Homotopy string links and the $[\kappa]$-invariant

F. Cohen\(^1\), R. Komendarczyk\(^2\), R. Koytcheff\(^3\), C. Shonkwiler\(^4\)

\(^1\) Department of Mathematics, University of Rochester, USA
\(^2\) Department of Mathematics, Tulane University, USA
\(^3\) Department of Mathematics, University of Massachusetts Amherst, USA
\(^4\) Department of Mathematics, Colorado State University, USA

Summary

Since helicity is an average asymptotic linking number, the search for higher helicities has long centered on trying to extend the scope of higher-order link invariants from links to vector fields. While perhaps not well-known to physicists, in the 1990s Koschorke conjectured a homotopy-theoretic interpretation of link homotopy which, if proved, would yield an abundance of numerical link invariants computed by iterated integrals. In this work we make progress towards proving Koschorke’s conjecture by showing that the analogous statement for string links is true.

As Moffatt\(^{16}\) first observed, the helicity of a vector field measures the extent to which its orbits wrap and coil around one another. Indeed, Arnold\(^{1}\) proved that helicity is the average asymptotic linking number of the orbits of the field and, although this is not the path Woltjer\(^{18}\) originally took, one could have arrived at the definition of the helicity integral by suitably generalizing the Gauss linking integral.

The linking number is the simplest possible linking invariant and, since helicity is an average asymptotic linking number, mathematicians have conjectured for many years that there should be “higher helicities” of vector fields which are average asymptotic versions of more sophisticated linking invariants. While this idea has had some success\(^2, 6\), no completely general higher helicity has ever been defined. Part of the problem is that interpretations of higher linking invariants which yield Gauss-type integral formulas have been rather hard to come by.

Recently, we gave an interpretation\(^5, 4\) of the triple linking number, which distinguishes the Borromean rings from the three-component unlink\(^{14}\), as a Hopf invariant and gave an explicit Gauss-type integral formula for it, setting the stage for a (unfortunately, only partial) definition of a third-order helicity\(^9, 10\). The key idea was to interpret a parametrized three-component link in $\mathbb{R}^3$ as a map from the three-dimensional torus $T^3$ to the configuration space $\text{Conf}(\mathbb{R}^3, 3)$ of 3 distinct points in $\mathbb{R}^3$, and to prove that the homotopy periods of this map record the link homotopy invariants (pairwise and triple linking numbers) of the link.

This phenomenon is a special case of a more general conjecture of Koschorke\(^{11, 12}\) relating link homotopy invariants of parametrized links to homotopy invariants of associated maps. More precisely, Koschorke observed that each parametrized link induces a map $\kappa : T^n \to \text{Conf}(\mathbb{R}^3, n)$, where $T^n$ is the $n$-dimensional torus and $\text{Conf}(\mathbb{R}^3, n)$ is the configuration space of $n$ distinct points on $\mathbb{R}^3$, and that link homotopies\(^4\) of the link induce homotopies of the map. Therefore, the homotopy class $[\kappa]$ of the map is a link homotopy invariant of the link, and Koschorke conjectured that it is actually a complete link homotopy invariant, meaning that links with different $[\kappa]$-invariants cannot be link homotopic. Koschorke’s conjecture is true when restricted to Brunnian links\(^{12, 3}\).

One of Koschorke’s primary motivations was to define an invariant which extended Milnor’s $\mu$-invariants\(^{14, 15}\) to higher-dimensional links, and indeed he and others (e.g.,\(^13, 17\)) have given very precise descriptions of the higher-dimensional situation.

Despite its rather abstract appearance, the $[\kappa]$-invariant gives a natural way of defining numerical link-homotopy invariants: the $[\kappa]$-invariant of a link is a homotopy class of maps, so the homotopy periods of the map\(^8\) are link-homotopy invariants of the link. Since link-homotopy invariants are

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\(^1\) A link homotopy is a deformation during which each component of the link may cross itself but distinct components must remain disjoint.
otherwise relatively hard to come by and since homotopy periods can naturally be computed as integrals, this is an important potential source for numerical link-homotopy invariants which may be extended to higher helicities.

In this work we define an analog $\tilde{\kappa}$ of $\kappa$ for homotopy string links (see Fig. 1) and show that $[\tilde{\kappa}]$ is a complete invariant of homotopy string links. This provides the platform for a potential proof of Koschorke’s conjecture: the next step is to show that the “closure” map $[\tilde{\kappa}] \mapsto [\kappa]$ is compatible with Habegger and Lin’s [7] classification of link homotopy classes of links based on a Markov-type theorem for closures of homotopy string links.

The key practical advantage of working with string links is that they form a group, and the essential technical step is to show that the map $\tilde{\kappa}$ is a homomorphism.

References