it has been shown that when the nonlinearity power μ is less than a critical value $\mu_{threshold}$ then a supercritical pitchfork bifurcation occurs; on the other side, when μ is large than $\mu_{threshold}$ then a subcritical pitchfork bifurcation associated to the appearance of saddle nodes occurs. The critical value is given by

$$\mu_{threshold} = \frac{1}{2}\left[3 + \sqrt{13}\right] \tag{4.9}$$

and it is an universal critical value in the sense that it does not depend on the spatial dimension as well as the shape of the double well.

As a final remark, we note that the analogy with the double well could be not a coincidence, indeed it is perhaps reminiscent of the fact that the δ' interaction can be approximated in the resolvent norm sense by a suitable scaled set of three δ interactions (see [29]) which can be considered, as above for the double well, a singular model of a multiple well.

5. Scattering of Solitons on Point Defects

A step beyond the stationary states in the analysis of dynamics of NLS with point perturbations is given by the analysis of the behaviour of traveling solitary waves. These solitary waves are exact solutions of the free NLS, as a consequence of translation invariance of the equation, but this symmetry is in fact destroyed by the defect, as already remarked above. In spite of this there are at least two valid motivations to study the evolution of traveling waves in the presence of a defect. The first is the obvious constatation that real media are not ideal, both in the case of BEC and in the other main field of application of NLS, i.e. nonlinear optics; the second is of more theoretical nature, and is the fact that a point defect is a first, quite primitive model of the by far more difficult and not yet well understood situation of the collision of two solitons. If one of the two solitons is much slower than the other, then its effect can be hopefully well approximated by a δ interaction. In this respect, in the last years in a series of papers ([37, 38]), Holmer, Marzuola, Zworski and Datchev have pointed out the remarkable behaviour of the cubic focusing NLS with a δ interaction, both repulsive and attractive. The NLS or GP equation

$$i\partial_t\psi = -\partial^2_{xx}\psi - |\psi|^2\psi$$

admits, as it is well known, the family of solitary solutions (where all the symmetries of the Galileo group show up)

$$\psi(t,x) = \sqrt{2}\omega e^{i\gamma(t)}\mathrm{sech}(\omega(x - x_0 - vt))$$

where $\gamma(t) = \gamma + \frac{v}{2}x + (\omega^2 - \frac{v^2}{4})t$ and $v, x_0, \omega, \gamma \in \mathbb{R}$.
Now, consider the same equation with a δ interaction, i.e.

$$i\partial_t\psi = H_\alpha\psi - |\psi|^2\psi.$$

Suppose in the remote past a soliton-like wave of the type above is approaching the defect, and at a certain time collides with it. For simplicity let us consider the remote initial data

$$\psi(0,x) = \sqrt{2}e^{i\frac{v}{2}x}\mathrm{sech}((x - x_0)) \ .$$

It is possible to show ([37, 20] for the repulsive and attractive defect respectively) that asymptotically in the future and in the limit of high v, the dynamics of the scattered wave can be described as a sum of a reflected and a transmitted solitary wave (with suitable frequency-amplitude relations) and a small amount of radiation, decaying with time. Precisely, one has the following result.
Let be given $\epsilon > 0$, and let us consider time interval

$$1 + \frac{|x_0|}{v} \leq t \leq \epsilon \log v$$

then, for $v \to +\infty$ one has

$$u(t,x) = u_T(t,x) + u_R(t,x) + \mathcal{O}_{L^2_x}(v^{\zeta+\epsilon}) + \mathcal{O}_{L^\infty_x}(t^{-\frac{1}{2}})$$

where $\zeta = -1$ for $\alpha > 0$ and $\zeta = -5/6$ for $\alpha < 0$ and the transmitted and reflected components u_T and u_R are solitons themselves:

$$u_T(t,x) = e^{i\phi_T}\ e^{ixv}\ e^{i(A_T - \frac{v^2}{4})t}\ A_T \mathrm{sech}(A_T(x - x_0 - vt)) \ ,$$

$$u_R(t,x) = e^{i\phi_R}\ e^{ixv}\ e^{i(A_R - \frac{v^2}{4})t}\ A_R \mathrm{sech}(A_R(x - x_0 + vt))$$

where the ϕ_T, ϕ_R, A_T, A_R depend on the parameters v and q but not on ϵ.
As regards the transmission rate of the system, defined here as the limit

$$T^s_\alpha(\psi) = \lim_{t\to\infty} \frac{\|\psi(t)|_{x>0}\|}{\|\psi(t)\|} \ ,$$

it exists and coincides, up to small correction for high velocity, with the transmission rate of the linear delta interaction (see [10] for these scattering properties of H_α):

$$T^s_\alpha = \frac{v^2}{v^2 + \alpha^2} + \mathcal{O}(v^{-\sigma})$$

for a certain $\sigma > 0$.
These facts, expected on the basis of numerical calculations, confirm in a rigorous way the view that the soliton wavefunction colliding with the potential, after a complicated interaction, splits in two solitons, one reflected and the other transmitted; these solitons conserve their form on a certain time, which at present status of the analysis is at least of the order $\log v$. Moreover the amount of soliton mass transmitted by the δ barrier is controlled by the scattering matrix of the linear operator H_α.
The proofs of these results are not easy, and we refer to the quoted literature. We remark that they rely heavily on asymptotic properties of the solutions of the cubic NLS free equation based on the inverse scattering method, so using the integrability structure of the equation. Up to now it seems that they are not extendible as such to a general nonlinearity, also of power type. There are however some recent announced results of Perelman G. (2009, private communications) showing for the NLS with nonlinearity close to cubic, that a fast soliton interacting with a stationary high mass soliton splits into two solitons which scatter through the scattering matrix of the high mass soliton. This result is in the direction of justify the substitution of one of the colliding solitons, the ”higher” with a δ interaction, as mentioned at the beginning of the paragraph. A recent extension of the results of Zworski and coworkers has been obtained in the case of fast solitons on star graphs, with various conditions at the vertex (see [3])
We mention moreover the recent paper [1], where the authors study the dynamics of *two* solitons of a non necessarily integrable NLS equation with an external potential. The system is somewhat different from that of Zworski and coworkers, but here too the timescale at which the two solitons preserve their identity and shape after the collision turn out to be again of the order of $\log v$, where v is the relative velocity of the initial solitons.

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