

## Delta Interactions and Electrodynamics of Point Particles

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**ABSTRACT.** We report on some recent work of the authors showing the relations between singular (point) perturbation of the Laplacian and the dynamical system describing a charged point particle interacting with the self-generated radiation field (the Maxwell–Lorentz system) in the dipole approximation. We show that in the limit of a point particle, the dynamics of the system is described by an abstract wave equation containing a selfadjoint operator  $H_m$  of the class of point interactions; the classical Abraham–Lorentz–Dirac third order equation, or better its integrated second order version, emerges as the evolution equation of the singular part of the field and is related to the boundary conditions entering in the definition of the operator domain of  $H_m$ . We also give the Hamiltonian structure of the limit model and, in the case of no external force, we study the reduced dynamics on the linear stable manifold.

*Dedicated to Sergio Albeverio*

### 1. Introduction

Over a century after the discover of the electron by Thompson and the first theoretical studies by Lorentz, a satisfying mathematical description of the interaction of point charged particles and electromagnetic field is still lacking. As it is well known, the Maxwell–Lorentz system, which correctly describes the bulk matter, looses its meaning in the case of a point particle, due to the elementary fact that the solutions of the field equations are not regular enough to be evaluated at a single point, as required by the particle equation. This classical singularity yields directly to the need of infinite mass renormalization and to the difficulties which plague the classical theory of the electron. This theory, as emerges in the work of Lorentz, Abraham and Dirac among the others, leads to a reduced equation for the particle alone, the so called Abraham Lorentz Dirac equation (ALD for short), a third order differential equation which embodies the interaction between particle and field in an inertial term (mass renormalization) and a third order term, describing radiation reaction. The mathematically dubious procedures involved in its deduction from Maxwell–Lorentz system, and the presence of unphysical solutions, the so called runaway solutions, surround the entire subject of a legitimate suspicion, and the usual way out is to invoke quantum electrodynamics for the solution of the problem. But, in turn, mathematical foundations of quantum electrodynamics

are far from being clear, and in the opinion of the present authors and many others a solution to the classical problem is a first necessary step toward a clarification of the quantum one. In the following (see [2,3] for complete proofs, more details and references) we describe the main results obtained in the last years by us in the so called dipole approximation of classical electrodynamics, a well known model which in essence amounts to linearize the interaction (see the Hamiltonian (3.1)) in the Maxwell-Lorentz system. For this model we show that a limit dynamical system describing the coupled dynamics of both particle and field indeed exists (theorem 3.3), and is an abstract wave equation in which the operator part is a point interaction, the elements of the domain of which have a well precise relation (see the system (3.5)) with the particle velocity. To simplify the presentation we introduce first the special case of a free particle and zero total momentum, where the standard delta potential appears, and then we give the general case in which an external force is present. To treat this second case a slight generalization of the point interaction is needed, where the linear boundary condition of standard delta interactions are replaced by an affine one in which the particle momentum enters (see definition 3.2). Such a boundary condition allows to interpret the solutions of the limit system as the solutions of the system in which a standard wave equation with a delta source is coupled with an ordinary second order equation. This equation is a low order version of the ALD equation.

Our description allows also an immediate Hamiltonian formulation of the limit dynamics, a result which solves an old problem in classical electron theory.

In the last paragraph we study, in the free case, the reduced dynamics on the stable linear manifold, also called, in the classical literature, the “non runaway dynamics”. In particular we show that the flow of the reduced Hamiltonian system is correlated, through a canonical transformation, to another Hamiltonian system which lives on the standard Hilbert space of the free wave equation, the symplectic form not being however the usual one.

## 2. Notations

- $L_*^2(\mathbb{R}^3)$  is the Hilbert space of square integrable, divergence-free, vector fields on  $\mathbb{R}^3$ .
- $M$  is be the projection from  $L_3^2(\mathbb{R}^3)$ , the Hilbert space of square integrable vector fields on  $\mathbb{R}^3$ , onto  $L_*^2(\mathbb{R}^3)$ .
- $\langle \cdot, \cdot \rangle$  ( $\| \cdot \|_2$  being the corresponding Hilbert norm) denotes the scalar products in  $L^2(\mathbb{R}^3)$ ,  $L_3^2(\mathbb{R}^3)$ ,  $L_*^2(\mathbb{R}^3)$  and also the obvious pairing between an element of  $L_3^2(\mathbb{R}^3)$  and one of  $L^2(\mathbb{R}^3)$  (the result being a vector in  $\mathbb{R}^3$ ).
- Given two functions  $f$  and  $g$  in  $L^2(\mathbb{R}^3)$ ,  $f \otimes g$  is the operator in  $L_3^2(\mathbb{R}^3)$  defined by  $f \otimes g(A) := f \langle g, A \rangle$ .
- $H^s(\mathbb{R}^3)$ ,  $s \in \mathbb{R}$ , is the usual scale of Sobolev-Hilbert spaces, and  $H_*^s(\mathbb{R}^3)$  and  $H_*^s(\mathbb{R}^3)$  are defined correspondingly.
- $\theta$  denotes the Heaviside function.
- $I(T)$  denotes the compact time interval  $[-T, T]$ .
- $\text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$  is the space of Lipschitz vector fields.
- Given a measurable non negative function  $\rho$ , its energy  $E(\rho)$  is defined as

$$E(\rho) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dx dy .$$

### 3. The Point Limit of the Maxwell-Lorentz System

Let us consider, on the symplectic vector space  $(H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3) \times \mathbb{R}^6, \Omega)$ ,  $\Omega((A_1, E_1, q_1, p_1), (A_2, E_2, q_2, p_2)) := \langle A_1, E_2 \rangle - \langle A_2, E_1 \rangle + q_1 \cdot p_2 - q_2 \cdot p_1$ , the Hamiltonian associated to the regularized Maxwell-Lorentz system in the dipole approximation with an external force  $F = -\nabla V$ , i.e.

$$(3.1) \quad \mathcal{H}_r(A, E, q, p) = 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} \|\nabla A\|_2^2 + \frac{1}{2m_r} \left| p - \frac{e}{c} \langle \rho_r, A \rangle \right|^2 + V(q),$$

where  $c$  is the velocity of light and the square integrable density  $\rho_r$  describes an extended particle with electric charge  $e$  and radius  $r$ . The corresponding Hamilton equations give rise to the Cauchy problem

$$(3.2) \quad \begin{cases} \dot{A}_r = 4\pi c^2 E_r \\ \dot{E}_r = \frac{1}{4\pi} \Delta A_r - \frac{e^2}{m_r c^2} M \langle \rho_r, A_r \rangle \rho_r + \frac{e}{m_r c} M p_r \rho_r \\ \dot{q}_r = \frac{1}{m_r} p_r - \frac{e}{m_r c} \langle \rho_r, A_r \rangle \\ \dot{p}_r = -\nabla V(q) \\ A_r(0) = A_0^r \in H_*^1(\mathbb{R}^3), \quad E_r(0) = E_0 \in L_*^2(\mathbb{R}^3), \\ q_r(0) = q_0, \quad p_r(0) = p_0. \end{cases}$$

Let us begin with the simplest situation, i.e.  $V = \text{const.}$  and  $p_0 = 0$ . In this case the fields equations decouple from the particle ones and one is led to study the convergence as  $r \downarrow 0$  of the self-adjoint operator

$$H_r := -\Delta + \frac{4\pi e^2}{m_r c^2} M \cdot \rho_r \otimes \rho_r.$$

This is a first rank perturbation of the Laplacian; its resolvent is readily calculated and it is given by

$$(H_r + z)^{-1} = (-\Delta + z)^{-1} + \Gamma_r(z)^{-1} M \cdot (-\Delta + z)^{-1} \rho_r \otimes (-\Delta + z)^{-1} \rho_r,$$

where  $\text{Im } z \neq 0$ ,

$$\Gamma_r(z) = -\frac{m_r c^2}{4\pi e^2} - \frac{2}{3} \langle (-\Delta + z)^{-1} \rho_r, \rho_r \rangle,$$

and one immediately obtains that, in order to obtain a non trivial (i.e. different from  $-\Delta$ ) limit, the mass must be renormalized according to the classical prescription

$$m_r := m - \frac{8\pi e^2}{3c^2} E(\rho_r),$$

where  $m$  is the phenomenological mass. With this definition of  $m_r$ ,  $(H_r + z)^{-1}$  converges in norm to

$$R_m(z) := (-\Delta + z)^{-1} + \Gamma_m(z)^{-1} M \cdot G_z \otimes G_z,$$

where

$$\Gamma_m(\lambda) = -\frac{m c^2}{4\pi e^2} + \frac{\sqrt{\lambda}}{6\pi},$$

and

$$G_z(x) := \frac{1}{4\pi} \frac{e^{-\sqrt{z}|x|}}{|x|}, \quad \text{Re } \sqrt{z} > 0.$$

It is not difficult to show then that  $R_m(z)$  is the resolvent of a self-adjoint operator. In more detail one has the following result, which is no more than an adaptation

to our situation of [1, §II.1.] as regards the operator aspects and of [4] as regards the form ones.

THEOREM 3.1. As  $\tau \downarrow 0$ , i.e as  $\rho_\tau(x) := \tau^{-3}\rho(\tau^{-1}x)$ ,  $\rho$  a spherically symmetric probability density with bounded support, weakly converges to  $\delta_0$ , the self-adjoint operator  $H_\tau$  converges in norm resolvent sense in  $L^2_*(\mathbb{R}^3)$  to the self-adjoint operator  $H_m$  so defined:

1.  $A \in D(H_m)$  if and only if

$$\exists Q_A \in \mathbb{R}^3 : A_\lambda := A - \frac{4\pi e}{c}MQ_A G_\lambda \in H^2_*(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0,$$

and the following boundary condition holds:

$$\lim_{r \downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left( A - \frac{4\pi e}{c}MQ_A G_0 \right) d\mu_r = -\frac{mc}{e}Q_A,$$

where  $S_r$  denotes the sphere of radius  $r$  and  $\mu_r$  is the corresponding surface measure.

$$(H_m + \lambda)A := (-\Delta + \lambda)A_\lambda.$$

Moreover

$$\sigma_{ess}(H_m) = \sigma_{ac}(H_m) = [0, +\infty), \quad \sigma_{sc}(H_m) = \emptyset,$$

and

$$\sigma_p(H_m) = \left\{ -\left( \frac{3mc^2}{2e^2} \right)^2 \right\} \equiv \{-\lambda_0\},$$

where  $-\lambda_0$  has a threefold degeneration and

$$X_j^0 = 2\sqrt{2\pi m} \frac{c}{e} M e_j G_{\lambda_0},$$

are the corresponding normalized eigenvectors, where  $\{e_j\}_1^3$  is an orthonormal basis.

If  $F_m$  is the quadratic form corresponding to  $H_m$  then

$$F_m(A, A) + \lambda \|A\|_2^2 = \|(-\Delta + \lambda)^{\frac{1}{2}} A_\lambda\|_2^2 + \left( \frac{4\pi e}{c} \right)^2 \Gamma_m(\lambda) |Q_A|^2, \quad \lambda > 0,$$

where the vector  $A$  is in the form domain  $D(F_m)$  if and only if

$$\exists Q_A \in \mathbb{R}^3 : A_\lambda := A - \frac{4\pi e}{c}MQ_A G_\lambda \in H^1_*(\mathbb{R}^3).$$

Finally, given  $A \in D(F_m)$ ,  $Q_A$  can be explicitly computed by the formula

$$Q_A = \frac{3c}{2e} \lim_{r \downarrow 0} r \frac{1}{4\pi r^2} \int_{S_r} A(x) d\mu_r.$$

The norm resolvent convergence of  $H_\tau$  to  $H_m$  implies, by using the explicit solution for abstract linear wave equations in terms of sine and cosine operator functions, that if

$$\lim_{r \downarrow 0} \|(H_\tau + \lambda)^{\frac{1}{2}} A_0^\tau - (H_m + \lambda)^{\frac{1}{2}} A_0\|_2 = 0, \quad \lambda > \lambda_0$$

(we used the natural distance in the energy norm between elements in the form domain of  $H_\tau$  and  $H_m$ ), then (see [2, corollary 2.9])

$$(3.3) \quad \lim_{r \downarrow 0} \sup_{|t| \leq \tau} \|(H_\tau + \lambda)^{\frac{1}{2}} A_\tau(t) - (H_m + \lambda)^{\frac{1}{2}} A(t)\|_2 = 0.$$

Here  $A(t)$  is the solution of the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \dot{A} = -H_m A_m \\ A(0) = A_0 \in D(F_m), \quad \dot{A}(0) = 4\pi c^2 E_0 \in L^2_*(\mathbb{R}^3) \end{cases}$$

This gives the limit field dynamics.

As regards the behaviour of the particle dynamics in the limit  $\tau \downarrow 0$ , by [3, lemma 2.4], relation (3.3) implies

$$\lim_{\tau \downarrow 0} \sup_{|t| \leq T} \left| -\frac{e}{m_r c} \langle \rho_r, A_r(t) \rangle - Q_{A(t)} \right| = 0.$$

This follows from (here  $X \in L^2_*(\mathbb{R}^3)$ )

$$\begin{aligned} -\frac{e}{m_r c} \langle \rho_r, (H_r + \lambda)^{-1/2} X \rangle &= -\frac{e}{c} \frac{\langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X \rangle}{m_r} \\ -\frac{2e}{3\pi c} \int_0^\infty \Gamma_r(x + \lambda)^{-1} \langle (-\Delta + x + \lambda)^{-1} \rho_r, X \rangle \frac{\langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} \frac{dx}{\sqrt{x}}, \end{aligned}$$

$$Q_{(H_m + \lambda)^{-1/2} X} = \left( \frac{4\pi e}{c} \right)^{-1} \frac{1}{\pi} \int_0^\infty \Gamma_m(x + \lambda)^{-1} \langle G_{x+\lambda}, X \rangle \frac{dx}{\sqrt{x}}$$

and

$$\lim_{\tau \downarrow 0} \Gamma_\tau(\lambda) = \Gamma_m(\lambda),$$

$$\lim_{\tau \downarrow 0} \left| \frac{\langle \rho_r, (-\Delta + \lambda)^{-\frac{1}{2}} X \rangle}{m_r} \right| = 0,$$

$$\lim_{\tau \downarrow 0} \frac{\langle (-\Delta + \lambda)^{-1} \rho_r, \rho_r \rangle}{m_r} = -\frac{3c^2}{8\pi e^2}.$$

Therefore

$$\lim_{\tau \downarrow 0} \dot{q}_r = \lim_{\tau \downarrow 0} -\frac{e}{m_r c} \langle \rho_r, A_r \rangle = Q_A$$

uniformly in time over compact intervals.

So, to summarize, the dynamics of the system corresponding to the limit of (3.2) is completely specified (in the case  $V = \text{const.}$ ,  $p_0 = 0$ ) by solving the abstract wave equation  $c^{-2} \dot{A} = -H_m A$  and then recovering the time evolution of the particle position from the relation  $\dot{q} = Q_A$ . Since the system is linear this can be done explicitly (see [2, §3]).

Let us now consider the general situation  $V \neq \text{const.}$ ,  $p_0 \neq 0$ . In this case the field equations and the particle ones no more decouple. However, considering the field equation with an assigned path  $t \mapsto p(t)$  not depending on the dynamics, one is led to study the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A}_r = -H_r A_r + \frac{4\pi e}{m_r c} M p \rho_r \\ A(0) = A_0 \in H^1_*(\mathbb{R}^3), \quad \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3). \end{cases}$$

As in the case  $p = 0$  one would like to show that the solution of the above system converges to the solution of the Cauchy problem

$$\begin{cases} \frac{1}{c^2} \ddot{A} = -H^p A \\ A(0) = A_0 \in D(F_m), \quad \dot{A}(0) = \dot{A}_0 \in L^2_*(\mathbb{R}^3), \end{cases}$$

where  $H_m^p$  is a (not necessarily linear) operator to be identified. To this end let us consider the system

$$\begin{cases} \frac{1}{2}\dot{A}_r = -(H_r + \lambda)A_r + \frac{4\pi e}{m_r c} M p \rho_r \\ A(0) = A_0^r \in H_*^1(\mathbb{R}^3), \quad \dot{A}(0) = \dot{A}_0 \in L_*^2(\mathbb{R}^3), \end{cases}$$

where the parameter  $\lambda > \lambda_0$  is inserted only to make  $H_r + \lambda$ , and successively  $H_m + \lambda$ , invertible. Its (mild) solution is readily found and, after an integration by parts, can be rewritten as

$$\begin{aligned} A_r(t) &= \cos(ct(H_r + \lambda)^{\frac{1}{2}})A_0^r + \sin(ct(H_r + \lambda)^{\frac{1}{2}})(H_r + \lambda)^{-\frac{1}{2}}\dot{A}_0 \\ &\quad - \frac{4\pi e}{m_r c} \int_0^t \cos(c(t-s)(H_r + \lambda)^{\frac{1}{2}})(H_r + \lambda)^{-1} M \dot{p}(s) \rho_r ds \\ &\quad + \frac{4\pi e}{m_r c} (H_r + \lambda)^{-1} M p(t) \rho_r - \frac{4\pi e}{m_r c} \cos(ct(H_r + \lambda)^{\frac{1}{2}})(H_r + \lambda)^{-1} M p(0) \rho_r. \end{aligned}$$

By [3, lemma 2.5]

$$\lim_{r \downarrow 0} \left\| \frac{4\pi e}{m_r c} (H_r + \lambda)^{-\frac{1}{2}} M p \rho_r + \frac{c}{e} \Gamma_m(\lambda)^{-1} (H_m + \lambda)^{\frac{1}{2}} M p G_\lambda \right\|_2 = 0$$

and so, by thm. 3.1, if

$$\lim_{r \downarrow 0} \|(H_r + \lambda)^{\frac{1}{2}} A_0^r - (H_m + \lambda)^{\frac{1}{2}} A_0\|_2 = 0,$$

then

$$\lim_{r \downarrow 0} \sup_{|t| \leq T} \|(H_r + \lambda)^{\frac{1}{2}} A_r(t) - (H_m + \lambda)^{\frac{1}{2}} A(t)\|_2 = 0,$$

where

$$\begin{aligned} A(t) &= \cos(ct(H_m + \lambda)^{\frac{1}{2}})A_0 + \sin(ct(H_m + \lambda)^{\frac{1}{2}})(H_m + \lambda)^{-\frac{1}{2}}\dot{A}_0 \\ &\quad + \frac{c}{e} \Gamma_m(\lambda)^{-1} \int_0^t \cos(c(t-s)(H_m + \lambda)^{\frac{1}{2}}) M \dot{p}(s) G_\lambda ds \\ &\quad - \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda + \frac{c}{e} \Gamma_m(\lambda)^{-1} \cos(ct(H_m + \lambda)^{\frac{1}{2}}) M p(0) G_\lambda. \end{aligned}$$

Finally, again integrating by parts, it is easily seen that

$$A_p(t) := A(t) + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(t) G_\lambda$$

solves the Cauchy problem

$$\begin{cases} \frac{1}{2}\dot{A}_p = -(H_m + \lambda)A_p + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \dot{p} G_\lambda \\ A_p(0) = A_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p(0) G_\lambda, \quad \dot{A}_p(0) = \dot{A}_0 + \frac{c}{e} \Gamma_m(\lambda)^{-1} M \dot{p}(0) G_\lambda, \end{cases}$$

and so  $A(t)$  solves the Cauchy problem

$$\begin{cases} \frac{1}{2}\dot{A} = -(H_m + \lambda)(A + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p G_\lambda) \\ A(0) = A_0 \in D(F_m), \quad \dot{A}(0) = \dot{A}_0 \in L_*^2(\mathbb{R}^3). \end{cases}$$

This induces us to define, on the domain

$$D(H_m^p) := D(H_m) - \frac{c}{e} \Gamma_m(\lambda)^{-1} M p G_\lambda,$$

the affine operator  $H_m^p$  according to

$$(H_m^p + \lambda)A := (H_m + \lambda)(A + \frac{c}{e} \Gamma_m(\lambda)^{-1} M p G_\lambda).$$

Alternatively (see [3, lemma 4.1])  $H_m^p$  can be defined in the following way:

DEFINITION 3.2. 1.  $A \in D(H_m^p)$  if and only if

$$\exists QA \in \mathbb{R}^3 : A_\lambda := A - \frac{4\pi e}{c} MQ_A G_\lambda \in H_*^2(\mathbb{R}^3), \quad -\lambda \in \rho(H_m), \quad \lambda > 0,$$

and the following boundary condition holds:

$$\lim_{r \downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left( A - \frac{4\pi e}{c} MQ_A G_0 \right) d\mu_r = -\frac{mc}{e} Q_A + \frac{c}{e} p.$$

2.

$$(H_m^p + \lambda)A := (-\Delta + \lambda)A_\lambda.$$

Note that, as it is evident, the affine operator  $H_m^p$  reduces, for  $p = 0$ , to the linear operator  $H_m$  describing the standard point interaction.

The same considerations leading to the definition of  $H_m^p$  give, coupled with a fixed point argument and estimates uniform in  $r$ , the following

THEOREM 3.3. (see [3, thm. 3.4]) Let  $V$  such that  $\nabla V \in \text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$ ,  $|\nabla V(x)| \leq K(1 + |x|)$ ,  $\lambda > \lambda_0$ , and  $E_0 \in L_*^2(\mathbb{R}^3)$ . Let  $A_0^r \in H_*^1(\mathbb{R}^3)$ ,  $A_0 \in D(F_m)$ , such that

$$(3.4) \quad \lim_{r \downarrow 0} \|(H_r + \lambda)^{\frac{1}{2}} A_0^r - (H_m + \lambda)^{\frac{1}{2}} A_0\|_2 = 0.$$

Then there exists  $T > 0$ , not depending on  $r$ , such that, denoting by

$$(A_r, E_r, q_r, p_r) \in C(I(T); H_*^1(\mathbb{R}^3)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^2(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3)$$

the unique mild solution of the Cauchy problem (3.2), one has

$$\begin{aligned} \lim_{r \downarrow 0} \sup_{|t| \leq T} \|(H_r + \lambda)^{\frac{1}{2}} A_r(t) - (H_m + \lambda)^{\frac{1}{2}} A(t)\|_2 &= 0, \\ \lim_{r \downarrow 0} \sup_{|t| \leq T} \|E_r(t) - E(t)\|_2 &= 0, \\ \lim_{r \downarrow 0} \sup_{|t| \leq T} |q_r(t) - q(t)| + \sup_{|t| \leq T} |q_r(t) - q(t)| &= 0, \\ \lim_{r \downarrow 0} \sup_{|t| \leq T} |p_r(t) - p(t)| + \sup_{|t| \leq T} |p_r(t) - p(t)| &= 0, \end{aligned}$$

where

$$(A, E, q, p) \in C(I(T); D(F_m)) \times C(I(T); L_*^2(\mathbb{R}^3)) \times C^1(I(T); \mathbb{R}^3) \times C^1(I(T); \mathbb{R}^3)$$

denotes the unique mild solution of the Cauchy problem

$$(3.5) \quad \begin{cases} \dot{A} = 4\pi c^2 E \\ \dot{E} = -\frac{1}{4\pi} H_m^p A \\ \dot{q} = Q_A \\ \dot{p} = -\nabla V(q) \\ A(0) = A_0 \in D(F_m), \quad E(0) = E_0 \in L_*^2(\mathbb{R}^3), \\ q(0) = q_0, \quad p(0) = p_0. \end{cases}$$

An alternative description of the limit dynamics defined by the previous system is provided by the following

THEOREM 3.4. (see [3, thm. 4.2]) Given  $V$  such that  $\nabla V \in \text{Lip}(\mathbb{R}^3; \mathbb{R}^3)$ , let  $(A, E, q, p) \in C^1([0, T]; D(F_m)) \times C^1([0, T]; L_*^2(\mathbb{R}^3)) \times C^2([0, T]; \mathbb{R}^3) \times C^1([0, T]; \mathbb{R}^3)$ ,  $A_0 \in D(H_m^{p_0})$ ,  $E_0 \in D(F_m)$ , be the unique strict solution of the Cauchy problem (3.5). Then

$$A(t) = A_f(t) + \frac{4\pi e}{c} MA_\delta(t),$$

where  $A_f(t)$  is the solution of the free wave equation with initial data  $A_0, E_0$  and  $A_\delta$  is the retarded potential of the source  $QA\delta_0$ , i.e.

$$A_\delta(t, x) = \frac{1}{4\pi} \frac{\theta(ct - |x|)}{|x|} Q_{A(t - |x|/c)}.$$

Moreover  $Q_A$  satisfies the equation

$$(3.6) \quad \dot{Q}_{A(t)} = c\sqrt{\lambda_0} Q_{A(t)} + \frac{3c^2}{2e} A_f(t, 0) - \frac{3c^3}{2e^2} p(t).$$

Let us sketch the proof of the above theorem. Given an arbitrary function  $Q(t)$ , consider the function

$$A(t, x) := A_f(t, x) + \frac{1}{4\pi} \frac{\theta(ct - |x|)}{|x|} Q(t - |x|/c).$$

It solves the distributional equation

$$\frac{1}{c^2} \ddot{A} = \Delta A + \frac{4\pi e}{c} MQ\delta_0.$$

Kirchoff formula shows that  $A_f$  gives no contribution to  $Q_A$  (see [1, §3]) and so  $Q_{A(t)} = Q(t)$ . Moreover, by [3, lemma 4.1] and [2, thm. 3.3] there follows

$$A - \frac{4\pi e}{c} MQG_\lambda \in H_*^2(\mathbb{R}^3).$$

Therefore if  $A(t) \in D(H_m^{p(t)})$ , since

$$\frac{1}{c^2} \ddot{A} = \Delta A + \frac{4\pi e}{c} MQ\delta_0 = \Delta \left( A - \frac{4\pi e}{c} MQG_\lambda \right) + \lambda \frac{4\pi e}{c} MQG_\lambda = -H_m^p A,$$

then the thesis will follow from unicity of the solution of (3.5). The conditions on  $Q(t)$  leading to  $A(t) \in D(H_m^{p(t)})$  are found as follows. By an elementary integration

$$\begin{aligned} & \lim_{r \downarrow 0} \frac{1}{4\pi r^2} \int_{S_r} \left( A - \frac{4\pi e}{c} MQG_0 \right) d\mu_r(x) \\ &= A_f(t, 0) + \frac{2}{3} \frac{e}{c^2} \lim_{r \downarrow 0} \left( \frac{Q(t - r/c) - Q(t)}{r/c} \right) \\ &= A_f(t, 0) - \frac{2}{3} \frac{e}{c^2} \dot{Q}(t) \end{aligned}$$

and so  $A$  satisfies the boundary condition in Definition 3.2 if and only if  $Q(t)$  solves (3.6).

Let us remark that the above theorem gives the connection with the traditional description: indeed the field variable satisfies a standard wave equation with a point source and the equation satisfied by  $Q_A$  (i.e.  $\dot{q}$ ) is the integrated version of the ALD equation

$$m\dot{q}(t) = \frac{2e^2}{3r^3} \ddot{q}(t) - \frac{e}{r} \dot{A}_f(t, 0) + F(q(t)).$$



Moreover the above argument also gives the converse statement: the solution of the distributional Cauchy problem

$$\begin{cases} \frac{1}{2} \dot{A} = \Delta A + \frac{4\pi e}{c} M q \dot{\delta}_0 \\ \ddot{q}(t) = c\sqrt{\lambda_0} \dot{q}(t) + \frac{3c^2}{2e} A_f(t, 0) - \frac{3c^3}{2e^2} p(t) \\ \dot{p} = -\nabla V(q) \\ A(0) = A_0 \in D(H_m^{p_0}), \quad \dot{A}(0) = 4\pi c^2 E_0 \in D(F_m), \\ q(0) = q_0, \quad \dot{q}(0) = Q_{A_0}, \quad p(0) = p_0 \end{cases}$$

solves (3.5). Also note that the equivalence between the two descriptions holds true if and only if the initial data for the field are chosen coherently with the particle's ones (i.e.  $\dot{q}_0 = Q_{A_0}$  and  $\dot{q}_0 = Q_{\dot{A}_0}$ ).

Let us now come to the Hamiltonian character of system (3.5). By definition 3.2, and by  $D(H_m^p) \subset D(F_m)$ , one can check that

$$\langle H_m^p A_1, A_2 \rangle = F_m(A_1, A_2) + 4\pi p \cdot Q_{A_2}.$$

Therefore equations (3.5) are nothing but the Hamilton equations corresponding to the (degenerate) Hamiltonian

$$\mathcal{H}_m(A, E, q, p) := 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} F_m(A, A) + p \cdot Q_A + V(q);$$

this is defined on the symplectic vector space  $(D(F_m) \times L_*^2(\mathbb{R}^3) \times \mathbb{R}^6, \Omega)$ . Moreover one has the following convergence result:

**THEOREM 3.5.** (see [3, thm.4.5]) *Let*

$$\mathcal{H}_r(A, E, q, p) = 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} \|\nabla A\|_2^2 + \frac{1}{2m_r} \left| p - \frac{e}{c} \langle \rho_r, A \rangle \right|^2 + V(q)$$

*be the Hamiltonian giving the equations (3.2), let  $E \in L_*^2(\mathbb{R}^3)$ ,  $(q, p) \in \mathbb{R}^6$ , and let  $A_r \in H_*^1(\mathbb{R}^3)$ ,  $A \in D(F_m)$  satisfy the condition (3.4). Then*

$$\lim_{r \downarrow 0} \mathcal{H}_r(A_r, E, q, p) = \mathcal{H}_m(A, E, q, p).$$

#### 4. The Non Runaway Dynamics

The negative eigenvalue in the spectrum of the operator  $H_m$  gives rise to unstable behaviour which corresponds, in classical electron theory, to the presence of the so called "runaway solutions". In this section we briefly describe, in the free case and, for simplicity of presentation, vanishing particle momentum, the reduced dynamics on the stable manifold.

Given any vector subspace  $\mathcal{V} \subseteq L_*^2(\mathbb{R}^3)$ , we define the corresponding "non runaway" subspace  $[\mathcal{V}]_{nr}$  by

$$[\mathcal{V}]_{nr} := \{ A \in \mathcal{V} : \langle A, X_j^0 \rangle = 0, j = 1, 2, 3 \},$$

$X_j^0$  being the eigenvectors corresponding to  $-\lambda_0$  (see thm. 3.1). Observe that, if  $A = A_\lambda + \frac{4\pi e}{c} MQ_A G_\lambda \in D(F_m)$ , then one has

$$\begin{aligned} \langle A, X_j^0 \rangle &= \langle A_\lambda, X_j^0 \rangle + \frac{4\pi e}{c} \langle MQ_A G_\lambda, X_j^0 \rangle \\ &= 2\sqrt{2\pi m} \frac{c}{e} \left( \langle A_\lambda^j, G_{\lambda_0} \rangle + \frac{8\pi e}{3c} Q_A^j \langle G_\lambda, G_{\lambda_0} \rangle \right). \end{aligned}$$

Therefore  $A \in [D(F_m)]_{\text{nr}}$  if and only if

$$Q_A = -\frac{3c}{8\pi e} \frac{\langle G_{\lambda_0}, A_\lambda \rangle}{\langle G_\lambda, G_{\lambda_0} \rangle},$$

so that any  $A \in [D(F_m)]_{\text{nr}}$  is univocally determined by its regular part  $A_\lambda$ . This implies, since  $\langle G_{\lambda_0}, G_{\lambda_0} \rangle^{-1} = 8\pi\sqrt{\lambda_0}$ , that the map

$$\Phi : H_*^1(\mathbb{R}^3) \rightarrow [D(F_m)]_{\text{nr}}, \quad \Phi A := A + MP_A G_{\lambda_0}, \quad P_A := -12\pi\sqrt{\lambda_0} \langle G_{\lambda_0}, A \rangle,$$

is bijective, with inverse given by

$$\Phi^{-1} : [D(F_m)]_{\text{nr}} \rightarrow H_*^1(\mathbb{R}^3), \quad \Phi^{-1} A := A - \frac{4\pi e}{c} MQ_A G_{\lambda_0} \equiv A_{\lambda_0}.$$

LEMMA 4.1.  $\Phi$  is a continuous bijection between  $H_*^1(\mathbb{R}^3)$  and  $[D(F_m)]_{\text{nr}}$ . Moreover

$$(4.1) \quad \Phi^{-1}([D(H_m)]_{\text{nr}}) = \{A \in H_*^2(\mathbb{R}^3) : A(0) = 0\}.$$

*Proof.* If  $\lambda > \lambda_0$  we can use  $\|A\|_\lambda := F_m^\lambda(A, A)$  as a norm on  $[D(F_m)]_{\text{nr}}$ . Then

$$\begin{aligned} \|\Phi A\|_\lambda^2 &= \|A + MP_A(G_{\lambda_0} - G_\lambda) + MP_A G_\lambda\|_\lambda^2 \\ &= \|(-\Delta + \lambda)^{\frac{1}{2}}(A + MP_A(G_{\lambda_0} - G_\lambda))\|_2^2 + \frac{2}{3}\Gamma_m(\lambda)|P_A|^2 \\ &\leq 2(\|(-\Delta + \lambda)^{\frac{1}{2}}A\|_2^2 + \|(-\Delta + \lambda)^{\frac{1}{2}}MP_A(G_{\lambda_0} - G_\lambda)\|_2^2) + \frac{2}{3}\Gamma_m(\lambda)|P_A|^2 \\ &\leq 2(\|(-\Delta + \lambda)^{\frac{1}{2}}A\|_2^2 + c_1|P_A|^2 \leq 2(\|(-\Delta + \lambda)^{\frac{1}{2}}A\|_2^2 + c_2\|A\|_2^2 \\ &\leq c_3\|(-\Delta + \lambda)^{\frac{1}{2}}A\|_2^2. \end{aligned}$$

Since, for any  $\lambda \neq \lambda_0$ ,  $A = A_\lambda + \Gamma_m(\lambda)^{-1}MA_\lambda(0)G_\lambda \in [D(H_m)]_{\text{nr}}$  if and only if  $A_\lambda(0) = \Gamma_m(\lambda)P_{A_\lambda}$ , one has

$$\begin{aligned} (\Phi^{-1}A)(0) &= A_\lambda(0) + (MP_A(G_\lambda - G_{\lambda_0}))(0) \\ &= A_\lambda(0) + \frac{2\sqrt{\lambda_0} - \sqrt{\lambda}}{4\pi}P_A \\ &= A_\lambda(0) - \Gamma_m(\lambda)P_{A_\lambda} = 0. \end{aligned}$$

Then  $\Phi^{-1}$  is continuous by bijectivity and by the open mapping theorem.  $\square$

Obviously the map  $\Phi$  can be extended to the whole  $L_*^2(\mathbb{R}^3)$ , giving rise the orthogonal projection onto  $[L_*^2(\mathbb{R}^3)]_{\text{nr}}$ .

Let us now consider the symplectic space  $([D(F_m)]_{\text{nr}} \times [L_*^2(\mathbb{R}^3)]_{\text{nr}}, \Omega_0)$ , where  $\Omega_0$  denotes the canonical symplectic form induced by  $\langle \cdot, \cdot \rangle$ , i.e.

$$\Omega_0((A_1, E_1), (A_2, E_2)) = \langle A_1, E_2 \rangle - \langle A_2, E_1 \rangle.$$

On  $([D(F_m)]_{\text{nr}} \times [L_*^2(\mathbb{R}^3)]_{\text{nr}}, \Omega_0)$  we have the non negative Hamiltonian

$$\begin{aligned} \mathcal{H}_m^+(A, E) &= 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} F_m(A, A) \\ &= 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} \left( \|\nabla A_{\lambda_0}\|_2^2 + \frac{\sqrt{\lambda_0}}{12\pi} \left( \frac{4\pi e}{c} \right)^2 |Q_A|^2 \right), \end{aligned}$$

with corresponding Hamiltonian vector field

$$X_{\mathcal{H}_m^+}(A, E) = \left( 4\pi c^2 E, -\frac{1}{4\pi} H_m A \right),$$

defined on the domain  $[D(H_m)]_{nr} \times [D(F_m)]_{nr}$ . If we pull-back  $\Omega_0$  and  $\mathcal{H}_m^+$  to  $H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3)$  by using the map

$$\Psi := \Phi \times \Phi : H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3) \rightarrow [D(F_m)]_{nr} \times [L_*^2(\mathbb{R}^3)]_{nr},$$

we obtain the following

**THEOREM 4.2.** 1.  $(H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3), \Omega_{nr})$  is a symplectic space, where the (weakly) nondegenerate symplectic form  $\Omega_{nr}$  is given by

$$\begin{aligned} \Omega_{nr}((A_1, E_1), (A_2, E_2)) &:= \Psi^* \Omega_0((A_1, E_1), (A_2, E_2)) \\ &= \Omega_0((A_1, E_1), (A_2, E_2)) - \frac{1}{12\pi\sqrt{\lambda_0}} (P_{A_1} \cdot P_{E_2} - P_{A_2} \cdot P_{E_1}). \end{aligned}$$

2. Defining

$$\begin{aligned} X_{\mathcal{H}_{nr}} : H_*^2(\mathbb{R}^3) \times H_*^1(\mathbb{R}^3) &\rightarrow H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3) \\ X_{\mathcal{H}_{nr}}(A, E) &:= \left( 4\pi c^2 E, \frac{1}{4\pi} \Delta A + \frac{3}{2} \sqrt{\lambda_0} M A(0) G_{\lambda_0} \right), \end{aligned}$$

one has, for any  $(A, E) \in H_*^2(\mathbb{R}^3) \times H_*^1(\mathbb{R}^3)$ ,

$$\frac{1}{2} \Omega_{nr}(X_{\mathcal{H}_{nr}}(A, E), (A, E)) = \mathcal{H}_{nr}(A, E),$$

where

$$\begin{aligned} \mathcal{H}_{nr} : H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3) &\rightarrow \mathbb{R} \\ \mathcal{H}_{nr}(A, E) &:= \Psi^* \mathcal{H}_m^+(A, E) \\ &= 2\pi c^2 \|E\|_2^2 + \frac{1}{8\pi} \|\nabla A\|_2^2 - \frac{c^2}{6\sqrt{\lambda_0}} |P_E|^2 + \frac{\sqrt{\lambda_0}}{96\pi^2} |P_A|^2. \end{aligned}$$

i.e.  $X_{\mathcal{H}_{nr}}$  is the (unique) Hamiltonian vector field corresponding to  $\mathcal{H}_{nr}$ .

3. The Hamiltonian vector fields  $X_{\mathcal{H}_m^+}$  and  $X_{\mathcal{H}_{nr}}$  are  $\Psi$ -correlated, i.e.

$$(4.2) \quad X_{\mathcal{H}_m^+} \circ \Psi = \Psi \circ X_{\mathcal{H}_{nr}},$$

and

$$(4.3) \quad U_m(t) \circ \Psi = \Psi \circ U_{nr}(t),$$

where  $U_m(t)$  and  $U_{nr}(t)$ ,  $t \in \mathbb{R}$ , denote the one parameter groups of canonical transformation of  $([D(F_m)]_{nr} \times [L_*^2(\mathbb{R}^3)]_{nr}, \Omega_0)$  and  $(H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3), \Omega_{nr})$  given by the flows of  $X_{\mathcal{H}_m^+}$  and  $X_{\mathcal{H}_{nr}}$  respectively. Moreover, if  $U_f(t)$ ,  $t \in \mathbb{R}$ , denotes the flow, on  $H_*^1(\mathbb{R}^3) \times L_*^2(\mathbb{R}^3)$ , given by solving the free wave equation, then

$$(4.4) \quad U_m(t)|_{[D(H_m)]_{nr} \times [D(F_m)]_{nr}} = \Psi \circ U_f(t) \circ \Psi^{-1}.$$

*Proof.* 1. A simple calculation shows that

$$\begin{aligned} \Omega_0((\Phi A_1, \Phi E_1), (\Phi A_2, \Phi E_2)) &= \langle \Phi A_1, \Phi E_2 \rangle - \langle \Phi A_2, \Phi E_1 \rangle \\ &= \Omega_0((A_1, E_1), (A_2, E_2)) - \frac{1}{12\pi\sqrt{\lambda_0}} (P_{A_1} \cdot P_{E_2} - P_{A_2} \cdot P_{E_1}), \end{aligned}$$

2. Since

$$\begin{aligned} F_m(\Phi A, \Phi A) &= \|(-\Delta + \lambda_0)^{\frac{1}{2}} A\|_2^2 - \lambda_0 \|\Phi A\|_2^2 \\ &= \|(-\Delta + \lambda_0)^{\frac{1}{2}} A\|_2^2 - \lambda_0 \langle A, \Phi A \rangle \\ &= \|\nabla A\|_2^2 + \frac{\sqrt{\lambda_0}}{12\pi} |P_A|^2, \end{aligned}$$

$$\|\Phi E\|_2^2 = \langle E, \Phi E \rangle = \|E\|_2^2 - \frac{1}{12\pi\sqrt{\lambda_0}} |P_E|^2,$$

posing  $4\pi H_{nr}A = -\Delta A - 6\pi\sqrt{\lambda_0}MA(0)G_{\lambda_0}$ , we need to verify the relation

$$-\langle \Delta A, A \rangle + \frac{\sqrt{\lambda_0}}{12\pi} |P_A|^2 = 4\pi \langle H_{nr}A, A \rangle - \frac{1}{12\pi\sqrt{\lambda_0}} P_{4\pi H_{nr}A} \cdot P_A.$$

Since

$$\begin{aligned} P_{\Delta A} &= -12\pi\sqrt{\lambda_0}(-\langle G_{\lambda_0}, (-\Delta + \lambda_0)A \rangle + \lambda_0 \langle G_{\lambda_0}, A \rangle) \\ &= 12\pi\sqrt{\lambda_0}A(0) + \lambda_0 P_A, \end{aligned}$$

and

$$P_{MA(0)G_{\lambda_0}} = -12\pi\sqrt{\lambda_0} \frac{2}{3} \frac{1}{8\pi\sqrt{\lambda_0}} A(0) = -A(0),$$

one has

$$\begin{aligned} 4\pi \langle H_{nr}A, A \rangle - \frac{1}{12\pi\sqrt{\lambda_0}} P_{4\pi H_{nr}A} \cdot P_A &= -\langle \Delta A, A \rangle - \frac{1}{2} A(0) \cdot P_A \\ &+ \frac{1}{12\pi\sqrt{\lambda_0}} \left( 12\pi\sqrt{\lambda_0}A(0) + \lambda_0 P_A \right) \cdot P_A - \frac{1}{2} A(0) \cdot P_A \\ &= -\langle \Delta A, A \rangle + \frac{\sqrt{\lambda_0}}{12\pi} |P_A|^2. \end{aligned}$$

3.  $X_{\mathcal{H}_{nr}}$  generates a group of continuous (w.r.t. the Hilbert norm on  $H_*^1(\mathbb{R}^3) \oplus L_*^2(\mathbb{R}^3)$ ) linear transformations since we can write

$$4\pi H_{nr} + \lambda_0 = (1 - 6\pi\sqrt{\lambda_0}G_{\lambda_0} \otimes G_{\lambda_0}) \circ (-\Delta + \lambda_0).$$

Moreover such transformations are symplectic, since  $X_{\mathcal{H}_{nr}}$  is Hamiltonian. Formula (4.2) follows from the definitions of  $\Omega_{nr}$  and  $\mathcal{H}_{nr}$ , and (4.3) follows from (4.2) by the unicity of generators. Finally, (4.4) follows from (4.1) and the definition of  $X_{\mathcal{H}_{nr}}$ .  $\square$

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