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On Classical Electrodynamics of Point Particles and Mass Renormalization: Some Preliminary Results

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Abstract. Apparently, no rigorous results exist for the dynamics of a classical point particle interacting with the electromagnetic field, as described by the standard Maxwell–Lorentz equations. Some results are given here for the corresponding linearized system (dipole approximation) in the presence of a mechanical linear restoring force. We consider a regularization of the system (Pauli–Fierz model), and explicitly solve the Cauchy problem in terms of normal modes. Then we study the limit of the particle's motion as the regularization is removed. We prove that the particle's motion corresponding to smooth initial data for the field has a well-defined limit if mass is renormalized, while the motion is trivial (i.e. the particle does not move at all) if mass is not renormalized. Moreover, the limit particle's motion corresponding to an interesting class of initial data satisfies exactly the Abraham–Lorentz–Dirac equation. Finally, for generic initial data the limit motion is runaway.

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1. Introduction

Concerning the interaction of a charged point particle with the electromagnetic field in classical electrodynamics, the following quotation from E. Nelson might be appropriate: "With suitable ultraviolet and infrared cutoffs, this is a dynamical system of finitely many degrees of freedom and we have global existence and uniqueness, etc. Is it an exaggeration to say that nothing whatever is known about the behavior of this system as the cutoffs are removed, and there is not one single theorem that has been proven?" [1, p. 65]. In this Letter, we give some preliminary results on the limit of the solution when the cutoffs are removed. The main limitation is due to the fact that, instead of the complete Maxwell–Lorentz system (namely Maxwell equations for the field with a current due to the particle's motion, and the relativistic Newton equation for the particle with Lorentz force due to the field), we consider the corresponding linearized, and thus nonrelativistic, version, i.e. the so-called dipole approximation. As is well known, in the Coulomb gauge the only unknowns are the vector potential $\mathbf{A}$ for the field (with $\mathrm{div}\mathbf{A} = 0$) and the particle's position $\mathbf{q}$; the formal equations of motion are

$$\frac{1}{c^2}\ddot{\mathbf{A}} - \Delta\mathbf{A} = \frac{4\pi}{c}\mathbf{j}_{\mathrm{tr}}, \qquad m_0\ddot{\mathbf{q}} = -\frac{e}{c}\dot{\mathbf{A}}(0) - \alpha\mathbf{q}, \tag{1.1}$$

where $\mathbf{j}_{\mathrm{tr}}$ is the transversal part of the current $\mathbf{j}(\mathbf{x}) := e\dot{\mathbf{q}}\delta(\mathbf{x})$, with $\delta(\mathbf{x})$ the usual delta function, while e and m_0 are, respectively, the particle's charge and bare mass; here we also added a linear restoring force $-\alpha\mathbf{q}$ (with $\alpha > 0$). This is indeed a model very much studied both in its classical and quantum versions, which presumably allows us to also give preliminary information on the complete nonlinear system. Moreover, many physical effects ranging from Thomson scattering to Lamb shift (see, e.g., [2, 3]) are well described within such an approximation.

In the classical literature, attention is usually addressed not to the complete Maxwell–Lorentz system, but rather to the so-called Abraham–Lorentz–Dirac (ALD) equation [4–6] for the particle's motion, which is obtained from it (in several ways) through formal manipulations. For a nonrelativistic harmonic oscillator, it has the form

$$m\tau_0 \dddot{\mathbf{q}} = m\ddot{\mathbf{q}} + \alpha\mathbf{q}, \tag{1.2}$$

which involves the characteristic time $\tau_0 := 2e^2/(3mc^3)$, and the renormalized mass m, namely the sum of the bare mass m_0 and of a suitably defined electromagnetic mass m_{em}, which is infinite in the point limit (see [7]). Doubts were also raised on the validity of (1.2) (see, e.g., [8]), which indeed has never been rigorously proved. So it seems that it is worthwhile to directly study the dynamics of (1.1).

We regularize the system by taking a form factor corresponding to a rigid extended particle. Moreover, we concentrate our attention just on the particle's motion, which, due to the nontrivial coupling with the field, depends also on the initial data for the field variables, and study the limit motion as the regularization is removed. We prove that, for initial data in a suitable function space, the particle's motion admits a limit (in the $H^2([-T,T],\mathbb{R}^3)$ topology) when the form factor tends to a delta function, and mass is renormalized. We also give an explicit formula for the limit particle's motion corresponding to some particularly interesting initial data, and show that in such cases it satisfies the ALD equation.

This result is obtained by using a representation formula for the solution (field plus particle) of the Cauchy problem for the regularized Maxwell–Lorentz system. Such a formula is presented here without proof; indeed a quantum analogue of it was previously obtained in [9] and, since the system is linear, the classical case can be deduced from the quantum case. A complete proof of the representation formula based on the introduction of the normal modes can be found in [10].

The rest of this Letter is divided into three sections and an appendix. Section 2 contains the representation formula for the solution of the Cauchy problem for the regularized system. The discussion of the point limit is contained in Section 3, while Section 4 contains a qualitative (and quantitative) description of the motion corresponding to some particular initial data. The Appendix contains an explicit form of all the constants and functions (eigenfunctions and normalization constants) used in this Letter. An enlarged version also containing the details of the proofs is given in [10].

2. Solution of the Cauchy Problem for the Regularized System

We regularize system (1.1) by substituting the δ function by a smooth normalized (in L^1) charge distribution ρ. This gives the (Hamiltonian) system

$$\frac{1}{c^2}\ddot{\mathbf{A}} - \Delta\mathbf{A} = \frac{4\pi e}{c}\Pi(\dot{\mathbf{q}}\rho), \qquad m_0\ddot{\mathbf{q}} = -\int_{\mathbb{R}^3}\frac{e}{c}\rho(x)\dot{\mathbf{A}}(x)\,\mathrm{d}^3x - \alpha\mathbf{q}, \tag{2.1}$$

where Π is the projector on the subspace of vector fields with vanishing divergence ($\Pi(\mathbf{j})$ is the so-called transversal part of the current $\mathbf{j}$ denoted above by $\mathbf{j}_{\mathrm{tr}}$).

We study system (2.1) for fields with finite energy, and this naturally leads to considering the phase space

$$\mathcal{P} := (H_*^{\{0\}} \oplus \mathbb{R}^3) \oplus (H_*^{\{1\}} \oplus \mathbb{R}^3) \ni ((\dot{\mathbf{A}}_0, \dot{\mathbf{q}}_0), (\mathbf{A}_0, \mathbf{q}_0)),$$

where the Hilbert space $H_*^{\{s\}}$ is defined as the completion in the norm $\|\,|\Delta|^{s/2}f\,\|_{L^2}$ of the space of C^∞ fields with vanishing divergence (see [11]).

Concerning ρ, we will assume that (i) it is C^∞, (ii) it decays at least exponentially fast at infinity, (iii) it is spherically symmetric, and (iv) its Fourier transform $\hat{\rho}$ is everywhere nonvanishing (i.e. $\hat{\rho}(\mathbf{k}) \neq 0, \forall\mathbf{k} \in \mathbb{R}^3$). Finally, in order to simplify the discussion of the point limit, we will assume that (v) ρ has the form

$$\rho(\mathbf{x}) = \rho_a(x) := \frac{1}{a^3}\mathcal{D}\left(\frac{x}{a}\right), \tag{2.2}$$

where $\mathcal{D}$ is a positive, normalized (in $\mathbf{L}^1$) function, and a is a positive parameter representing the 'radius' of the particle. The point limit is then obtained by letting a tend to zero.

It will be shown below that, due to the spherical symmetry of the form factor, only a part of the field actually interacts with the particle, while the rest evolves according to the free wave equation. A precise formulation requires the introduction of a suitable space $\mathcal{F}_*$ of symmetric fields, as follows. Denoting by $\mathcal{F}_0$ the subset of $\mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ (Schwartz space) constituted by the fields depending only on the distance $x = |\mathbf{x}|$ of the point from the origin, namely

$$\mathcal{F}_0 := \{\mathbf{A} \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3) : \mathbf{A}(\mathbf{x}) = \mathbf{A}(x)\}, \tag{2.3}$$

we define $\mathcal{F}_*$ by

$$\mathcal{F}_* := \Pi(\mathcal{F}_0), \tag{2.4}$$

i.e. as a set of vector fields which are the divergence free part of some spherically symmetric (in the above sense) vector field. We will denote by Π_m the orthogonal projector (in the L^2 metric) of $\mathcal{S}$ onto $\mathcal{F}_*$ (essentially Π_m extracts the *monopole* part of a vector field). It is easy to see that Π_m extends to a continuous operator from $H_*^{\{s\}}$ to $H_*^{\{s\}}$.

We are now ready to give the representation formula for the solution of system (2.1). In order to simplify its reading, we point out that the solution will appear as a superposition of simple harmonic oscillations corresponding to the normal modes of the system; in the case of negative bare mass (which is needed for the point limit) the solution contains a part with real exponentials corresponding to normal modes with imaginary frequencies (runaway modes). The functions $\mathbf{A}_\omega^l$ and $\mathbf{A}_r^l$ appearing in the forthcoming theorem are just the field components of the improper and of the proper eigenvectors of the linear operator describing the Maxwell–Lorentz system. Furthermore, the function $C(\omega, m)$ plays the role of a normalization constant for the improper eigenfunction corresponding to the frequency ω, while the constant C_r is the normalization constant for the runaway eigenvector. Moreover, as pointed out above, the fields having no monopole component evolve according to the free wave equation. For this reason, the evolution operator S_t of the wave equation is present in the formula below.

THEOREM 2.1. *Consider the Cauchy problem for system (2.1) with $\alpha > 0$ and initial data $((\dot{\mathbf{A}}_0, \dot{\mathbf{q}}_0)), (\mathbf{A}_0, \mathbf{q}_0)) \in \mathcal{P}$, and let $\mathbf{u}_l, l = 1, 2, 3$ be a fixed orthonormal basis of $\mathbb{R}^3$. Then there exists a family of functions $\{\mathbf{A}_\omega^l\}_{\omega\in[0,\infty)}^{l=1,2,3}$, $\mathbf{A}_\omega^l \in L^2_{\mathrm{loc}}$, and a function $C(\omega, m), \omega \in \mathbb{R}$, such that the solution of the Cauchy problem is given by*

$$
\begin{aligned}
\mathbf{q}(t) &:= \sum_{l=1,2,3} \mathbf{u}_l \int_0^\infty C(\omega, m_0 + m_{\mathrm{em}}) \times \\
&\quad \times[-\xi_\omega^l \cos(\omega t) + \omega\eta_\omega^l \sin(\omega t)]\,\mathrm{d}\omega + \theta(-m_0)\mathbf{q}_r(t), \\
\mathbf{A}(t) &:= \sum_{l=1,2,3} \int_0^\infty C(\omega, m_0 + m_{\mathrm{em}}) \left[\eta_\omega^l \cos(\omega t) + \frac{\xi_\omega^l}{\omega}\sin(\omega t)\right] \times \\
&\quad \times \mathbf{A}_\omega^l\,\mathrm{d}\omega + \theta(-m_0)\mathbf{A}_r(t) + S_t((\mathbb{1} - \Pi_m)\mathbf{A}_0, (\mathbb{1} - \Pi_m)\dot{\mathbf{A}}_0).
\end{aligned}
\tag{2.5}
$$

Here S_t is the evolution operator of the free wave equation,

$$
m_{\mathrm{em}} := \frac{32}{3}\pi^2\frac{e^2}{c^2}\int_0^\infty |\hat{\rho}(k)|^2\,\mathrm{d}k, \tag{2.6}
$$

is the well-known electromagnetic mass, θ is the usual step function, while $\mathbf{q}_r(t)$, $\mathbf{A}_r(t)$, which define the 'runaway part' of the solution (that is present only in the case of negative bare mass), are given by

$$
\begin{aligned}
\mathbf{q}_r(t) &:= \sum_{l=1,2,3} \mathbf{u}_l C_r[-\xi_r^l \operatorname{Ch}(\nu_r t) - \nu_r\eta_r^l \operatorname{Sh}(\nu_r t)], \\
\mathbf{A}_r(t) &:= \sum_{l=1,2,3} C_r \left[\eta_r^l \operatorname{Ch}(\nu_r t) + \frac{\xi_r^l}{\nu_r}\operatorname{Sh}(\nu_r t)\right]\mathbf{A}_r^l;
\end{aligned}
\tag{2.7}
$$

$\mathbf{A}^l_\tau$ are functions of the space point $\mathbf{x}$, while C_τ and $\lambda_\tau = \nu^2_\tau$ are real constants depending on the form factor, on m_0 and on e. Moreover, the functions ξ^l_ω, η^l_ω, can be expressed in terms of the initial data by

$$\begin{aligned}\xi^l_\omega &:= \frac{1}{4\pi c^2}\langle \mathbf{A}^l_\omega, \dot{\mathbf{A}}_0\rangle - \alpha \mathbf{q}_0 \cdot \mathbf{u}_l,\\ \eta^l_\omega &:= \frac{1}{4\pi c^2}\langle \mathbf{A}^l_\omega, \mathbf{A}_0\rangle + m_0 \mathbf{u}_l \cdot \dot{\mathbf{q}}_0 + \frac{e}{c}\mathbf{u}_l \cdot \int_{\mathbb{R}^3} \rho(x)\mathbf{A}_0(x)\,\mathrm{d}^3x,\end{aligned} \tag{2.8}$$

where $\langle .,. \rangle$ denotes the L^2 scalar product, and ξ^l_τ, η^l_τ are given by the same expressions with $\mathbf{A}^l_\tau$ in place of $\mathbf{A}^l_\omega$. The explicit expressions for $\lambda_\tau, C(\omega, m), C_\tau, \mathbf{A}^l_\omega$ and $\mathbf{A}^l_\tau$ are given in the Appendix.

Strictly speaking, (2.5) and (2.8) are well defined only on a subset of $\mathcal{P}$. However, they define the action of a unitary operator, and therefore the application that associates to an initial datum its time t evoluted phase point can be extended to the whole $\mathcal{P}$.

We remark that Theorem 2.1 holds only for strictly positive α, so that the case of a 'free' particle ($\alpha = 0$) is not covered, so that the representation formula (2.5) might fail when $\alpha = 0$. Therefore it cannot be used in naive way for the study of the dynamics of a point-like 'free particle'. The limit $\alpha \to 0$ will be briefly discussed below, after the point limit $a \to 0$.

As anticipated in the introduction, the above representation formula can be deduced from that for the quantum case which was proved in [9]. Indeed, the system is linear and therefore the evolution commutes with the quantization. So we omit the proof. In order to explicitly deduce our theorem from the main result of [9], we have just to pay attention to the fact that singular integrals appearing in Theorem 2.1 are regularized by taking the principal value, while those appearing in [9] are regularized by the 'retarded prescription'. Moreover, in [9] is not pointed out that the nonsymmetric component of the fields does not interact with the particle. This can be easily proved by using the Hamiltonian formulation of system (2.1).

3. Well Posedness of the Dynamics of a Point Particle

We use now formulae (2.5), (2.7), (2.8) in order to study the point limit of the particle's motion. To begin with, we consider the special class of initial data with the particle initially at rest in some position $\mathbf{q}_0 \neq 0$, and vanishing initial field.

PROPOSITION 3.1. *Having fixed $m_0 > 0$, for each $a > 0$ denote by $\mathbf{q}^a(t)$ the solution of (2.1) corresponding to initial data $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \dot{\mathbf{q}}_0 = 0$, and $\mathbf{q}_0 \neq 0$; for all positive T the function $\mathbf{q}^a(.)$ converges in $C([-T, T], \mathbb{R}^3)$ to the constant $\mathbf{q}_0$.*

This shows rigorously what was well known heuristically, namely that a point particle with positive bare mass is unaffected by the presence of a force $(-\alpha \mathbf{q}_0)$

no matter how large, so that it behaves as if its effective mass were infinite. Thus mass renormalization is necessary.

Proof. The convergence of the solution is obtained by the energy estimate together with the Ascoli-Arzelá theorem. Then, exploiting formula (2.5), we obtain that the Fourier transform of $\mathbf{q}^a$ converges to a singular distribution concentrated at the origin. So the weak limit of such a Fourier transform is a finite linear combination of the δ function and of its derivatives. The thesis follows using again energy conservation. □

So we renormalize mass, i.e. study the limit of the particle's motion by taking the bare mass m_0 as a function of a, given precisely by

$$m_0(a) := m - m_{\mathrm{em}}(a) = m - \frac{1}{a}\left[\frac{32}{3}\pi^2\frac{e^2}{c^2}\int_0^\infty |\hat{\mathcal{D}}(k)|^2\,\mathrm{d}k\right], \tag{3.1}$$

where m is a fixed parameter to be identified with the physical particle's mass.

PROPOSITION 3.2. *Consider the Cauchy problem for system (2.1) with $m_0(a)$ given by (3.1), and initial data $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \dot{\mathbf{q}}_0 = 0$, with $\mathbf{q}_0 \neq 0$. For each $a > 0$, let $\mathbf{q}^a(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$ the function $\mathbf{q}^a(\cdot)$ converges in $H^2([-T, T], \mathbb{R}^3)$ to a nonconstant function.*

The explicit form of the limit motion will be given below (see Proposition 4.1). We point out that the convergence, being H^2, is also C^1. Moreover, the particle's acceleration converges almost everywhere.

Proof. First notice that, using the explicit form of $C(\omega, m)$ (see (A.2) below), for $a > 0$ the nonrunaway part of the motion can be written as

$$\mathbf{q}_0\int_0^\infty \frac{2}{\pi}\frac{8\pi^3\left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2\omega^2\cos(\omega t)}{\frac{1}{\gamma}\left(\frac{m_{\mathrm{tot}}}{\alpha}\omega^2 - 1\right)^2 + \gamma\left(8\pi^3\left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2\right)^2\omega^6}\,\mathrm{d}\omega, \tag{3.2}$$

where $\gamma = 2\,e^2/(3c^3)$. We want to exchange the limit $a \to 0$ with the integral, so we look for a uniform L^1 majorant for the argument of the integral. We claim that, provided a is small enough, there exist positive constants $K_1, K_2, \bar{\omega}, M$ such that the wanted majorant is given by

$$g(\omega) := \begin{cases} M, & \text{if } \omega \in [0, \bar{\omega}], \\ \max\left\{\frac{1}{(K_1\omega^2 - \alpha)\omega}, \frac{1}{K_2\omega^4}\right\}, & \text{if } \omega > \bar{\omega}. \end{cases}$$

For $\omega \leqslant \bar{\omega}$, this is obvious since m_{tot} is bounded in this set. Using the standard inequality $\gamma a^2 + b^2/\gamma \geqslant 2ab$, we obtain

$$C(m, \omega) \leqslant \frac{2}{\pi}\frac{1}{(m_{\mathrm{tot}}\omega^2 - \alpha)\omega} \leqslant \frac{2}{\pi}\frac{1}{(K_1\omega^2 - \alpha)\omega}, \tag{3.3}$$

which holds in the set where $|m_{\text{tot}}| \geqslant k_3$ for some positive k_3. However, $m_{\text{tot}}(\omega)$ has a zero which tends to infinity as $a \to 0$, so (3.3) holds on a set smaller than $(\bar{\omega}, \infty)$. We will show that, when (3.3) fails, the quantity $|\hat{\mathcal{D}}(a\omega/c)|^2$ is uniformly away from zero so that either (3.3) or the inequality

$$C_a(\omega) \leqslant \frac{2}{\pi} \frac{1}{\gamma 8\pi^3 \left|\hat{\mathcal{D}}\left(\frac{a\omega}{c}\right)\right|^2 \omega^4} \leqslant \frac{1}{K_2 \omega^4}. \tag{3.4}$$

hold. To this end, make the change of variables $(\omega, a) \mapsto (b, a)$, with $b = \omega a/c$, and study the inequality $|m_{\text{tot}}(b, a)| \leqslant k_1$. Using the explicit form (A.4) of m_{tot} this gives

$$(m - k_1)a \leqslant \frac{16}{3}\pi^2 \frac{e^2}{c^2} b \int_{-\infty}^{\infty} \frac{|\hat{\mathcal{D}}(b)|^2}{s - b}\, ds \leqslant (m + k_1)a. \tag{3.5}$$

But the integral is proportional to the Hilbert transform of $|\hat{\mathcal{D}}(b)|^2$ and, therefore, it is $O(b^{-1})$ when $b \to \infty$. It follows that the middle term in (3.5) has a finite nonvanishing limit as $b \to \infty$. So we can conclude that there exist $\bar{a}$ and $\bar{b}$ such that, for any $0 < a < \bar{a}$ and any $|b| > \bar{b}$, (3.5) is false. It follows that $\hat{\rho}_a(\omega/c) = \hat{\mathcal{D}}(b)$, where $\hat{\mathcal{D}}$ is a function vanishing only at infinity, is uniformly and strictly positive when (3.3) fails, so that either (3.3) or (3.4) are true. Therefore, (3.2) converges to

$$\mathbf{q}_0 \omega_0^2 \tau_0 \frac{2}{\pi} \int_0^{\infty} \frac{\omega^2 \cos(\omega t)}{(\omega^2 - \omega_0^2)^2 + \omega^6 \tau_0}\, d\omega,$$

where $\omega_0^2 := \alpha/m$. The convergence of the runaway part is very simple and is omitted. □

We consider now the case where the initial particle's velocity too is different from zero. This is a nontrivial generalization. Indeed we have

PROPOSITION 3.3. *Consider the Cauchy problem for system (2.1) with m_0 given by (3.1), and initial data $\mathbf{A}_0 = \dot{\mathbf{A}}_0 = \mathbf{q}_0 = 0$, with $\dot{\mathbf{q}}_0 \neq 0$. For each $a > 0$, let $\mathbf{q}^a(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$ we have*

$$|\mathbf{q}^a(t)| \to \infty, \quad \forall t \in [-T, T] \backslash \mathcal{N},$$

when $\mathcal{N}$ is a finite (possibly empty) set.

Proof. Just notice that, as $m_0 \to -\infty$, in agreement with mass renormalization, the coefficient η_ω^l defined by (2.8) diverges. □

On the other hand, the result of Proposition 3.3 is not astonishing. Indeed, it is well known that a particle moving with uniform velocity $\mathbf{v}$ carries a field (see, e.g. [12–14]), which in the dipole approximation satisfies the equation

$$\Delta \mathbf{X} = -4\pi \frac{e}{c} \Pi(\rho \mathbf{v}), \tag{3.6}$$

and vanishes at infinity. We denote such a field by $\mathbf{X}_v$. Notice that in the point limit, $\mathbf{X}_v$ has a singularity at the origin.

It is now easy to prove that the particle's motion corresponding to 'adapted' initial data, i.e., data of the form $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0}, 0)$, has a limit when $a \to 0$. A more general result concerning the case of initial fields which differ from $\mathbf{X}_{\dot{q}_0}$ only for a smooth one, is covered by the following theorem.

THEOREM 3.4. *Consider the Cauchy problem for system (2.1) with m_0 given by (3.1), and initial data*

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0} + \mathbf{A}'_0, \dot{\mathbf{A}}_0)$$

with

$$(\mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3).$$

For each $a > 0$, let $\mathbf{q}^a(t)$ be the corresponding particle's motion. Then, for any $T > 0$, as $a \to 0$ the function $\mathbf{q}^a(\cdot)$ converges in $H^2([-T, T], \mathbb{R}^3)$. Moreover, the limiting particle's motion depends continuously on

$$(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}'_0, \dot{\mathbf{A}}_0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3) \times \mathcal{S}_*(\mathbb{R}^3, \mathbb{R}^3).$$

So, even if (due to mass renormalization) the Maxwell–Lorentz system has no point limit, the above theorem shows that the dynamics of a point particle is well defined, at least for regular initial data for the field. Moreover, the Cauchy problem is well posed in the sense of Hadamard.

4. On the Solutions Corresponding to 'Adapted' Initial Data, and the Abraham–Lorentz–Dirac Equation

In the case of 'adapted' initial data, i.e. of the form $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0}, 0)$, it is possible to calculate explicitly the point–limit of the solution. Define

$$\omega_0^2 := \frac{\alpha}{m}, \qquad \tau_0 := \frac{2}{3} \frac{e^2}{mc^3},$$

consider the equation

$$\tau_0 \nu^3 - \nu^2 - \omega_0^2 = 0, \tag{4.1}$$

and denote by $\nu_r, \nu_+ = \nu_3 + i\nu_2, \nu_- = \nu_3 - i\nu_2$ its three solutions $(\nu_2, \nu_3 > 0)$. Then we have the following theorem.

THEOREM 4.1. *The point limit of the particle's motion corresponding to the solution of the Cauchy problem for the Maxwell–Lorentz system with initial data* $(\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{A}_0, \dot{\mathbf{A}}_0) = (\mathbf{q}_0, \dot{\mathbf{q}}_0, \mathbf{X}_{\dot{q}_0}, 0)$ *is given by*

$$\mathbf{q}(t) = \begin{cases} e^{-\nu_3 t}[\mathcal{A}_1^+ \cos(\nu_2 t) + \mathcal{A}_2^+ \sin(\nu_2 t)] + \mathcal{A}_3^+ e^{\nu_r t}, & \text{if } t > 0, \\ e^{\nu_3 t}[\mathcal{A}_1^- \cos(\nu_2 t) + \mathcal{A}_2^- \sin(\nu_2 t)] + \mathcal{A}_3^- e^{-\nu_r t}, & \text{if } t < 0, \end{cases} \tag{4.2}$$

where $\mathcal{A}_1^\pm, \mathcal{A}_2^\pm, \mathcal{A}_3^\pm$ *are real vector constants depending on the initial data, and on* e, m, ω_0. *Moreover, we have the following asymptotics*

$$\begin{aligned} &\mathcal{A}_1^\pm = \mathbf{q}_0 + O(\varepsilon^2), \qquad \mathcal{A}_2^\pm = \frac{\dot{\mathbf{q}}_0}{\omega_0} + O(\varepsilon^2), \qquad \mathcal{A}_3^\pm = O(\varepsilon), \\ &\nu_r = \frac{\omega_0}{\varepsilon} + O(\varepsilon), \qquad \nu_2 = \omega_0 + O(\varepsilon^2), \qquad \nu_3 = \omega_0\varepsilon/2 + O(\varepsilon^2), \end{aligned} \tag{4.3}$$

as $\varepsilon \to 0$, *in terms of the dimensionless parameter* $\varepsilon := \omega_0\tau_0$.

Sketch of the proof. First, we have to calculate the integral, and this is obtained through the method of the residues. It turns out that the integral is the sum of two terms, the first describing a damped oscillation, and the second being a real decreasing exponential. This last term exactly cancels the decreasing exponential present in the hyperbolic functions. The actual verification of this requires a long but straightforward calculation. □

Thus, according to the Maxwell–Lorentz system, in the dipole approximation the motion of a point charged harmonic oscillator is the superposition of a damped oscillation and of a runaway motion, at least when the field is of adapted type (or as one could say, the initial 'free' field vanishes). Moreover, it is possible to show that the particle's motion corresponding to generic initial data is runaway. However, it turns out that it is possible to select the initial field in such a way that the corresponding particle's motion is nonrunaway. A detailed discussion of this point is beyond the aim of this Letter, and is deferred to forthcoming work.

We come to a brief remark on the case of a free particle ($\alpha = 0$). It is easy to see that when $\alpha \to 0$, the motion (4.2) has a limit which is $\mathbf{q}(t) = \mathbf{q}_0 + \dot{\mathbf{q}}_0 t$, as it should be for a particle which is not subjected to mechanical forces. Concerning the case of more general initial data, we can study the limit $\lim_{\alpha\to 0}\lim_{a\to 0}\mathbf{q}^a(t)$. However, this is nontrivial. Indeed, our formulae are based on the spectral resolution of the linear operator describing the dipole approximation. In order to recover the zero eigenvalue which appears in the free particle case (due to translational invariance), the spectral mesure has to concentrate when $\alpha \to 0$. We expect this to cause some difficulties, so we postpone a more detailed discussion to future work.

Concerning the Abraham–Lorentz–Dirac equation, we have the following corollary.

COROLLARY 4.2. *The point limit (4.2) of the particle's motion corresponding to adapted initial field satisfies the differential identities*

$$-m\tau_0 \dddot{\mathbf{q}} = m\ddot{\mathbf{q}} + \alpha\mathbf{q}, \quad t < 0; \qquad m\tau_0 \dddot{\mathbf{q}} = m\ddot{\mathbf{q}} + \alpha\mathbf{q}, \quad t > 0. \tag{4.4}$$

Two remarks are in order. First, there are two distinct identities for positive and negative times, differing for the sign of the 'dissipative' term. But this is not surprising, due to the reversible character of the complete system. Indeed, a simple and enlightening mechanical model where this phenomenon appears was given by Lamb at the beginning of the century (see [15]). Secondly, the identities (4.4) could be considered as equations for the particle's motion only if we could dispose freely of the initial particle's acceleration, while, the dynamical point of view considered here, where the complete system is involved, the initial particle's acceleration should be deduced from the initial data of the complete system. We defer a discussion of this problem and of the related 'runaway problem' to future work.

Appendix

We give here the constants and the functions appearing in Theorem 2.1, which describe the normal modes of the system.

The constant $\lambda_r = \nu_r^2$ is the only positive solution of the equation

$$m_0 + \frac{32}{3}\pi^2 \int_0^\infty e^2|\hat{\rho}(k)|^2 \frac{k^2}{c^2k^2 + \lambda_r}\,\mathrm{d}k = -\frac{\alpha}{\lambda_r}. \tag{A.1}$$

Then we have

$$\begin{aligned} C(\omega, m_0 + m_{\mathrm{em}}) := {} & \frac{2}{\pi}\frac{1}{m_{\mathrm{tot}}(\omega)}\tau 8\pi^3 \left|\hat{\rho}\left(\frac{\omega}{c}\right)\right|^2 \times \\ & \times \frac{\omega^2}{(\omega^2 - \Omega^2)^2 + \left(8\pi^3 \left|\hat{\rho}\left(\frac{\omega}{c}\right)\right|^2\right)^2 \omega^6\tau^2}, \end{aligned} \tag{A.2}$$

where

$$\Omega^2 = \Omega^2(\omega) := \frac{\alpha}{m_{\mathrm{tot}}(\omega)}, \qquad \tau := \frac{2}{3}\frac{e^2}{m_{\mathrm{tot}}(\omega)c^3}, \tag{A.3}$$

and

$$m_{\mathrm{tot}}(\omega) := m_0 + m_{\mathrm{em}} + \frac{16}{3}\pi^2\frac{e^2}{c^2}\omega \int_{-\infty}^{\infty} \frac{|\hat{\rho}(k)|^2}{\omega_k - \omega}\,\mathrm{d}k, \tag{A.4}$$

the integral being just a symbol for the Hilbert transform. The functions are extended to $(-\infty, 0)$ by symmetry, and $\omega_k := ck$. In addition,

$$C_r := \left[\alpha + \frac{32}{3}\pi^2 e^2 \lambda_r^2 \int_0^\infty \frac{k^2|\hat{\rho}(k)|^2}{(\omega_k^2 + \lambda_r)^2}\right]^{-1}. \tag{A.5}$$

Finally,

$$\mathbf{A}_r^l := \Pi(\mathbf{u}_l A_{r,s}), \qquad \mathbf{A}_\omega^l := \Pi(\mathbf{u}_l A_{\omega,s}^o) + \Pi(\mathbf{u}_l A_{\omega,s}^e),$$

where

$$A_{r,s}(x) = -\frac{e}{c}\nu_r^2 \int_{\mathbb{R}^3} \frac{e^{-\nu_r y/c}}{y}\rho(|x-y|)\,d^3y, \tag{A.6}$$

$$A_{\omega,s}^e(x) = \frac{e}{c}\omega^2 \int_{\mathbb{R}^3} \frac{\cos(\omega y/c)}{y}\rho(|x-y|)\,d^3y, \tag{A.7}$$

and $A_{\omega,s}^{o,l}$, in terms of Fourier transform, is given by

$$\hat{A}_{\omega,s}^o(k) := -\frac{m_{\text{tot}}(\omega)\omega^2 - \alpha}{\frac{8}{3}\pi\frac{e\omega^2}{c^4}\hat{\rho}^*\left(\frac{\omega}{c}\right)}\delta(\omega_k - \omega). \tag{A.8}$$

Notice also that the Fourier transform of $A_{\omega,s}^e$ is given by

$$\hat{A}_{\omega,s}^{e,l}(k) = 4\pi ec\frac{\omega^2\hat{\rho}(k)}{\omega_k^2 - \omega^2}, \tag{A.9}$$

and that, in the point limit, $A_{\omega,s}^o$ is given by

$$A_{\omega,s}^o(x) = -\frac{e}{\tau_0 c}\frac{\omega^2 - \omega_0^2}{\omega}\frac{\sin(\omega x/c)}{x}. \tag{A.10}$$

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