Recent Progress
in Topological Fluid Dynamics

Renzo L. Ricca
Department of Mathematics & Applications, U. Milano-Bicocca
renzo.ricca@unimib.it

Course contents

1. Topological interpretation of helicity
2. Vortex knots dynamics and momenta of a tangle
3. Magnetic knots and groundstate energy spectrum
4. Topological transition of soap films
5. Helicity change under reconnection: the GPE case
6. Topological decay measured by knot polynomials
Lecture 1

- **Topological interpretation of helicity:**
  - Coherent structures and topological fluid mechanics
  - Diffeomorphisms and topological equivalence
  - Kinetic and magnetic helicity of flux tubes
  - Gauss linking number
  - Călugăreanu invariant and geometric decomposition

- **Selected references**

  **Barenghi, CF, Ricca, RL & Samuels, DC 2001** How tangled is a tangle? *Physica D* 157, 197.
Coherent structures

- Vortex filaments in fluid mechanics
  *(Kleckner & Irvine 2013)*

- Vortex tangles in quantum systems
  *(Villois et al. 2016)*

- Magnetic fields in astrophysical flows and plasma physics
  *(TRACE mission 2002)*
150 years of topological dynamics

Linking number formula
(Gauss 1833)

Knot tabulation
(Tait 1877)

Applications to magnetic fields
(Maxwell 1867)

Applications to vortices
(Kelvin 1867)

“topological dynamics”

- Knotted solutions to Euler’s equations
- Energy relaxation methods
- Dynamical systems and \( \mu \)-preserving flows
- Change of topology

Applications to magnetic fields

- 3-D fluid topology
- vortex solutions
- fluid invariants
- topological stability

- magnetic knots
- “charged” knots
- groundstate energy

- \( \exists \) Theorems for vector fields
- closed and chaotic orbits
- Hamiltonian structures

- reconnection mechanisms
- singularity formation
Diffeomorphisms of frozen fields

- ideal, incompressible perfectly conducting fluid in $\mathbb{R}^3$: \[ u = u(X,t) \begin{cases} \nabla \cdot u = 0 \quad \text{in} \quad \mathbb{R}^3 \\ u = 0 \quad \text{as} \quad X \to \infty \end{cases} \]

- frozen field evolution:

\[ B(X,t) \in \left\{ \frac{\partial B}{\partial t} = \nabla \times (u \times B) \land \nabla \cdot B = 0; \quad L_2 - \text{norm} \right\} \]

- topological equivalence class:

\[ B_i(X, t) = B_j(X_0, 0) \frac{\partial X_i}{\partial X_{0j}} : \quad B(X_0, 0) \sim B(X, t) \]
The concept of topological equivalence and invariants

- Re-arrangement of internal structure

- Linked pretzel

- Knotted pretzel
Change of topology

reconnection via local surgery
(dissipative effects)
Physical knots and links as tubular embeddings

Let \( \mathcal{T}_i = S_i \otimes C_i \) and \( V_i = V(\mathcal{T}_i) \):

\[ \mathcal{T}_i \to \mathcal{K}_i \quad \text{in} \quad \mathbb{R}^3 \]

- **physical embedding:**
  
  \[ \mathcal{K}_i := \text{supp}(B) \]

  by a standard foliation \( \mathcal{F}_{\{p_i, q_i\}} \) of the \( B \)-lines, such that \( B \cdot \hat{\nu} = 0 \) on \( \partial \mathcal{T}_i \) (material surface).

- **Definition:** A physical knot/link is a smooth immersion into \( \mathbb{R}^3 \) of finitely many disjoint standard solid tori \( \mathcal{T}_i \), such that

  \[ \text{supp}(B) := \bigcup_i \mathcal{K}_i \to \mathcal{L}_n \quad (i = 1, \ldots, n) \]

- **volume and flux-preserving diffeomorphism:**

  \[ V = V(B), \quad \Phi_i = \int_{A(S_i)} B \cdot \hat{\lambda} \, d^2x \; ; \quad \text{signature} \quad \{V, \Phi_i\} \quad \text{constant.} \]
Analogies between Euler equations and magnetohydrodynamics

\[ \omega = \nabla \times \mathbf{u} \]
\[ \mathbf{u} \]
\[ \frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega) \]
\[ H = \int_{V_\omega} \mathbf{u} \cdot \omega \; d^3x \]

\[ \mathbf{B} = \nabla \times \mathbf{A} \]
\[ \mathbf{u} \]
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \]
\[ H = \int_{V_B} \mathbf{A} \cdot \mathbf{B} \; d^3x \]

\[ \omega = \nabla \times \mathbf{u} \]
\[ \mathbf{u} \times \omega = \nabla h \]
\[ H = \int_{V_\omega} \mathbf{u} \cdot \omega \; d^3x \]

\[ \mathbf{J} = \nabla \times \mathbf{B} \]
\[ \mathbf{J} \times \mathbf{B} = \nabla p \]
\[ H = \int_{V_J} \mathbf{J} \cdot \mathbf{B} \; d^3x \]

\[ h = p + \frac{1}{2} \mathbf{u}^2 \]

\[ \mathbf{B}_0 \rightarrow \mathbf{B}_E \]
\[ \mathbf{u}_0 \rightarrow \mathbf{u}_E \]
Helicity and linking numbers

- **Helicity** $H(t)$:
  \[ H(t) = \int_{V(B)} A \cdot B \, d^3X \]
  where \( B = \nabla \times A \), with \( \nabla \cdot A = 0 \) in \( \mathbb{R}^3 \).

- **Theorem** (Woltjer 1958; Moreau 1961). Under ideal conditions helicity is a conserved quantity (frozen in the flow), that is
  \[ \frac{dH(t)}{dt} = 0 \quad \rightarrow \quad H = \text{constant}. \]

- **Theorem** (Moffatt 1969; Moffatt & Ricca 1992). Let \( \mathcal{L}_n \) be an essential physical link in an ideal fluid. Then, we have
  \[
  H = \int_{V(B)} A \cdot B \, d^3X = \sum_{i \neq j} Lk_{ij} \Phi_i \Phi_j + \sum_i Lk_i \Phi_i^2 \\
  = \sum_{i \neq j} Lk_{ij} \Phi_i \Phi_j + \sum_i (Wr + Tw) \Phi_i^2 .
  \]
Helicity in terms of Gauss linking number

- **Field** $\mathbf{B}$ *entirely localized in* $N$ unlinked, unknotted flux tubes
- **No contribution to helicity from individual flux tubes**

Following **Moffatt 1969** ($Lk_i = 0$) we have:

$$ H = \int_{V(B)} A \cdot B \, d^3X = \sum_{i \neq j} Lk_{ij} \Phi_i \Phi_j. $$

**Proof.** By Stokes’ theorem we have:

$$ K_1 = \oint_{C_1} A \cdot dl = \int_{S_1} B \cdot dS $$

$$ K_1 = \begin{cases} 0 & \text{if } C_1 \text{ and } C_2 \text{ are not linked;} \\ \pm \Phi_2 & \text{if } C_1 \text{ and } C_2 \text{ are singly linked.} \end{cases} $$

For $N$ tubes multiply linked with each other, we have

$$ K_i = \oint_{C_j} A \cdot dl = \sum_j \alpha_{ij} \Phi_j $$
Written in integral form, we have
\[ \Phi_i K_i = \oint_{C_i} \mathbf{A} \cdot \Phi_i \, dl = \oint_{V_i} \mathbf{A} \cdot \mathbf{B} \, dV \]
and summing over all tubes, we have an invariant integral over the whole \( \mathbf{B} \)-field:
\[ H = \sum_i \Phi_i K_i = \sum_{i \neq j} \alpha_{ij} \Phi_i \Phi_j = \int_{V(\mathbf{B})} \mathbf{A} \cdot \mathbf{B} \, dV \]
The \( \mathbf{A} \) field is given by the Biot-Savart law due to \( \mathbf{B} \)-field, i.e.
\[ \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int_{V(\mathbf{B})} \frac{\mathbf{R} \times \mathbf{B}(\mathbf{x}')}{R^3} \, dV' \]
\[ \mathbf{R} = \mathbf{x}_i - \mathbf{x}_j \quad (R = |\mathbf{R}|) \]
\[ \mathbf{x}_i \in C_i \quad \mathbf{x}_j \in C_j \]
hence
\[ H = \frac{1}{4\pi} \iint \frac{\mathbf{R} \cdot [\mathbf{B}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}')]}{R^3} \, dV \, dV' \]
that for discrete tubes becomes
\[ H = \sum_{i \neq j} \alpha_{ij} \Phi_i \Phi_j = \sum_{i \neq j} \Phi_i \Phi_j \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{\mathbf{R} \cdot d\mathbf{l}_i \times d\mathbf{l}_j}{R^3} = \sum_{i \neq j} \Phi_i \Phi_j Lk_{ij} \]
where
\[ Lk_{ij} = Lk_{ji} \quad \text{is Gauss linking number.} \]
**First topological invariants**

- **Number of components:** $N$
- **Minimum number of crossings:** $c_{\text{min}} = \min(\#)$
- **(Gauss) linking number between components:** $Lk = \frac{1}{2} \sum_r \varepsilon_r$

\[
\varepsilon_r = \pm 1
\]

$N = 4$, $c_{\text{min}} = 0$, $Lk = 0$

$N = 2$, $c_{\text{min}} = 2$, $Lk = +1$
Computations by hand: some examples

\[ Lk(A, B) = \frac{+2}{2} = +1 \]

\[ Lk(A, B) = \frac{-2}{2} = -1 \]

\[ Lk(A, B) = \frac{+4}{2} = +2 \]

\[ Lk(A, B) = \frac{0}{2} = 0 \]
Helicity in terms of Călugăreanu invariant

- Suppose now that B-field lines inside individual flux tubes contribute to helicity:

- internal winding
- tube axis knottedness

- Suppose there is no other contribution to helicity.

Following Moffatt & Ricca 1992 ($Lk_{ij} = 0$) we have:

$$H = \int_{V(B)} A \cdot B \, d^3X = \sum_i Lk_i \Phi_i^2$$

and introduce the concept of ribbon $R(C,C^*)$:

$C_1 \rightarrow C: \ x = x(s)$

$C_2 \rightarrow C^*: \ x^* = x(s) + \varepsilon N(s)$
Călugăreanu-White invariant

Let us re-consider the Gauss linking formula:

\[ Lk_{12} = Lk(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(x_1 - x_2) \cdot dx_1 \times dx_2}{|x_1 - x_2|^3} \]

and take the limit: \( \lim_{\varepsilon \to 0} Lk_{12}(\varepsilon) = Lk(C, C^*) \)

- Călugăreanu-White invariant:

\[ Lk(C, C^*) = Wr(C) + Tw(C, C^*) \quad \text{(as} \ \varepsilon \ \to 0 \ \text{)} \]

- Writhing number: \( Wr(C) = \frac{1}{4\pi} \oint_{C} \oint_{C} (x - x^*) \cdot dx \times dx^* \)

- Total twist number:

\[ Tw(C, C^*) = \frac{1}{2\pi} \oint_{C} \theta(x, x^*) \, ds = \begin{cases} T(C) = \frac{1}{2\pi} \oint_{C} \tau(s) \, ds \\ + \\ N(C, C^*) = \frac{1}{2\pi} [\Theta(x, x^*)]_R \end{cases} \]
Properties of $Lk$, $Wr$, and $Tw$

- **Călugăreanu invariant**: $Lk = Lk(C, C^*)$
  - $Lk(C, C^*)$ is a topological invariant of the ribbon;
  - it is an integer;
  - under cross-switching $(\mp \rightarrow \pm )$: $\Delta Lk = \pm 2$

- **Writhing number** $Wr = Wr(C)$:
  - $Wr(C)$ is a geometric measure of the curve $C$;
  - it is a conformational invariant;
  - under cross-switching $(\mp \rightarrow \pm )$: $\Delta Wr = \pm 2$

- **Total twist number** $Tw = Tw(C, C^*)$:
  - $Tw(C, C^*)$ is a geometric measure of the ribbon $R(C, C^*)$;
  - it is a conformational invariant;
  - it is additive: $Tw(A) + Tw(B) = Tw(A + B)$
Topological crossing number versus average crossing number

- 2-component oriented link with topological crossing number \( c_{\text{min}} = 4 \)

\[ C_1 \quad \begin{array}{c}
\text{minimal projection} \\
C_2
\end{array} \]

\[ c = c_{\text{min}} = 4 \]

\[ Lk = +2 \]

\[ C_1 \quad \begin{array}{c}
\text{generic projection} \\
C_2
\end{array} \]

\[ c = 10 \]

\[ Lk = +2 \]
**Signed crossings and complexity measure**

- **Writhing number in terms of signed crossings:**

\[
Wr = Wr(C) = \left< n_- (\nu) - n_+ (\nu) \right> = \left< \sum_{r \in C} \varepsilon_r \right>
\]

(Fuller 1971)

- **Average crossing number:**

\[
\overline{C} = \overline{C}(C) = \left< \sum_{r \in C \cap C} |\varepsilon_r| \right>
\]

(Freedman & He 1991)

\[
\overline{C} = 3:
\]

\[
Wr = -3 \quad \text{Wr} = +3 \quad \text{Wr} = +1
\]
Tangle analysis by indented projections

Let \( \Pi_i = \Pi(\hat{T}_i) \) be the “indented” \( \hat{T}_i \)-projection of the oriented tangle component \( \chi_i \); assign the value \( \varepsilon_r = \pm 1 \) to each apparent crossing in \( \Pi_i \).

- **writhing:**
  \[
  W_{r_i} = \Wr(\chi_i) = \left< \sum_{r \in \chi_i} \varepsilon_r \right> , \quad W_r = \Wr(\mathcal{T}') = \left< \sum_{r \in \mathcal{T}'} \varepsilon_r \right> ;
  \]

- **linking:**
  \[
  L_{kj} = \Lk(\chi_i, \chi_j) = \frac{1}{2} \sum_{r \in \chi_i \cap \chi_j} \varepsilon_r , \quad L_{k_{tot}} = \sum_{r \in \mathcal{T}'} |L_{kj}| ;
  \]

- **average crossing number:**
  \[
  \overline{C}_{ij} = \overline{C}(\chi_i, \chi_j) = \left< \sum_{r \in \chi_i \cap \chi_j} \varepsilon_r \right> , \quad \overline{C} = \sum_{r \in \mathcal{T}'} \overline{C}_{ij} ;
  \]

- **estimated values:**
  \[
  \overline{W}_r = \left( \sum_{r \in \mathcal{T}'} \varepsilon_r \right) , \quad \overline{C} = \left( \sum_{r \in \mathcal{T}'} |\varepsilon_r| \right) .
  \]
Energy-complexity relation: a 16 years-old test case

- ABC-flow field super-imposed on an initial circular vortex ring
Energy-complexity relation (Barenghi et al. 2001)

\[ E(t) \sim \sqrt{\bar{C}(t)} \]