3. Vortex dynamics, knots and links

3.1 Elements of vortex dynamics: Navier-Stokes and Euler equations, localized vorticity, Poisson equation, Biot-Savart law, dynamics of thin filaments, vortex ring solution and localized induction approximation (LIA).

3.2 Vortex knots and links: vortex links, Thomson’s stability results, torus knot solutions under LIA, Kida class and stability criterion, direct numerical simulations.

3.3 Conservation laws and topology: Helmholtz conservation laws, scalar and vector invariants (Lagrangian quantities, frozen-in fields, Frobenius invariants), kinetic helicity and linking numbers.

3.4 Articles included:

Further reading: a good introduction to vortex dynamics is:
The role of invariants in ideal fluid mechanics is discussed in:
A very first discussion of the invariance of kinetic helicity is given by:
A review on the role of kinetic helicity in real fluids is given by:
3.1 Elements of vortex dynamics

**Fluid motion and vortex dynamics**

Fluid domain $D \subseteq \mathbb{R}^3$ (possibly unbounded)

- Fluid particles move with velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$;
- Vorticity $\omega(\mathbf{x}, t) = \nabla \times \mathbf{u}(\mathbf{x}, t)$, localised in $D$.

**Navier–Stokes' equations: real fluids**

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \mathbf{u}$$

- Convection
- Dissipation

*Fluid evolution is dominated by convection; non-linear interaction dominant on large scales; dissipation negligible compared to dynamics; then: Euler's equations: ideal fluids*

$$\frac{\partial \mathbf{u}}{\partial t} = \nabla \times (\mathbf{u} \times \omega)$$
Dynamics of localized structures

- Localized vorticity

Definition: A vector field $\mathbf{B}(x)$ is localized in $D$ if there exists $R > 0$ such that

$$\lim_{|x| \to \infty} e^{\frac{|x|}{R}} \cdot |\mathbf{B}(x)| = 0$$

i.e. $|\mathbf{B}|$ decreases super-exponentially outside a bounded region $|x| = R > 0$.

Let vorticity be localized.

Corollary: If vorticity $\omega(x, t)$ is localized at an initial time $t_0$, then it remains localized at any instant $t > t_0$.

- Poisson's equation

$\omega = \nabla \times \mathbf{u}$, $\nabla \cdot \mathbf{u} = 0$:

If a vector potential $\mathbf{A}(x, t)$ such that

$\mathbf{u} = \nabla \times \mathbf{A}$, $\nabla \cdot \mathbf{A} = 0$ ($\mathbf{A} \to \text{const. as } x \to \infty$)

Then

$$\omega = \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

so

$$\nabla^2 \mathbf{A} = -\omega(x, t) \quad \text{(Poisson's equation)}$$
**Biot–Savart Law**

Let \( \omega = \omega(z^*, t) \) and \( r = |z - z^*| \). A solution to the Poisson equation is given by

\[
A(z,t) = \frac{1}{4\pi} \int_{\Omega} \frac{\omega(z^*, t)}{r} \, d\Omega(z^*) \quad \forall t \in I,
\]

where \( \Omega = \Omega_0 + \partial \Omega \) is the domain of definition of vorticity, with measure \( \mu(\Omega) = \mathcal{U} \), and \( G(z, z^*) = 1/r \) is the Green function. So

\[
\mathbf{u}(z, t) = \nabla \times A = \frac{1}{4\pi} \int_{\Omega} \frac{\omega(z^*, t) \times (z - z^*)}{|z - z^*|^3} \, d\Omega(z^*) \quad \forall t \in I,
\]

is the Biot–Savart law of vortex dynamics.

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**Properties of Biot–Savart (BS) integral**

1) \( \mathbf{u}(z, t) \) is a non-local linear functional of \( \omega(z^*, t) \).
2) \( \mathbf{u}(z, t) \) depends on the distribution of \( \omega(z^*, t) \) on \( \Omega \).
3) \( \Omega \) (in particular \( \partial \Omega \)) is not known a priori.
4) the Green function (hence \( \mathbf{u}(z, t) \)) becomes singular as \( z \to z^* \).
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Vortex ring homeomorphic to a torus

Is there any steady solution to BS?
- reduce the problem from 3D to 2D ($\mathbb{R}^2$);
- look for $u(\xi, t) = U$ in $D_2$.

Theorem (Fraenkel & Berger, 1974): There exists a region $\Omega = \mathbb{R} + \Omega_2$: $\omega = 0$ in $D_2/\Omega_2$ and $U, \omega$ on $\Omega_2$ such that $\psi$ = cst. on $\Omega_2$ & $\psi = 0$ at $\infty$.

\[ U = \frac{\Phi}{4\pi R} \left( \log \frac{ER}{a} + F(\omega) \right) + O \left( \frac{Ea^4}{R^8} \log \frac{PR}{a} \right) \]

(Kelvin, 1867; Hicks, 1885; Dyson, 1893; Hill, 1894; ....)
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Vortex filament motion under the Localized Induction Approximation (LIA)

Let \( \omega \) be localized. We have

\[
\mathbf{u}(x, t) = \frac{\Phi}{4\pi} \int \frac{\hat{E}(x^*, t) \times (x - x^*)}{r^3} \, dx^* = f(u_t, u_n, u_\theta)
\]

- Thin tube approximation \( R = \frac{1}{a} \ll a \)
- No self-intersections \( \frac{|x(s_1) - x(s_3)|}{|s_1 - s_3|} = O(1) \)

Localized Induction Approximation (LIA):

\[
\mathbf{v} = \dot{x} = \dot{x}' \times \dot{x}'' = c \hat{b}
\]

Intrinsic kinematics under LIA:

\[
C = (s, \theta) \text{ curvature; } \tau(s, \theta) \text{ torsion.}
\]

\[
C(t) = \begin{cases} 
\dot{c} = - (c \tau)' - c' \tau \\
\dot{\tau} = \left( \frac{c'' - c \tau^2}{c} \right)' + c' c
\end{cases}
\]

(Da Rios, 1906 – Betchov, JFM 22, 1965)

(see also Ricca, Fluid Dyn. Res. 18, 1996)
3.2 Vortex knots and links

\[ r^2 = \tilde{R} = f_1 (..., J, ...) \]
\[ \alpha = f_2 (..., \varepsilon, ...) \]
\[ \varepsilon = f_3 (..., \Pi, ...) \]

Small amplitude perturbations from circular solution

Torus knot solutions to LIA as small amplitude perturbations from circular solution:

\[ \begin{align*}
\dot{x} &= y'z'' - z'y'' \\
\dot{y} &= z'x'' - x'z'' \\
\dot{z} &= x'y'' - y'x''
\end{align*} \]

or

\[ \dot{x}(s,t) = \tilde{x}'x'' \]

\[ \tilde{x} = \frac{r'd'' - 2r'\dot{z}' - r\dot{z}'' z'}{r''} \]

\[ \tilde{z} = 2r'^2 + rr'\dot{\alpha}' - rr''\dot{\alpha}' + r^2 \dot{\alpha}'^3 \]

\[ \text{fully non-linear régime} \]

\[ \text{circular solution at } \tilde{R} = \tilde{R}_0, f(\tilde{R}_0) \]
Vortex torus knot solutions under LIA:

Look for "travelling-wave"-like solutions of type

\[ f(s, t) = f(s) \quad \xi = s - \kappa t \quad \kappa = \text{cst.} \quad \text{(wave speed)} \]

in the form of torus knots \( \tau_{p,q} : p, q \) co-prime

- \( p \) (\( \geq 1 \)) longitudinal wraps
- \( q \) (\( \geq 1 \)) meridional wraps

with winding no. \( W = \frac{q}{p} \).

Linear perturbation from the circular solution:

\[
\begin{align*}
\varepsilon & = r_0 + \varepsilon r_1 \\
\alpha & = s + \varepsilon \alpha_1 \\
\xi & = \frac{c}{r_0} + \varepsilon \xi_1
\end{align*}
\]

\[
\begin{align*}
\varepsilon & < 1, \quad \hat{\iota}_1 = \frac{\kappa}{\kappa} \\
\dot{\theta}_1 & = -\frac{1}{r_0^2} Z_1 \\
\dot{Z}_1 & = -r_1 + \frac{e}{r_0^2} r_1 + 3 \alpha_1
\end{align*}
\]

\[
K^2 = \frac{1}{r_0^2} (W^2 - 1)^2
\]

\[
\begin{align*}
\tau_{p,q} : & \quad \alpha = \frac{s}{r_0} + \varepsilon \frac{Kr}{r_0} \cos \left( \frac{W \frac{\xi}{r_0} + \beta_k}{r_0} \right) \\
& \alpha (Kr, \beta_k = \text{cst.}) \\
& \xi = \hat{\xi} + \varepsilon \frac{Kr}{r_0} \left( W^2 - 1 \right)^{1/2} \cos \left( \frac{W \frac{\xi}{r_0} + \beta_k}{r_0} \right)
\end{align*}
\]

\[ (\text{Ricca, Chaos 3, 1993 & Erismann, Chaos 1, 1995}) \]
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Vortex torus knots under LIA

\[(\text{zeroth-order}) \ LIA : \ \mathbf{u}^{(0)} = c \hat{z}\]

Theorem (Kida, 1981): Let \( K_v(\pi) \) denote the embedding of a knotted vortex filament in \( \mathbb{D} \). If \( K_v \) evolves under (zeroth-order) LIA, then there exists a class of steady solutions in the shape of torus knots \( K_v = \mathcal{T}_{p,q} \) where \( p > 1, q > 1 \) are co-prime integers.

\[\mathcal{T}_{2,3} \sim \mathcal{T}_{3,2}\]

Theorem (Ricca, 1993): Let \( \mathcal{T}_{p,q} \) (\( p > 1, q > 1 \); \( p, q \) co-prime integers) be the embedding of a ‘small-amplitude’ vortex torus knot \( K_v \) under (zeroth-order) LIA. \( \mathcal{T}_{p,q} \) is steady and stable under linear perturbations iff \( q > p \) (\( \omega = q/p > 1 \)).
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\[ \zeta_{2,3} \text{ under L1A} \]

Time \( t_0 \)

Time \( t_1 > t_0 \)

Time \( t_2 > t_1 \)

(R. Samuels & Barenghi, JFM 391, 1999)
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$E_{3,2}$ under LIA

Time $t_0$

Time $t_1 > t_0$

Time $t_2 > t_1$

(R., Samuels & Barenghi, JFM 391, 1999)
Links of thin core vortex rings

(J.J. Thomson, Adams Prize, 1882)

- Two-component link

\[ \lambda = \max \{ |x_i - x_i^*|, \{x_i, x_i^*\} \in \mathcal{C} \} \]
\[ \delta = \min \{ |x_i - x_i^*|, x_i \in \mathcal{C}_1, x_i^* \in \mathcal{C}_2 \} \]
\[ \Gamma = \text{circulation (or flux)} \]

\[ \delta \ll R \]

\[ \begin{align*}
\text{angular velocity} \quad \Omega &= \frac{\Gamma}{\pi d^2} \\
\text{propagation velocity} \quad V &= \frac{\Gamma}{4nR} \log \frac{c_4 R^2}{a^2}
\end{align*} \]

Theorem (Thomson, 1883): A link of two vortex rings of equal flux \( \Gamma \), embedded and equally spaced on \( \Gamma \) in \( D \) with linking \( L_k \), are steady and stable iff

\[ \frac{M(2\pi \gamma \Gamma)^{1/2}}{L_k \rho^{3/2}} < 1 \]

where \( M \) is the resultant of the angular momentum, \( \rho \) the fluid density and \( \Gamma \) the linear momentum.
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Theorem (Thomson, 1883): A link of \( n \) equal vortex rings (\( \Pi \)), embedded and equally spaced on a torus \( \Pi \) in \( \mathbb{R}^3 \), with relative linking \( L_k \), is steady and stable iff \( n \leq 6 \).

The period of vibration of a 2-component link is given by

\[
T = \frac{2 \pi}{\Pi \left( \frac{2}{d^2} - \frac{(2L_k^2 - 1)}{4a^2} \log \frac{d}{a} \right)}
\]
3.3 Conservation laws and topology

- Helmholtz’s laws and conservation laws for localized vorticity (in $\mathbb{R}^3$)

I. For any two simply closed curves $C_1, C_2$ encircling a vortex tube, we have

$$\oint_{C_1} \mathbf{u} \cdot d\ell = \oint_{C_2} \mathbf{u} \cdot d\ell = \int_{\Sigma} \mathbf{\omega} \cdot \hat{n} dS = \Phi,$$

where $\hat{n}$ is the unit normal to the vortex cross-section $\Sigma$ of area $S$.

II. Fluid particles on a vortex line at any instant will be on a vortex line at all subsequent times.

III. The strength $\Phi$ of a vortex tube does not vary with time during the motion of the fluid.

- Kinetic energy: $T = \frac{1}{2} \int_{\Omega} \rho \mathbf{u}^2 dV = \text{cst.}$

- Linear momentum: $P = \frac{1}{2} \int_{\Omega} \rho \mathbf{x} \times \mathbf{\omega} dV = \text{cst.}$

- Angular momentum: $M = \frac{1}{3} \int_{\Omega} \rho \mathbf{x} \times (\mathbf{x} \times \mathbf{\omega}) dV = \text{cst.}$

- Helicity: $H = \int_{\Omega} \mathbf{u} \cdot \mathbf{\omega} dV = \text{cst.}$
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Invariants in ideal fluid mechanics

(See, for example, Tur & Yanovsky, TFM, 1993)

Classification:
- Local → point wise
- Global → integral
- Topological → non-metric

Local invariants

- I type: conserved quantity $f$ (mass)
- II type: Lagrangian invariant $I$ (passive scalar)
- III type: frozen-in field $\psi$ (vorticity)
- IV type: Frobenius invariant $g$ (momentum)

I type: conserved quantities $f$:

Example: $f = \frac{m}{V}$ mass density

 Governing eq.: $\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{u}) = 0$ (I)

II type: Lagrangian invariants $I$:

Example: $I = \text{ink}$

 Governing eq.: $\frac{dI}{dt} = \frac{\partial I}{\partial t} + (\mathbf{u} \cdot \nabla)I = 0$ (II)
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• **III type:** frozen-in fields \( J \):

Example: \( J = \nabla \times u = \omega \) vorticity

Governing eq.: \[ \frac{\partial J}{\partial t} = \nabla \times (u \times J) \]  \( \text{(III)} \)

• **IV type:** Frobenius invariants \( S \):

Example: \( S = \) momentum of a vortex ring

Governing eq.: \[ \frac{dS}{dt} = (S \times \nabla) \times u \]  \( \text{(IV)} \)

• Physical/geometrical meaning of governing eqs.:

I type: governing eq. as a "balance" conservation law for a scalar quantity of fluid flow;

II type: governing eq. for scalar quantities advected "Lagrangian invariantly" by the flow;

III type: governing eq. for vector quantities advected along the integral curves \[ \int dx = \int u \, dt \] (streamlines) of the flow.
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IV type: governing eq. for vector quantities advected on surfaces defined by
\[ \Sigma (x, t) \cdot d\Sigma = 0. \]

The surface orthogonal to the vector field \( \Sigma \) is frozen in the flow.

Remarks:

i) Integral trajectories and streamlines always exist in a fluid flow;

ii) Integral surfaces orthogonal to an \( \Sigma \)-field exist if the Frobenius condition
\[ \Sigma \cdot \nabla \times \Sigma = 0 \]
is satisfied. If it is satisfied at time \( t=0 \), it will be satisfied at all times \( t>0 \).

Geometric interpretation of local invariants

i) Local dynamic invariants are geometric objects, i.e. invariant differential forms;

ii) There exist only four types of local invariants corresponding to \( p \)-forms \( (p=0, 1, 2, 3) \);

iii) All governing eqs. reduce to a single differential form equation.
Helicity and linking numbers

Definition: The helicity $\mathcal{H}(L)$ of a physical link (knot) $L$ is defined by

$$\mathcal{H}(L) = \int_A \mathbf{B} \cdot \mathbf{n} \, d\Sigma$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, $\nabla \cdot \mathbf{A} = 0$ and $d\Sigma$ is a closed, orientable, material surface in $\mathbb{D}$.

\[ L = U_i L_i \]

Theorem (Woltjer, 1958; Moffatt, 1969; Arnold, 1974): The helicity $\mathcal{H}(L)$ is a fluid flow invariant, i.e.

$$\frac{d}{dt} \mathcal{H}(\gamma_t L) = 0.$$ 

Theorem (Moffatt, 1969; Berger & Field, 1984; Moffatt & Ricca, 1992): Let $L$ be a physical link (knot) in $\mathbb{D}$. Then

$$\mathcal{H}(L) = \sum_{i} L_{k_i} \Phi_i^2 + 2 \sum_{i,j \neq i} L_{k_{ij}} \Phi_i \Phi_j$$

where $L_{k_i}$ is the Călugăreanu-White linking number of $L_i$ with respect to the framing induced by the $\mathbf{B}$-field within $L_i$, and $L_{k_{ij}}$ is the Gauss linking number between $L_i$ and $L_j$. 

\[ \text{---} \]
Example

i) identify components:
\[ L = L_1 \cup L_2 \]

ii) identify orientation:
\[ B_1 \leftrightarrow \pi_1 \]
\[ B_2 \leftrightarrow \pi_2 \]

iii) assign fluxes: \( \Phi_1, \Phi_2 \)

iv) evaluate \( \text{lk}_{ij} \):
\[ \text{lk}_{12} = \frac{1}{2} \sum \varepsilon_{ij} = +1 \]

v) evaluate \( \text{lk}_i = \text{wr}_i + \text{tw}_i = \text{wr}_i + z_i + n_i \)

1: \( \text{wr}_1 = \langle n_+ - n_- \rangle = 0, \text{tw}_1 = 0 \implies \text{lk}_1 = 0 \)

2: \( \text{wr}_2 = 0, \text{tw}_2 = +3 \implies \text{lk}_2 = +3 \)

\[ \therefore \mathcal{H}(L) = 3\Phi_2^2 + 2\Phi_1\Phi_2 \]

\[ \approx \approx \]
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Methods of "Regularized vortex filaments":

The degree of knottedness of tangled vortex lines

By H. K. Moffatt

Department of Applied Mathematics and Theoretical Physics,
St John's College, Cambridge

(Received 17 May 1968)

Let \( u(x) \) be the velocity field in a fluid of infinite extent due to a vorticity distribution \( \omega(x) \) which is zero except in two closed vortex filaments of strengths \( \kappa_1, \kappa_2 \). It is first shown that the integral

\[
I = \int u \cdot \omega \, dV
\]

can be equated to \( x_0 \kappa_1 \kappa_2 \), where \( x_0 \) is an integer representing the degree of linkage of the two filaments; \( x_0 = 0 \) if they are unlinked, \( x_0 = \pm 1 \) if they are singly linked. The invariance of \( I \) for a continuous localized vorticity distribution is then established via barotropic inviscid flow under conservative body forces. The result is interpreted in terms of the conservation of linkages of vortex lines which move with the fluid.

Some examples of steady flows for which \( I \neq 0 \) are briefly described; in particular, attention is drawn to a family of spherical vortices with swirl (which is closely analogous to a known family of solutions of the equations of magneto-statics); the vortex lines of these flows are both knotted and linked.

Two related magneto-hydrodynamic invariants discovered by Weltjer (1938a, b) are discussed in §8.

1. Introduction: discrete vortex fields

Consider any flow \( u(x, t) \) under conservative body forces, of an inviscid fluid whose density is either uniform or a function of pressure only. Under these conditions, the circulation round any circuit \( C \) moving with the fluid,

\[
K = \oint_C u \cdot dl,
\]

is constant.

In the particular circumstance that the vorticity \( \omega = \nabla \times u \) is zero except inside two closed vortex filaments \( C_1, C_2 \) of strengths \( \kappa_1, \kappa_2 \) each of which moves with the fluid, we may choose \( C \) to be one of these, say \( C_1 \). If \( C_1 \) is unknotted, so that it can be spanned by a surface \( S \), which does not intersect itself, then Stoke's theorem gives

\[
K_1 = \oint_{C_1} u \cdot dl = \int_S \omega \cdot dS,
\]
and so, since the flux of vorticity across $S_1$ is simply that due to the filament $C_i$, 

$$K_1 = \begin{cases} 
0 & \text{if } C_1 \text{ and } C_2 \text{ are not linked}, \\
\pm \kappa_2 & \text{if } C_1 \text{ and } C_2 \text{ are singly linked,}
\end{cases}$$

(figure 1), the $\pm$ referring to the two possible relative orientations of the two filaments. More generally, the filament $C_2$ may wind an integral number of times round $C_1$ in which case

$$K_1 = \alpha_{12} \kappa_2$$

(3)

where $\alpha_{12} (= \alpha_{21})$ is an integer which may be positive or negative (the \textquote{winding number} of the curves $C_1$ and $C_2$).\footnote{The term \textquote{winding number} (anzahl der umschlingungen) and the expression given below for it, equation (11), can be traced to a paper by Gauss (1833) which was concerned with the magnetic field produced by two or more electric current circuits. It is the simplest (but by no means the only) topological invariant of two linked curves (see, for example, Crockett & Fox, 1964, and the references given therein).

The possibility of linked and knotted vortex lines was conceived by Kelvin (1868), then Sir William Thomson in his celebrated paper, \textit{On Vortex Motions}, in which the \textquote{circulation theorem} was established. The simplest knots were subsequently catalogued by Tait (1885, pp. 273-277) in increasing order of complexity. The development of knot theory as a recognizable branch of modern topology received considerable stimulus from these investigations.}

A simple vortex line that is knotted may be decomposed into two (or more) linked but unlinked vortex lines by the insertion of a pair (or pairs) of equal and opposite vorticity segments. For example, if the vorticity field is zero except in a vortex filament of strength $\kappa$ having the shape $C$ in figure 2 (the trefoil knot), then

$$\oint_C \mathbf{u} \cdot d\mathbf{l} = 4\oint_{C_1} \mathbf{u} \cdot d\mathbf{l} + \oint_{C_2} \mathbf{u} \cdot d\mathbf{l} = 2\kappa.$$

For a more complicated knot in a vortex filament $C$,

$$\oint_C \mathbf{u} \cdot d\mathbf{l} = 2\pi \kappa,$$

where $\kappa$ is an integer representing the degree of knottedness of $C$, the \textquote{self-winding} number of $C$. All knots will be supposed in what follows to be dealt with in this manner.

If there are a unknotted filaments $C_1, C_2, \ldots, C_n$ then a simple generalization of the result (3) is

$$K_i = \oint_{C_i} \mathbf{u} \cdot d\mathbf{l} = \sum_j \alpha_{ij} \kappa_j$$

(4)

where $\alpha_{ij}$ is the winding number of $C_i$ and $C_j$.

The quantity $\kappa_i K_i$ (not summed) may be written in the form of an integral over the volume $V_i$ occupied by the vortex filament $C_i$. Since $d\mathbf{l}$ is parallel to $\mathbf{u}$ in the filament, $\kappa_i d\mathbf{l}$ may be replaced by $\mathbf{u} \cdot dV$ so that

$$K_i = \oint_{C_i} \kappa_i d\mathbf{l} \cdot \mathbf{u} = \int_{V_i} \mathbf{u} \cdot dV.$$

(5)

If we sum over all the filaments, we obtain an invariant integral over the whole vorticity field:

$$I = \sum_i \kappa_i K_i = \sum_{i,j} \alpha_{ij} \kappa_i \kappa_j = \int_V \mathbf{u} \cdot dV.$$

(6)
where \( V \) is the volume occupied by all the filaments, or equivalently (as far as the integral in (6) is concerned) the total volume occupied by the fluid. It should be noted that \( J \) is determined solely by the vorticity field; this dependence may be made explicit by writing

\[
\mathbf{u} = \mathbf{u}_1(\mathbf{x}) + \nabla \phi,
\]

where

\[
\mathbf{u}_1(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\mathbf{R} \times \omega(\mathbf{x}')}{R^2} dV' \quad (\mathbf{R} = \mathbf{x} - \mathbf{x}').
\]

\( \phi \) is the streamfunction. The choice of sign in (b), (c) is determined by the relative orientation of the two filaments.

The potential contribution to \( u \) (which is certainly present if the fluid is enclosed by a rigid boundary) makes no contribution to \( I \) since

\[
\int_V \nabla \phi \cdot \omega dV = \int_V \nabla (\omega \phi) dV - \int_S n \cdot \omega \phi dS = 0.
\]

Substitution of (8) in (6) then gives

\[
I = \frac{1}{4\pi} \iiint \frac{\mathbf{R} \cdot [\omega(\mathbf{x}) \times \omega(\mathbf{x}')] dV dV'}{R^2}.
\]

If this is re-expressed in terms of line integrals, we obtain an explicit expression for \( z_{ij} \) in terms of the relative geometry of the circuits \( C_i \) and \( C_j \):

\[
z_{ij} = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{\mathbf{R} \cdot d\mathbf{l}_i \times d\mathbf{l}_j}{R^3} = \sigma_{ij},
\]

where

\[
\mathbf{R} = \mathbf{x}_i - \mathbf{x}_j, \quad \mathbf{x}_i \in C_i, \quad \mathbf{x}_j \in C_j.
\]
2. Continuous vorticity fields

Consider now a continuous localized distribution of vorticity in an infinite expanse of inviscid fluid. In general the vortex lines will not be closed; a single vortex line may cover a surface, or it may even pass arbitrarily close to any point of a closed volume, if followed far enough. (The instantaneous vorticity field of a turbulent 'blob' is likely to have this latter character in general.) The simple considerations of the preceding section are no longer directly applicable. However, it might be expected that the integral $I$ defined in (8), might still be an invariant for a continuous vorticity blob and that, if so, it may give a useful generalization of the concept of 'degree of knottedness' to a continuous solenoidal vector field.

Let us first obtain an equation for the local rate of change of the quantity $u \cdot \omega$. Under the barotropic condition $p = \rho(\rho)$, the equation of motion may be written

$$D (\rho) dt = -\nabla (h - \Omega),$$

(12)

where $h = \int d\rho/\rho$ and $\Omega$ is the potential of any conservative body forces. Under the same conditions, the vorticity equation takes the well-known form

$$\frac{D}{Dt} \left( \frac{\omega}{\rho} \right) = -\frac{\omega}{\rho} \nabla (h + \Omega) + u \left( \frac{\partial \omega}{\partial \rho} \right) u,$$

(13)

Hence

$$\frac{D}{Dt} \left( \frac{u \cdot \omega}{\rho} \right) = -\frac{\omega}{\rho} \nabla \left( \frac{u \cdot \omega}{\rho} \right) + u \left( \frac{\partial \omega}{\partial \rho} \right) u,$$

(14)

$$= \frac{\omega}{\rho} \nabla \left( \frac{u \cdot \omega}{\rho} \right) + u \left( \frac{\partial \omega}{\partial \rho} \right) u,$$

where

$$\Omega = u \cdot u.$$

Now let $S$ be any surface enclosing a volume $V$ and moving with the fluid, and let

$$I = \int V u \cdot \omega dV,$$

(15)

Since

$$\frac{D}{Dt} \left( \rho dV \right) = 0,$$

(16)

it follows that

$$\frac{dI}{dt} = \int \left( \frac{\partial}{\partial t} \left( \frac{u \cdot \omega}{\rho} \right) \right) dV$$

$$= \int \left( \nabla \cdot \left( \frac{u \cdot \omega}{\rho} \right) \right) dV$$

$$= \int \left( \nabla \cdot \left( \frac{u \cdot \omega}{\rho} \right) \right) dV - \int \left( \frac{u \cdot \omega}{\rho} \right) dS,$$

(17)

†For example, if the flow is steady, and the body forces have a potential $\Omega$, then

$$\omega \times u = -\nabla \Phi$$

where $\Phi = \int \frac{u \cdot \omega}{\rho} dV$, and the $\omega$-lines lie on the surfaces $\Phi = \text{constant}$. 
Chapter 3 – Vortex dynamics, knots and links

The degree of linkedness of tangled vortex lines

using \( \nabla \cdot \omega = 0 \) and the divergence theorem. Hence the condition \( \nabla \cdot \omega = 0 \) on \( S \) is sufficient to ensure that

\[ I = \text{constant}. \]

If the fluid is of infinite extent, and \( S \) is taken to be the surface ‘at infinity’, then the condition \( \omega = o(R^{-1}) \) as \( R = |x| \to \infty \) is likewise sufficient to ensure the invariance of \( I \).

If the vortex lines of the field \( \omega \) are all closed then there is a separate invariant \( I \) for each closed vortex filament in the field (the volume \( V \) in (15) being then the volume occupied by the filament). In the limit as the cross-section of each filament is decreased to zero, we have a doubly infinite family of invariants. If the vortex lines cover surfaces, then there is an invariant \( I \) for each ‘vorticity layer’ in the neighbourhood of each surface, and as the thickness of the layers is decreased to zero, we have a singly infinite family of invariants. If the vortex lines of the field are ‘space-filling’, there is only one invariant \( I \) for each subdomain of \( V \) that is filled by a vortex line.

The quantity \( u \cdot \omega \) admits a simple, essentially kinematical, interpretation. The fluid particles in any small volume element \( dV \) undergo, at any instant, a superposition of three motions: the (uniform) velocity \( u \) of any representative point \( \zeta \) of the element, an irrotational uniform strain \( \nabla \phi \) relative to \( \zeta \), and a rigid body rotation \( 2\omega \), where \( \omega \) is the vorticity at \( \zeta \). The streamlines of the flow \( u - \nabla \phi \) passing near \( \zeta \) are (locally) helices about the streamline through \( \zeta \), and the contribution:

\[ u \cdot \omega dV = u \cdot \omega dV \]

to \( I \) from \( dV \) is positive or negative according as the screw of these helices is right-handed or left-handed. The term helicity is used in particle physics for the scalar product of the momentum and spin of a particle, and it would seem to be a natural candidate in the present context to describe the quantity \( u \cdot \omega dV \); the quantity \( u \cdot \omega \) may then be described as the helicity per unit volume of the flow. Equation (17) then expresses the result that the total helicity within any closed vortex surface (on which \( \omega \cdot n = 0 \) is constant.

3. The effect of the presence of solid boundaries

An inviscid flow in the presence of a solid boundary \( S_b \) need not satisfy the condition \( \nabla \cdot \omega = 0 \) on \( S_b \), since \( \nabla \cdot u \) may vary from one point to another on the boundary. It would therefore appear that the value of \( I \) may then change according to (17), and this is at first sight surprising in view of the interpretation given above of the invariance of \( I \) in terms of the conservation of linkages of vortex lines; vortex lines are still frozen in the fluid when rigid boundaries are present, so these should not affect the invariance of \( I \).

The explanation lies in the fact that if \( n \cdot \omega = 0 \) on \( S_b \), and if \( S_b \) is at rest, then there exists a vortex sheet on \( S_b \), and the vortex lines of the fluid interior must be imagined to be continued and completed within this sheet. (If \( S_b \) is rotating, there is the further complication that the vortex lines actually continue into the solid.) We should therefore expect invariance of the quantity \( I \) defined in (15).
only if it is supplemented by a finite contribution from the vortex sheet on the
surface, and possibly a contribution from the interior of the solid surroundings.

The surface contribution depends on the structure of the vortex sheet on \( S_b \).
The thickness of the vortex sheet (or boundary layer) is controlled by viscous
forces, and it is physically unrealistic to ignore these in any treatment of the
surface layer. Suppose, for simplicity, that \( S_b \) is at rest, and that the fluid has a
small kinematic viscosity \( \nu \). Then \( \mathbf{u} = 0 \) on \( S_b \), so that \( \mathbf{u} \cdot \mathbf{n} = 0 \) on \( S_b \). However,
we must now include viscous terms \( \nu \nabla \mathbf{u} \) and \( \rho \nu \nabla \mathbf{\omega} \) on the right-hand sides of
(12) and (13), and this leads to

\[
\frac{dI}{dt} = -2\nu \int_V \mathbf{\omega} \cdot (\nabla \times \mathbf{\omega}) dV,
\]

(18)

where \( I \) is still defined as in (15) (so that it now includes a “surface contribution”
distributed through the boundary layer) and \( V \) is the total volume occupied by
the fluid.

Let \( L \) be the scale of variation of \( \mathbf{u} \) in the tangential directions (on \( S_b \)), and let
\( q_b \) be the scale of \( |\mathbf{u}| \) just outside the boundary layer. Then the thickness of the
layer is (in general)

\[
\delta = O(q_b \nu \delta),
\]

(19)

and the normal and tangential components of vorticity in the layer have orders of
magnitude

\[
\omega_n, |\omega \cdot \mathbf{n}| = O(q_b \nu \delta), \quad \omega_t = |\omega \times \mathbf{n}| = O(\nu \delta) .
\]

Hence

\[
|\omega_n \cdot \nabla \times \omega| = O(\nu \delta) = O(\omega, \delta^2 \delta),
\]

(20)

and so the contribution to \( dI/dt \) from unit area of the boundary layer is, from
(18), of order \( \omega_n \delta \) and this is independent of \( \nu \) in the limit \( \nu \to 0 \). The structure of
the boundary layer is therefore of critical importance as \( \nu \to 0 \) in determining not
only the value of \( I \), but also its rate of change \( dI/dt \).

4. A simple consequence of the invariance of \( I \) in an incompressible fluid

Henceforth we restrict attention to vorticity blobs in an inviscid incompressible fluid with \( \mathbf{u} \cdot \mathbf{n} = 0 \) on all solid boundaries. The integrals

\[
I = \int u \cdot \mathbf{\omega} dV, \quad E = \frac{\mathbf{\omega} \cdot \mathbf{\omega}}{\rho}, \quad \Omega = \int \omega^2 dV,
\]

(22)

satisfy the Schwarz inequality

\[
I^2 \leq E \Omega, \quad \Omega \geq \frac{I^2}{E},
\]

(23)

† This may be contrasted (in the incompressible case) with the behaviour of the energy

\[
T = \frac{\mathbf{\omega} \cdot \mathbf{\omega}}{\rho} = \frac{\mathbf{\omega} \cdot \mathbf{\omega}}{\rho} \int \mathbf{u} \cdot (\nabla \times \mathbf{\omega}) dV,
\]

for a localized vorticity blob, which satisfies

\[
\frac{dT}{dt} = -2\nu \int \mathbf{\omega} \cdot \mathbf{\omega} dV
\]

when \( S_b \) is stationary. As \( \nu \to 0 \), \( dT/dt = O(\nu) \) and \( T \to \text{const.} \), independent of the
boundary layer structure.
and since both \( I \) and \( E \) are invariants, \( \Omega \) has a fixed lower bound, which is evidently attained only if \( \omega = \alpha \xi \) where \( \alpha \) is constant.

To understand the physical significance of this result, consider the following situation. Suppose that a vortex ring propagates along an axisymmetric duct of decreasing cross-section, the axis of symmetry of the duct and of the ring coinciding. Evidently, the value of \( \Omega \) for this vorticity field may become arbitrarily small if the radius of the duct becomes sufficiently small; but since \( \omega \cdot \omega = \alpha \xi \) for this flow, \( I = 0 \) and there is no contradiction with (23). Suppose now instead that a blob of vorticity for which \( I = 0 \) is so disposed as to propagate into a similar contraction; is it then physically conceivable that the value of \( \Omega \) for the blob can be made to decrease without limit by choosing a contraction of suitable geometry? The answer is negative, consistently with (29), for the following reason. Since \( I = 0 \), there must exist knots or links in the vortex lines of the blob. No single Cartesian component of the vorticity field can then be identically zero (since curves confined to a plane cannot be knotted or linked).

Since the volume of the blob is constant, any decrease in the components of vorticity perpendicular to the axis of the duct is then necessarily accompanied by an increase (through stretching) of the vorticity component parallel to the axis, and it is therefore evident that \( \Omega \) cannot decrease indefinitely.

5. Relation with the magnetohydrodynamic invariants of Woltjer (1958a, b)

Let \( B = \nabla \times A \) and \( E = -\varepsilon A/\partial t - \nabla \phi \) be the magnetic field and electric field in a perfectly conducting fluid; then, since \( E + u \times B = 0 \),

\[
\varepsilon A/\partial t = u \times (\nabla \times A) - \nabla \phi, \tag{24}
\]

and

\[
\beta B/\partial t - \nabla \times (u \times B). \tag{25}
\]

Hence

\[
\frac{D}{Dt} \left( \frac{A \cdot B}{\rho} \right) - \frac{B}{\rho} \cdot \nabla (A \cdot u + \phi), \tag{26}
\]

which may be compared with (14). It follows that

\[
\int_A B \, dV = \text{const.} \tag{27}
\]

provided \( B, u = 0 \) on the surface \( S \) of \( V \). This result was proved (under slightly more restrictive conditions) by Woltjer (1958c). The interpretation of the invariant in terms of conservation of knottedness of magnetic lines of force (which are frozen in the fluid) is immediate. Note that the value of

\[
\int_A B \, dV
\]

† The term 'blob' of vorticity will be used to indicate a vorticity distribution \( \omega(x) \) that is entirely confined within some closed surface \( S \) of finite extent, i.e., \( \omega \equiv 0 \) outside \( S \).
is independent of the choice of gauge of $\mathbf{A}$; it is determined uniquely by the field $\mathbf{B}$ and the volume $V$.

The equation of motion is now

$$\rho(D\mathbf{u}/Dt) = -\nabla p + \mathbf{j} \wedge \mathbf{B},$$

(28)

where $\mathbf{j} = \nabla \wedge \mathbf{B}$. From (28) and the induction equation in the form

$$\frac{D}{Dt} \left(\frac{\mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho} \cdot \nabla (\mathbf{u} + \mathbf{a}^\prime),$$

(29)

we may deduce (as for the case of vorticity) that

$$\frac{D}{Dt} \left(\frac{\mathbf{B}, \mathbf{u}}{\rho}\right) = \frac{\mathbf{B}}{\rho} \cdot \nabla (\mathbf{u} + \mathbf{a}^\prime).$$

(30)

Hence

$$\int_V \mathbf{u} \cdot \mathbf{B} dV = \text{const.}$$

(31)

provided again that $\mathbf{B}, \mathbf{u} = 0$ on $S$ (cf. Wortel 1938a).

It is evident, from (7) and (8), and similar formulae for $\mathbf{A}$ in terms of $\mathbf{B}$, that

$$\int_V \mathbf{u} \cdot \mathbf{B} dV = -\frac{1}{4\pi} \sum \int_{\beta_i} \frac{\mathbf{R} \cdot \omega(x') \wedge \mathbf{B}(x)}{R^3} dV - \int_V \mathbf{A} \cdot \omega dV.$$

(32)

The integrals are determined by the fields $\omega$ and $\mathbf{B}$ (and of course by the volume of integration $V$); in order to emphasize this fact, it may be useful to introduce the notation

$$F[\omega, \mathbf{B}] = \int_V \mathbf{u} \cdot \mathbf{B} dV = F[\mathbf{B}, \omega].$$

(33)

Then also,

$$\int_V \mathbf{A} \cdot \mathbf{B} dV = F[\mathbf{B}, \mathbf{B}] = f[B]$$

(34)

If $\mathbf{B} = 0$ except in flux filament $C_1, C_2, \ldots, C_n$ with strengths $\Phi_1, \ldots, \Phi_n$, then

$$F[\omega, \mathbf{B}] = \sum \int_{C_i} \mathbf{u} \cdot d\ell - \sum \Phi_i K_i$$

(35)

where $K_i$ is the flux of vorticity through $C_i$. This quantity is constant, because, although Kelvin's theorem does not now hold for an arbitrary curve (the Lorentz force $\mathbf{j} \wedge \mathbf{B}$ being, in general, rotational) it does hold if $C$ is a closed $\mathbf{B}$-line; for then

$$\oint_C \mathbf{j} \wedge \mathbf{B} \cdot d\ell = 0,$$

(Shercliff 1955, problem 6.7). Hence again the integral $F[\omega, \mathbf{B}]$ may be interpreted as a measure of the degree of mutual knottedness of the two fields $\omega$ and $\mathbf{B}$; this remains constant even though the vortex lines are no longer frozen in the fluid.
6. Some examples of flows for which $I \neq 0$

The only situations considered so far which definitely give a non-zero value for $I$ are those of §1 in which discrete vortex filaments are linked or knotted. Such a configuration may seem physically artificial and unlikely and one might be tempted to conclude that flows with continuous vorticity and with $I \neq 0$ are unlikely to occur naturally. Such a conclusion would not however be justified.

The influence of viscosity near solid boundaries in causing changes in $I$ has already been mentioned. A blob of vorticity may be generated by the sudden acceleration of part of a solid boundary surrounding a fluid. If, say, a right-handed impulsive wrench is applied to an immersed body, then it is more than likely that some of the helicity imparted to the body will be transferred to the fluid via the vorticity shed from the boundary during the initial stages of the motion. The vortex lines then become linked during the shedding process and remain linked thereafter. The spiral trailing vortex system behind an advancing propeller provides perhaps the best example. Any advancing rotating body must likewise leave a helical vorticity distribution in its wake.

Some examples of flows with particular symmetries will help to clarify the character of the linkages that are likely to occur in situations of practical interest.

(a) Two-dimensional incompressible flow

For a velocity field of the form

$$\mathbf{u} = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, w \right),$$

(36)

where $\psi = \psi(x, y, t)$, $w = w(x, y, t)$, we have

$$\mathbf{u} \cdot \mathbf{\omega} = \nabla \psi \cdot \nabla w - w \nabla^2 \psi.$$  

(37)

Provided the flow is localized in the $(x, y)$-plane, $(|\mathbf{\omega}| = 6(x^2+y^2)$ as $r^2 = x^2+y^2 \to \infty$ is certainly a sufficient restriction, we may take $V$ to be the volume between any two planes $z = \text{const.},$ at unit distance apart, the contribution to the surface integral (17) from these two planes then cancelling, and the invariant $I$ degenerates to an integral over the $(x, y)$-plane,

$$I = \iint_{\mathbb{R}^2} (\nabla \psi \cdot \nabla w - w \nabla^2 \psi) \, dx \, dy = -2 \iint_{\mathbb{R}^2} w \nabla^2 \psi \, dx \, dy.$$  

(38)

The conditions for steady flow are

$$w = C(\psi), \quad \psi + \frac{1}{2} \rho^2 = E(\psi),$$

(39)

and

$$\nabla^2 \psi = \frac{\partial H}{\partial \psi} - C \frac{\partial C}{\partial \psi} = f(\psi) \quad \text{say.}$$

(40)

In this case,

$$I = -2 \iint_{\mathbb{R}^2} f(\psi) C(\psi) \, dx \, dy.$$  

(41)

$^\dagger$ i.e., an impulsive force $F$ and couple $G$ with $F \cdot G > 0.$
The simplest explicit example is perhaps that of the rectilinear vortex with an axial motion confined to the vortex:

\[ f(\psi) = \begin{cases} -\alpha & (r < a), \\ 0 & (r > a), \end{cases} \]

where \( r = (x^2 + y^2)^{1/2} \). In this case, \( I = 2\alpha Q \).

\[ Q = \iint w(x, y) dx dy. \]

(If it is assumed that \( Q \) is finite, then the vortex lines are closed at \( r = \infty \) is then unmaterial.) Thus \( I \) is 0 according as \( \alpha_i \) and \( Q \) have the same or opposite sign, that is, according as the sense of the net screw of the vortex is right-handed or left-handed.

The vortex lines of a steady flow of this type are helices which spiral round on the cylindrical surfaces \( H = \text{const.} \), or equivalently \( \psi = \text{const.} \). There must exist at least one point \( (x_0, y_0) \) at which \( \nabla \psi = 0 \), and the line \( x = x_0, y = y_0 \) is itself a vortex line, lying on a degenerate member of the family of surfaces \( \psi = \text{const.} \). This line may be termed a "vortex axis"; all vortex lines in the neighbourhood of a vortex axis spiral round it.

(b) Axisymmetric incompressible flow

Suppose now that, relative to cylindrical polar co-ordinates \((x, r, \phi)\), the velocity has components

\[ u = \left( \frac{1}{r} \frac{\partial \psi}{\partial x}, -\frac{1}{r} \frac{\partial \psi}{\partial z}, w \right). \]

where \( \psi = \psi(x, r, t) \), \( w = w(x, r, t) \).

Then

\[ u \cdot \omega = r^{-2} (V \nabla \psi \cdot \nabla (rw) - rw \partial \partial \psi), \]

where

\[ D^2 = \frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} + r^{-2} \frac{\partial^2}{\partial \phi^2}. \]

and for an axisymmetric blob of vorticity,

\[ I = 2\pi \iiint \left[ \frac{1}{r} \nabla \psi \cdot \nabla (rw) - w D^2 \psi \right] dx dr. \]

Provided \( |\nabla \psi| \) is everywhere finite, and \( o(R^{-3}) \) as \( R = (x^2 + y^2)^{1/2} \rightarrow \infty \), this may be transformed by means of the divergence theorem to

\[ I = -4\pi \int w D^2 \psi \, dV. \]

Hence the integrated product of the swirl and the azimuthal vorticity \(-r^2 D^2 \psi\) is invariant in any axisymmetric unsheared inviscid flow. The result may be regarded as a particular consequence of the fact that for any material toroidal filament, \( w \) and \( r^{-2} \partial \omega \psi - r^{-2} D^2 \psi \) are invariant; integration of the product \( r^{-1} w D^2 \psi \) over the volume of fluid, with \( dV = 2\pi r dr \, dx \) then gives the integral (48).

The conditions for steady flow in this case (Batchelor 1967, §7.5) are

\[ mw = C(\psi), \quad p/\rho + \frac{1}{2} v^2 = H(\psi), \]

(50)
The degree of incoherence of tangled vortex lines

\[ H^2 \psi = r^2 \frac{dH}{\psi} - C \frac{d\psi}{d\phi}. \]  (51)

An interesting family of solutions, each of which represents a blob of vorticity confined to the sphere \( R < a \), exists when \( G(\psi)^{\alpha} \) and \( H(\psi) \) have the simple forms

\[ H = H_0 + \lambda \psi, \quad C = \pm \alpha \psi, \]  (52)

where \( H_0, \lambda \) and \( \alpha \) are constants. In this case, (51) becomes

\[ D^2 \psi = \lambda \psi^2 - \alpha \psi, \]  (53)

and this admits the solution in spherical polar \((R, \theta, \phi)\),

\[ \psi = R^a \sin^a \theta \left[ \frac{\lambda}{x^2} + A \left( \frac{a}{R} \right)^2 J_2(\pi R) \right], \]  (54)

where \( A \) is a constant. There are other solutions of (53) with more complicated dependence on \( a \), but the interest of solutions of the form (54) is that they can be matched, by suitable choice of the constants \( \lambda, \alpha \) and \( A \) to an irrotational stream, represented by

\[ \psi = \frac{1}{2} U (R^a - (a^2 R^2)) \sin^2 \theta, \]  (55)

for \( R > a \). We have to satisfy

\[ \frac{\partial \psi}{\partial R} = 0, \quad \frac{\partial \psi}{\partial \theta} = \frac{1}{2} U a \sin^2 \theta, \quad \text{on} \quad R = a. \]  (56)

These ensure that the surface \( R = a \) is a stream-surface, and that the velocity is continuous across it; continuity of pressure, given by (59), can then also be satisfied. These conditions give respectively

\[ \frac{\lambda}{x^2} = -A J_2(a), \quad U = \frac{1}{2} A J_2(a), \]  (57)

and a doubly infinite family of solutions is obtained by varying the parameters \( A \) and \( x \). \( U \) is the speed at which the vortex propagates relative to the fluid at infinity.

Two possibilities deserve particular comment. If \( J_2(a) = 0 \), then \( \lambda = 0 \), and (52) and (53) together imply that \( \omega = \pm 2 \nu \); the resulting velocity field is then exactly analogous to the "force-free" magnetic field obtained (among others) by Chandrasekhar (1956). Secondly, if \( J_2(\pi a) = 0 \), then \( U = 0 \), and so the fluid is at rest outside the sphere \( R = a \); this is exactly analogous to the magnetostatic solution proposed by Poynting (1957) as a model for the equilibrium structure of a magnetic star (and described by Roberts 1967, §4.7).

The total helicity of the vortex described by (52), (54) is given by (49), i.e.

\[ I = -4\pi \int \left( \lambda \psi^2 - \alpha \psi \right) \sin \theta \cos \theta \, d\phi \, d\psi. \]  (58)

1 The governing equations in the magnetostatic problem are \( \mathbf{j} \wedge \mathbf{B} = \nabla \times \mathbf{u} \), \( \mathbf{j} = \nabla \times \mathbf{A} \), \( \nabla \cdot \mathbf{B} = 0 \), and the analogy with the situation under consideration here is that between the variables

\[ \mathbf{a} \rightarrow \mathbf{B}, \quad \omega \rightarrow \mathbf{j}, \quad R \rightarrow \mu. \]
After straightforward manipulation, this reduces to
\[
I = \pm \frac{1}{4} \rho \alpha^2 A_2 J_2(\alpha),
\]
where
\[
f(z) = \frac{1}{2} \left( z J_2(z)^2 - z J_2(z) J_2(z) - z J_1(z) J_1(z) \right).
\]
The function \( f(z) \) has the asymptotic behaviour
\[
f(z) \sim \begin{cases} 
\frac{n a^2}{1500} & \text{as } z \to 0, \\
\frac{\pi z^2}{4} & \text{as } z \to \infty.
\end{cases}
\]

Figure 3. Stream-surfaces, streamlines and vortex lines for the spherical vortices derived by (54), (52). In (b) the vortex is viewed from the direction of the stream at infinity.

Figure 4. One of the knotted vortex lines of the spherical vortex represented by the stream function (54) and the condition (52).

The choice of sign in (58) corresponds to the choice in (52); both 'right-handed' and 'left-handed' vortices are possible.

The surfaces \( \psi = \) constant for \( R < a \) consist in all cases of a family of nested tori, the sphere \( R = a \) itself being the limiting outer member of the family. The innermost member of each family of tori degenerates into a circle which is both a streamline and a vortex line; it is located on the plane \( \theta = \frac{1}{2} \pi \) at a point where \( \delta \psi / \delta R = 0 \), i.e. \( \psi = \psi_{\text{max}} \), say, (at least one such point exists). The surfaces \( \psi = \) const. are sketched in figure 3(a) for the simplest case in which there is only one such 'vortex axis' within the sphere. The streamlines and the vortex lines lie on these surfaces, as indicated in figure 3(b), which is a view of the eddy from along its axis of symmetry.

If any one vortex line is followed in the direction of increasing \( \psi \) the value of \( z \) on that line varies periodically; the pitch \( \psi \) of the vortex line may conveniently
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be defined as twice the increase in $\phi$, between successive zeros of $z$. This quantity clearly increases continuously from zero to infinity as $\phi$ increases from zero (on $R = r_0$) to $\phi_{\text{ext}}$ (on the vortex axis). If $p = 2m/n$, where $m$ and $n$ are integers prime to each other, then the vortex line will close on itself after traversing the smaller circumference of the torus $2n$ times and the larger circumference $2m$ times. Such vortex lines are self-knotted if $m \geq 2$, $n \geq 3$; the corresponding knot is known as the torus knot of type $m/n$. For example, if $m = 2$ and $n = 3$, so that $p = \frac{2}{3}$, the vortex line is in the form of the trefoil knot, as indicated in figure 4.

It is interesting that every torus knot is represented once and only once among all the vortex lines of each member of the family of flows represented by the stream function (54), together with the circulation (52).

It is a pleasure to acknowledge the stimulus of several discussions with Professor G. K. Batchelor on the topic of this paper. It was he in particular who recognized the physical significance of Wolter's second invariant, equation (31). I am also indebted to Dr K. J. Whiteman who drew my attention to the family of torus knots referred to in the last paragraph.

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The Asymptotic Hopf Invariant and Its Applications*

V. I. Arnold

The classical Hopf invariant distinguishes among the homotopy classes of continuous mappings from the three-sphere to the two-sphere and is equal to the linking number of the two curves that are the preimages of any two regular points of the two-sphere.

The asymptotic Hopf invariant is an invariant of a divergence-free vector field on a three-dimensional manifold with given volume element. It is invariant under the group of volume-preserving diffeomorphisms, and describes the "helicity" of the field, i.e., the mean asymptotic rotation of the phase curves around each other. The asymptotic Hopf invariant coincides with the classical Hopf invariant for the unitary vector field that is tangent to the Hopf bundle. In the general case the asymptotic Hopf invariant can have any real value (whereas the classical Hopf invariant is always an integer).

The asymptotic Hopf invariant can also be considered as a quadratic form on the Lie algebra of the volume-preserving diffeomorphisms of the three-dimensional manifold that is invariant under the adjoint action of the group on the algebra.

In this paper we present the definition and simplest properties of the asymptotic Hopf invariant, as well as some of its applications to an unusual variational problem that arises in magnetohydrodynamics which was called to the author's attention by Ya. B. Zel'dovich. In connection with this problem there arise a whole series of unsolved mathematical problems, some of which appear to be difficult. The main object of this paper is to discuss these unsolved problems; all the theorems in the paper are obvious.

Attention was first called to the problems considered here by Yudifjer [7] in connection with magnetohydrodynamics. Applications to ordinary hydrodynamics were given by Moffatt [4], [5] and Kraichnan [3].

1. The problem of the minimum magnetic energy of a frozen-in field

Let $M$ be a three-dimensional closed Riemannian manifold, and $\xi$ a divergence-free vector field on $M$. The energy of the field is the integral

$$E = \frac{1}{2} \langle \xi, \xi \rangle = \frac{1}{2} \int_M \langle \xi, \xi \rangle \, dv.$$ 

We are to find the minimum energy for fields obtained from a given field under the action of volume-preserving diffeomorphisms of the manifold $M$.

Here the action of a volume-preserving diffeomorphism $g: M \to M$ associates with a divergence-free field $\xi$ on $M$ another divergence-free field $g \ast \xi$ such that the flux of the field $\xi$ across any surface $\sigma$ is equal to the flux of $g \ast \xi$ across $g\sigma$. In other words, the field is frozen into a covering of $M$ by an incompressible fluid; the vector field can be thought of as drawn on the elements of the fluid and expanding as these elements expand.

The two-dimensional analog can be formulated as follows.

To find a function that minimizes the Dirichlet integral

$$E = \frac{1}{2} \langle \nabla u, \nabla u \rangle = \frac{1}{2} \int_M \langle \nabla u, \nabla u \rangle \, dv$$

among all functions $u$ on the closed two-dimensional Riemannian manifold $M$ obtained from a given function $u_0$ by the action of an area-preserving diffeomorphism on itself:

$$u(x) = u_0(g^{-1}x).$$

It is clear that similar problems arise for manifolds with boundary, for example, for functions $u(x, y)$ in an ordinary Euclidean disk. The mathematical aspects of investigations of these problems have been highly unsatisfactory.

1.1. The Euler equation

**Theorem.** The extremals of the problem stated above are divergence-free fields that commute with their curl.

**Proof.** Let $\eta$ be any divergence-free field. The variation of a field $\xi$ under the infinitesimal diffeomorphism defined by $\eta$ is given by the Poisson bracket $\delta \xi = \{\eta, \xi\}$ (in terms of the coordinates, $(\eta, \xi) = (\xi \nabla) \eta - (\eta \nabla) \xi$).

Consequently $\delta E = \langle \xi, \delta \xi \rangle = \langle \xi, \{\eta, \xi\} \rangle$. But, by a formula from vector analysis, $\text{curl} \{\eta, \xi\} = \{\eta, \xi\} - \eta \text{ div } \xi - \xi \text{ div } \eta$ on any three-dimensional Riemannian manifold. Since our fields $\xi$ and $\eta$ are divergence-free, $0 = \delta E = \langle \xi, \text{ curl} \{\eta, \xi\}\rangle = \langle \text{curl} \xi, \{\eta, \xi\}\rangle = \langle \eta, \{\xi, \text{ curl} \xi\} \rangle$. Since $\eta$ is divergence-free,
the vector product \([\xi, \text{curl} \, \xi]\) is orthogonal to all divergence-free fields. Consequently it is a gradient:

\[ [\xi, \text{curl} \, \xi] = \text{grad} \, \alpha, \]

whence, taking the curl of both sides, we obtain

\[ [\xi, \text{curl} \, \xi] = 0, \]

as was to be proved.

Remark 1. In the two-dimensional case we obtain the equation

\[ [\nabla u, \nabla \Delta u] = 0, \]

which says that the gradient of the extremal function is collinear with that of its Laplacian.

Remark 2. A similar calculation leads to the following expression for the second variation:

\[ \delta^2 E = \langle \langle \eta, \xi \rangle, \langle \eta, \xi \rangle \rangle + \langle \langle \eta, \xi \rangle, [\text{curl} \, \xi, \eta] \rangle \]

(where \(\xi\) is an extremal whose first and second variations are given by the formula

\[ \xi(t) = \xi(0) + \epsilon [\eta, \xi] + \frac{\epsilon^2}{2} \langle \eta, [\eta, \xi] \rangle + \cdots, \quad \epsilon \to 0, \]

in terms of a divergence-free vector field \(\eta\)).

1.2. Study of the extremal fields

Let \(\xi\) be a divergence-free field on a three-dimensional closed orientable Riemannian manifold \(M\) for which \([\xi, \text{curl} \, \xi] = \text{grad} \, \alpha\). All such fields are extremals for our problem. It turns out that the field lines of \(\xi\) have a very special topology.

Theorem. Every noncritical level set of the function \(\alpha\) is diffeomorphic to a torus (or union of tori). In a neighborhood of such a torus we can define coordinates \(\varphi_1, \varphi_2, z\) (mod \(2\pi\)), \(z\), such that \(\varphi\) is the angular coordinate on the torus, \(z\) indexes the torus, and the field \(\xi\) (as well as the field curl \(\xi\)) has components

\[ \xi = \omega_1(z) \frac{\partial}{\partial \varphi_1} + \omega_2(z) \frac{\partial}{\partial \varphi_2}; \quad \text{curl} \, \xi = \omega'_1(z) \frac{\partial}{\partial \varphi_1} + \omega'_2(z) \frac{\partial}{\partial \varphi_2}. \]

Here the coordinate \(z\) can be chosen so that the volume element has the form \(d\varphi_1 \times d\varphi_2 \times dz\).

Remark. The coordinates \(\varphi_1, \varphi_2, z\) are analogs of the action-angle variables of classical mechanics. The theorem means, in particular, that both the field lines of \(\xi\) and of curl \(\xi\) lie on the tori \(\alpha = \text{const}\). These lines are either closed (if the relative frequency of \(\omega\) is rational) or dense on the torus.
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For the proof see [1]. It follows from the theorem that, for example, in the analytic case, when \( a \neq 0 \) the manifold \( M \) is divided by the critical level sets of \( a \) into a finite number of cells, homeomorphic to the product of the torus by intervals in each of which the fields \( \xi \) and \( \text{curl} \ \xi \) are tangent to the torus and generate periodic or conditionally periodic windings of the torus. Consequently, we obtain an explicit description of the topology of the field \( \xi \) (or \( \text{curl} \ \xi \)).

It remains to consider the case when \( \text{grad} \ a = 0 \). In this case \( [\xi, \text{curl} \ \xi] = 0 \), i.e., the fields \( \xi \) and \( \text{curl} \ \xi \) are colinear at each point. Such fields are called force-free fields in magnetohydrodynamics.

If a force-free field \( \xi \) is non-zero, then \( \text{curl} \ \xi = \psi \), where \( \psi: M \to \mathbb{R} \) is a smooth function. But \( \text{div} \ \text{curl} \ \xi = 0 \); consequently, \( (\text{grad} \ c, \xi) = 0 \), i.e., the function \( c \) is a first integral of the field \( \xi \) (and also of \( \text{curl} \ \xi \)). Hence it follows that the connected components of the nonsingular level surfaces of \( c \) are tori, and the field lines of \( \xi \) are windings on these tori (in the corresponding coordinates \( \varphi_1, \varphi_2, \varphi_3 \), the constants along the field lines of \( \xi \) will be the frequency ratios, \( \omega_1/\omega_2 = \kappa(z) \)). Therefore even in the case of a force-free field the field lines lie on two-dimensional tori, provided that the field does not have zeroes and \( c \) is not constant.

A force-free field with \( \text{curl} \ \xi = \lambda \xi \), where \( \lambda \) is a constant, can have a much more complicated topology. An example of such a field on the three-dimensional torus \( \{x, y, z, \text{mod} \ 2\pi\} \) is given by the components

\[
\xi_x = A \sin x + C \cos y, \quad \xi_y = B \sin x + A \cos y, \quad \xi_z = C \sin y + B \cos z.
\]

The topology of these field lines was investigated experimentally by Henon [2], using the computer at the Astrophysical Institute at Paris. As a result he discovered a set of tori filled out by field lines ("magnetic surfaces") together with whole domains of three-dimensional space whose field lines, as far as one can tell from the experimental data, are ergodic, or everywhere dense.

1.3. Discussion

Returning to our extremal problem, we see that a field of minimum energy in a given class of frozen-in fields must either have a very special topology (the field lines fill out tori), or be force-free fields of a special kind. But the topological properties of the field lines are invariant under diffeomorphisms, and therefore if the original field is a general one, then every field obtained from it by a diffeomorphism has the same property. Consequently, a field of minimum energy either does not exist (in the class of smooth fields to which the preceding analysis applies) or is a force-free field of special type.

But force-free fields with \( \text{curl} \ \xi = \lambda \xi \) are source; they are eigenvectors of the field of the operator \( \text{curl} \) on the space of divergence-free fields. Hence
we must assume that our variational problem apparently does not always admit a smooth solution.

In this connection we consider the following example. Let $M$ be a sphere in three-dimensional Euclidean space, and let the field lines of $\xi$ be horizontal circles with centers on the vertical axis. According to Zeldovich, the energy of such a field can be made arbitrarily small by means of a suitable deformation which preserves volumes and is fixed in a neighborhood of the boundary. In fact, let us divide the whole sphere into a number of slender solid tori (doughnuts) formed from the circles of the field, and a remainder of small volume. Then let us deform (preserving its volume) each solid torus (violating the axial symmetry of the field) so that it becomes fat and small, with the hole decreasing almost to zero. Then the field energy in the solid tori is decreased (since the field lines are shortened). It can be seen that the whole construction can be carried out in such a way that the field energy in the remaining small volume is not increased by too much, as a result the total energy remains arbitrarily small.

It would be of interest to carry out this construction precisely.

In connection with this example, there arises the question of whether it is possible to reduce the energy of an arbitrary field to an arbitrarily small value by an appropriate volume-preserving deformation. We shall see below that this is not the case. An obstacle to the complete annihilation of the energy can be constructed by considering two linked doughnuts of field lines. In this case the shortening of the field lines in one doughnut, shrinking its hole, induces a lengthening of the field lines in the other, so that there is an obstacle to the decrease of the energy. The asymptotic Hopf invariant, which measures the linking of the field lines (not necessarily closed) lets us give a qualitative expression for this situation in the form of a lower bound for the energy.

1.4. Magnetohydrodynamic discussion

In magnetohydrodynamics the role of $\xi$ is played by the magnetic field $H$, frozen into a fluid of finite viscosity, but of infinite conductivity, which fills $M$. With an appropriate choice of units, the velocity field $u$ and the magnetic field $H$ satisfy the system of equations

$$\frac{\partial u}{\partial t} + (\nabla p, u) = \text{grad} p - \nu \Delta u + |\nabla H|,$$

$$\text{div} u = 0,$$

$$\frac{\partial H}{\partial t} = [u, H], \quad \text{div} H = 0, \quad \text{curl} H = j.$$

The magnetic field $H$ and the velocity field $u$ are prescribed at the initial time. In the course of time, the kinetic energy is dissipated because of the viscosity, and the motion ceases "in the end," since each particle approaches
some terminal position. The magnetic field, being frozen in, then attains
some terminal value. The energy of this terminal field must be a minimum;
otherwise the magnetic energy would have been converted into kinetic
energy and, on account of the Lorentz force, the fluid would move until it
dissipated the excess of magnetic energy above the minimum.

This sort of description of the behavior of solutions of the system
presented above is usually given by physicists. Unfortunately, the preceding
analysis of the topology of the extremal fields holds out little hope that this
description is correct for any general initial conditions: in fact, the initial
magnetic field can be taken without having magnetic surfaces, and then the
limiting field, if there is one, must be a force-free field of special type, but
such fields are too scarce, and one would hardly find a field with the
prescribed lines of force among them.

It appears that for a correct description of the actual process it is necessary
to take account of the magnetic viscosity, which violates the assumption
that the field is frozen in, and was not taken into account in our system of
equations.

The question of the extent to which one can use the extremal field to
study the behavior of $H$ over an extended period of time during which the
ordinary viscosity succeeds in extinguishing the motion of the fluid, but the
magnetic viscosity does not extinguish $H$, is an interesting unsolved problem.

Zel'dovich proposed the problem of the minimum magnetic field in
connection with the question of the evolution of the magnetic field of a
star. In this case $M$ is a sphere in three-dimensional Euclidean space, and
the field is propagated over the whole space with the boundary conditions

$$\text{curl } H = 0 \text{ outside } M, \quad \text{div } H = 0 \text{ outside } M,$$

$$(H, n) \text{ is continuous on } \partial M,$$

and the condition of decrease at infinity. Consequently, the volume-
 preserving diffeomorphism of $M$ acts on the field $H$ throughout the whole
space. It is necessary to minimize the total energy of the field $H$ (i.e., the
integral over all space). The minimizing field must provide a minimum of
the magnetic energy inside $M$ with respect to fields obtained from the given
diffeomorphism and stationary near the boundary.

We will not discuss the question of how close this simple model is to
reality. In what follows we restrict ourselves to a more simple system, in
which $M$ is a manifold without boundary.

2. Definition of the invariant

We begin with a dogmatic presentation: we consider an ad hoc definition
of the invariant, and establish its simplest properties. The interesting
meaning of the invariant (and an explanation of how the invariant was found)
will be discussed in the following sections.
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Let $M$ be a three-dimensional manifold that is closed (compact, without boundary), oriented, and connected, and let $v$ be the volume element (i.e., a 3-form defining the correct orientation) on $M$. It will be convenient to assume that the total volume of $M$ is $1$. Notice that we are given a volume element on $M$, but we are not given any particular Riemannian metric.

2.1. Notation

Every vector field $\xi$ on $M$ generates a differential 2-form $\omega_\xi$ according to the formula

$$\omega_\xi(\eta, \eta') = v(\xi, \eta, \eta')$$

for all $\eta, \eta'$. The correspondence $\xi \mapsto \omega_\xi$ is an isomorphism of the linear spaces of fields and 2-forms. The derivative of the form $\omega_\xi$, as for every 3-form, can be written in the form

$$d\omega_\xi = \varphi \eta$$

where $\varphi : M \to \mathbb{R}$ is a smooth function. The function $\varphi$ is called the divergence of the field $\xi$:

$$\varphi = \text{div} \ \xi.$$

The velocity field of a flow that preserves the volume element on $M$ is divergence-free; and, conversely, every flow with divergence $0$ on $M$ is the velocity field of an incompressible flow (i.e., of a flow that preserves the volume element $v$ on $M$).

A divergence-free vector field $\xi$ on $M$ is said to be homologous to zero if the 2-form $\omega_\xi$ corresponding to it is the total differential of a 1-form $\alpha$ on $M$:

$$\omega_\xi = d\alpha.$$

The 1-form $\alpha$ will be called a form-potential. A field is homologous to zero if and only if its flux across every closed surface is zero.

Remark. If a Riemannian metric is given on $M$ then the 1-form $\alpha$ can be identified with the vector field $a$ for which

$$\alpha(\eta) = (a, \eta) \quad \text{for every } \eta.$$

In this case $\xi = \text{curl} \ a$, and the vector field $a$ is called the vector potential of $\xi$. However, it is essential to observe that the forms $\omega$ and $\alpha$ (in contrast to the field $a$) do not depend on the Riemannian metric, but only on the choice of the volume element $v$.

2.2. Definition

The (mean) Hopf invariant of a field $\xi$ that is homologous to zero on the three-dimensional manifold $M$ with volume element $v$ is the integral of the
product of the form $\omega_\xi$ and its form potential, i.e., the number

$$I(\xi) = \int_M \alpha \wedge \delta a, \quad \text{where} \quad \omega_\xi = \delta a.$$ 

Let us show that this definition is consistent, i.e., that the value of $I$ does not depend on the particular choice of the form potential $\alpha$, but only on the field $\xi$.

In fact, if $\beta = \alpha + \gamma$ is another form potential, then $d\gamma = 0$, and therefore

$$\int_M \alpha \wedge \delta a - \beta \wedge \delta b = \int_M \gamma \wedge \delta a = \int_M \delta (\gamma \wedge a)$$

$$= \int_M \gamma \wedge a = 0.$$ 

Remark. If a Riemannian metric with volume element $v$ is given on $M$, then

$$I(\xi) = \int_M (\xi, a) dv = (\xi, \text{curl}^{-1} \xi),$$

where $a$ is any vector potential of $\xi$. Therefore $I$ is the scalar product of the field with its vector potential. It is essential to observe, however, that the Riemannian metric does not enter into the definition of $I$.

2.3. Invariance

Corollary. Every volume-preserving diffeomorphism $\varphi: M \to M$ carries every field $\xi$ that is homologous to zero into a field with the same Hopf invariant.

In particular, on a Riemannian manifold the scalar product of a divergence-free field and its vector potential is preserved when the field is acted on by a volume-preserving diffeomorphism.

Consequently the invariance of $I$ under diffeomorphisms that preserve the volume element follows from the fact that $I$ can be defined by using no structures other than the smooth structure of $M$ and the volume element $v$.

Remark. The question of whether $I$ is preserved under homeomorphisms that preserve the volume element (transforming the phase flow of $\xi$ into the phase flow of another field $\xi'$) is an interesting unsolved problem, as is the closely related problem of whether one can define the invariant $I$ directly for one-parameter groups of homeomorphisms that preserve the volume element.

Remark. In the case when $M$ is a manifold with boundary, the number $I$ is preserved under all volume-preserving diffeomorphisms that are stationary in a neighborhood of the boundary. If, however, $\xi$ is tangent to the boundary, then $I$ is preserved under all volume-preserving diffeomorphisms provided that $M$ is simply connected. The question of whether one can define an
invariant analogous to $I$ for general divergence-free fields on a manifold with boundary (including a surface term in $I$) is an interesting unsolved problem.

2.4. Examples

If we take $\xi$ to be a magnetic field, we arrive at the conclusion that the Hopf invariant of a magnetic field frozen into an incompressible fluid that fills a closed manifold does not change during any flow of the fluid.

If we interpret the field $\xi$ as the vorticity field of a perfect fluid, we obtain the result that in the flow of a perfect fluid on a closed three-dimensional manifold, the scalar product of the velocity field and the vorticity field does not change with time.

If we consider the field $\xi$ as an element of the Lie algebra of the group $S \text{Diff} M$ of volume-preserving diffeomorphisms of the three-dimensional manifold $M$, we obtain the result that on the Lie algebra of the group $S \text{Diff} M$ there is a symmetric bilinear form that is invariant with respect to the corresponding action of the group on the algebra. If we give a Riemannian metric on $M$ then

$$I(\xi, \eta) = \langle \xi, \text{curl}^{-1} \eta \rangle,$$

where $\text{curl}^{-1} \eta$ is the vector potential of the field $\eta$. In particular, for every divergence-free field $\eta$ we have

$$\langle \xi, \eta \rangle, \text{curl}^{-1} \xi \rangle = 0,$$

which is, of course, easily verified by direct calculation.

For a two-dimensional manifold $M$ we obtain a skew-symmetric form instead of a symmetric form.

3. Asymptotics of the coefficient of linking with a curve

Let $M$ be a closed connected oriented and simply connected three-dimensional manifold with volume element $\nu$, let $\gamma$ be a smoothly embedded closed orientable curve in $M$, and let $\xi$ be a divergence-free vector field on $M$. We define an asymptotic coefficient of the linking of the field lines of $\xi$ that issue from the point $x$ with the curve $\gamma$. Let $\{g^t: M \to M\}$ be the phase flow of the field $\xi$. Select a 2-chain $\sigma$ (of smooth simplexes) for which $\partial \sigma = \gamma$.

3.1. Asymptotic linking coefficient

For every pair of points $x, y$ of $M$ we introduce a "short curve" $\Delta(x, y)$ that joins these points and has the following properties:

1. If $x$ and $y$ do not belong to $\gamma$, then $\Delta$ does not intersect $\gamma$. 

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(2) The number of intersections of $\Delta(x, y)$ with $\gamma$ is bounded by a constant independent of $x$ and $y$.

It is easy to construct such a system of "short curves;" the dependence of $\Delta$ on $x$ and $y$ can be made measurable (and even piecewise smooth).

We fix a system of curves $\Delta$ and consider the segment of the orbit $g^t x$ of $x$ corresponding to $0 \leq t \leq T$. We join the last point $g^t x = y$ with the first by $\Delta(y, x)$; then we have a closed curve $\Gamma_T(x)$. We assume that this curve does not intersect $\gamma$.

Let $N_T(x)$ denote the linking coefficient of $\Gamma_T(x)$ with $\gamma$ (i.e., the index of the intersection of $\Gamma_T(x)$ with $\alpha$).

**Theorem.** For almost all $x$ in $M$ the limit

$$\lim_{T \to \infty} \frac{1}{T} N_T(x) = \lambda(x)$$

exists (where $T$ runs through the values for which $\Gamma_T(x)$ does not intersect $\gamma$). This limit belongs to $L_1(M, \nu)$ and, as an element of $L_1$, is independent of the system of curves $\Delta$.

The limit $\lambda(x)$ is called the asymptotic linking number of the orbit $g^t x$ with the curve $\gamma$.

To prove the theorem it is convenient to give a different definition of the asymptotic linking number, and then prove that it is equivalent to the definition given above.

### 3.2. Second definition of the asymptotic linking number

On the manifold $M - \gamma$ we can construct a closed differential 1-form $\alpha$ with the following properties:

(1) The linking number with $\gamma$ of every closed curve $\delta$ in $M - \gamma$ is equal to the integral of $\alpha$ over $\delta$.

(2) There is a diffeomorphic embedding $u : S^1 \times D^2 \to M$ of the direct product of a circumference and a disk into $M$ such that the circumference $S^1 \times 0$ maps to $\gamma$ and the form $\alpha$ induces, on the complement of this circumference, the standard form $\alpha = (1/2\pi) \arctan(y/x)$ (where $x, y$ are the coordinates in $D^2$).

We select a form $\alpha$ with these properties, and consider the limit

$$\lambda(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha \left( \frac{d}{dt} g^t x \right) dt$$

**Theorem.** The limit exists for almost all $x$ and is independent of the choice of the 1-form $\alpha$ satisfying hypotheses (1) and (2).
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**Proof.** Consider the function

\[ f(x) = \alpha(\xi(x)), \quad \text{where} \quad \xi(x) = \frac{d}{dt} \bigg|_{t=0} g^t x. \]

This function belongs to the space \( L_1(M, \nu) \) by condition (2) on \( \alpha \). By Birkhoff's ergodic theorem, the time average of \( f \) exists almost everywhere. This establishes the first part of the theorem, since \( \tilde{\lambda}(x) \) is the time average of \( f \).

To prove the second part, we observe that \( \alpha \) is defined on \( M - \gamma \) up to a differential of a single-valued function. If \( \psi \) is a smooth function on \( M - \gamma \), then

\[ \int_0^T \psi \left( \frac{d}{dt} g^t x \right) dt = \psi(g^T x) - \psi(x). \]

Now we observe that if \( g^t x \) does not approach \( \gamma \) asymptotically as \( t \to \infty \), we can choose a sequence \( T_n \to \infty \) such that the distance of the points \( g^T_n x \) from \( \gamma \) remains bounded below. But \( \phi \) is bounded above by a constant \( C \epsilon \) outside an \( \epsilon \)-neighborhood of \( \gamma \). Consequently, for all points \( x \) that are not asymptotic to \( \gamma \) there is a sequence \( T_n \to \infty \) along which \( \psi(g^T_n x) \) is bounded above. Therefore the limit \( \tilde{\lambda}(x) \) is the same for any two 1-forms \( \alpha \) for all points \( x \) except those asymptotic to \( \gamma \) (and those points for which one of the limits does not exist). But the points asymptotic to \( \gamma \) form a set of measure 0 (since the field \( \xi \) is divergence-free), and we have established that \( \tilde{\lambda} \) is independent of \( \alpha \) for almost all \( x \).

### 3.3. Equivalence of the definitions

The theorem of Section 3.1 is a consequence of the following theorem.

**Theorem.** For almost all \( x \), the limit \( \lambda(x) \) exists and is equal to \( \tilde{\lambda}(x) \).

**Proof.** By property (1) of the form \( \alpha \), it is enough to prove that for almost all \( x \)

\[ \lim_{T \to \infty} \frac{1}{T} \int_{\Delta(x)} \alpha(\xi) \, dt = 0. \]

But since the number of intersections of \( \Delta \) and \( \sigma \) is bounded (see property (2) of the curves \( \Delta \)), it follows that the integrals of \( \alpha \) along \( \Delta \) are uniformly bounded; consequently, the limit (over a sequence of values of \( T \) for which \( \Delta \) does not intersect \( \gamma \)) is zero; this establishes the theorem.

**Remark.** We have simultaneously proved that \( \lambda(x) \) is independent of the family of short curves \( \Delta \).
Remark. It is obvious from the theorem that the asymptotic linking number is invariant under volume-preserving diffeomorphisms, in the sense that if a diffeomorphism $h$ carries the system $(M, v, \gamma, \xi, x)$ to $(M', v', \gamma', \xi', x')$ then

$$\lambda_{\xi, \gamma}(x) = \lambda_{\xi', \gamma'}(x').$$

The question of whether the asymptotic linking number is invariant under volume-preserving homeomorphisms is an unsolved problem, as is the related question of the possibility of defining an asymptotic linking number with a topological curve $\gamma$ for a one-parameter group of volume-preserving homeomorphisms.

3.4. The mean linking number with a curve

Let $\{g^t\}$ be the phase flow of a divergence-free field $\xi$ on a simply connected three-dimensional manifold $M$ with volume element $v$. Let $\gamma = \partial \sigma$ be an oriented smooth curve in $M$, and let $\sigma$ be a piecewise smooth 2-chain. The mean linking number of $\{g^t\}$ with $\gamma$ is the average of the asymptotic linking number with respect to $M$:

$$\lambda = \int_M \lambda(x) v.$$

Theorem. The mean linking number $\lambda$ is equal to the flux of the field $\xi$ through the surface $\sigma$.

Proof. The number $\lambda(x)$ is the time average of $f(x) = c(\xi(x))$. Consequently, the space averages of $f$ and $\lambda$ are the same, i.e.,

$$\lambda = \int_M c(\xi) v = \int_M \sigma \wedge \omega.$$

Now the theorem follows from the homology of the 2-chain $\sigma$ and the 1-form $\omega$ as de Rham flows in $M - \gamma$ (strictly speaking, we should consider not $M - \gamma$, but the complement in $M$ of an $\epsilon$-neighborhood of $\gamma$, and then let $\epsilon \to 0$).

Remark. One can obtain similar results for the case when $\gamma$ is not smoothly embedded, but is a piecewise smooth curve. In addition, one can assume that $M$ is $n$-dimensional and that the chain $\gamma$ is $(n-2)$-dimensional.

4. Asymptotic linking number of a pair of trajectories

Let $M$ be a three-dimensional closed simply connected manifold with volume element $v$, let $\xi$ be a divergence-free field on $M$, and let $\{g^t\}$ be its phase flow.
4.1. Definition of the asymptotic linking number of a pair of trajectories

We consider a pair $x_1, x_2$ of points of $M$. We are going to associate with this pair of points a number that characterizes the “asymptotic linking” of the trajectories of $\{x^t\}$ that issue from them. For this purpose we first join any two points of $M$ by a “short path” connecting the points (the conditions imposed on a short path were described above and are satisfied for “almost any” choice of the short path).

We select two large numbers $T_1$ and $T_2$ and close the segment $g^t x_1$ ($0 \leq t \leq T_2$) of the trajectories issuing from $x_1$ and $x_1$ by short paths $\Delta g^T x_1, x_2$ ($k = 1, 2$) so that we obtain two closed curves $\Gamma_1 = g_{T_1}, x_1$). We assume that these curves do not intersect (which is true for almost all pairs $x_1, x_2$ for almost all $T_1, T_2$). Then the linking number $N_{\Gamma_1, \Gamma_2}(x_1, x_2)$ of $\Gamma_1$ and $\Gamma_2$ is defined as follows.

**Definition.** The asymptotic linking number of the pair of trajectories $g_{T_1}, g_{T_2}$ is defined as the limit

$$\lambda(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{N_{\Gamma_1, \Gamma_2}(x_1, x_2)}{T_1, T_2}$$

($T_1$ and $T_2$ are to vary so that $\Gamma_1$ and $\Gamma_2$ do not intersect).

We are going to prove that this limit exists almost everywhere and is independent of the system of short paths (as an element of $L_1(M \times M)$).

4.2. Digression on Gauss's formula

It will be useful to have the formula given by Gauss for the linking number of two closed curves in three-dimensional Euclidean space. There is also a similar formula for a simply connected manifold; see de Rham's book *Variétés différentiables*.

In order to state Gauss's formula, we introduce the following notation.

Let $x_1: S^1 \to \mathbb{R}^3$ and $x_2: S^1 \to \mathbb{R}^3$ be smooth mappings of a circumference in three-dimensional Euclidean space, with disjoint images. Let $t_1 (\mod T_1)$ and $t_2 (\mod T_2)$ be coordinates on the first and second circumference; then we denote by $x_1 = x_1(t_1)$ the velocity vector of the flow on the first, and by $x_2 = x_2(t_2)$ that on the second.

We assume that the circumferences are oriented by the choice of the parameters $t_1$ and $t_2$, and we fix an orientation for $\mathbb{R}^3$. Then we can define vector products and triple scalar products in $\mathbb{R}^3$.

**Gauss's Theorem.** The linking number of the closed curves $x_1(S^1)$ and $x_2(S^2)$
is equal to

\[ N_{1,2} = -\frac{1}{4\pi} \int_{t_1}^{t_2} \frac{t_1(x_1, x_2) - x_2(x_1)}{|x_1 - x_2|} \, dt_1 \, dt_2. \]

**Proof.** Consider the mapping

\[ f: T^2 \to S^3 \]

of the torus on the sphere, making a pair of points on our circumferences correspond to the vector of unit length directed from \( x_2(t_2) \) to \( x_1(t_1) \):

\[ f = \frac{F}{\|F\|}, \text{ where } F(t_1, t_2) = x_1(t_1) - x_2(t_2). \]

We orient the sphere by the inner normal and the torus by the coordinates \( t_1, t_2 \). The degree of the mapping is equal to the linking number \( N_{1,2} \). In fact, this is true for widely separated small circumferences: both the linking number and the degree of the mapping \( f \) are 0. Furthermore, it is easy to verify that under a deformation of a curve by any passage of one curve through another both the linking number and the degree of the mapping change by 1, in the same direction. Now the equation \( N_{1,2} = \deg f \) is established, in view of the connectedness of the set of smooth mappings \( S^1 \to \mathbb{R}^3 \).

Let us show that the degree of the mapping \( f \) is given by the integral formula of Gauss. In fact, by the definition of the degree,

\[ \deg f = -\frac{1}{4\pi} \int f \cdot \omega^2, \]

where \( \omega^2 \) is the area element on the unit sphere. By the definition of \( f \), the value of the form \( f \cdot \omega^2 \) on the pair of vectors \( \xi_1, \xi_2 \), tangent to the torus, is equal, at \( t \), to its triple scalar product with the vector \(-f = -f(t)\) (we oriented the sphere by the inner normal),

\[ \omega^2(f \cdot \xi_1, f \cdot \xi_2, f \cdot \xi_3) = (f \cdot \xi_1, f \cdot \xi_2, f \cdot \xi_3, -f). \]

Differentiating \( f \), we obtain \( f \cdot \xi = F \cdot \xi / \|F\| + o(\xi) \), and therefore

\[ \omega^2(f \cdot \xi_1, f \cdot \xi_2, f \cdot \xi_3) = (F \cdot \xi_1, F \cdot \xi_2, F \cdot \xi_3, -F)/\|F\|^3. \]

Since \( F = x_1 - x_2 \), we obtain, for an element of the spherical image of the torus, the expression

\[ f \cdot \omega^2 = -(x_1, x_2, x_1 - x_2) \|x_1 - x_2\|^3 \, dt_1 \, dt_2, \]

as was to be shown.

### 4.3. A second definition of the asymptotic linking number

Let \( \phi' \) be a phase flow, defined by a divergence-free field \( \xi \) in a three-dimensional compact Euclidean domain \( M \). The field is assumed to be
tangent to \( M \) on the boundary of \( M \). We set

\[
\lambda(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} \frac{1}{4\pi} \left[ \int_0^{t_1} \frac{(z(t_1), z(t_2), z(t_1) - z(t_2))}{\|x_i(t) - x_j(t)\|^3} \, dt_1 \, dt_2 \right] dt_1 \, dt_2.
\]

We shall show that:

1. the limit \( \lambda(x_1, x_2) \) exists almost everywhere on \( M \times M \);
2. the number \( \lambda(x_1, x_2) \) coincides (or almost all \( x_1, x_2 \), with the number \( \lambda(x, y) \) defined above.

To prove the first statement it is enough to verify that \( \lambda \) is the “time average” of a summable function on the manifold \( M \times M \), on which the commutative group \( \{g^s\} \times \{g^s\} \) acts.

The integrand

\[
\Phi(x_1, x_2) = \left( \frac{d}{dt} \right)_{t=0} \left( g^{t}x_1 \right)
\]

has a singularity on the diagonal of \( M \times M \) of order no higher than \( r^{-2} \) (where \( r \) is the distance to the diagonal); since the codimension of the diagonal is 3, the function \( \Phi \) belongs to the space \( L_1(M \times M) \), as was to be proved.

To compare \( \lambda \) with \( \lambda \) we represent the linking coefficient of the curves \( \Gamma_{T_1, x_1} \) and \( \Gamma_{T_2, x_2} \) by Gauss’s integral with \( 0 \leq t_1 \leq T_1, 0 \leq t_2 \leq T_2 \), and using the value of the parameter \( t \) from \( T_k \) to \( T_{k+1} \), for parametrizing the “short path” that joins \( g^{t}x_1 \) to \( x_2 \).

**Definition.** A system of short paths joining the points \( x, y \in M \) is a system of paths, depending in a measurable way on \( x \) and \( y \), such that the integrals of Gauss type for every pair of nonintersecting paths of the system, and also for any nonintersecting pairs (paths of the system, segments of phase curves \( g^{t}x_0 \mid 0 \leq t \leq \tau < 1 \)), are bounded independently of the paths by a constant \( c \).

It is easy to verify that systems of short paths exist (it is useful to keep in mind that an integral of Gauss type for a pair of straight-line segments remains bounded when the segments approach each other).

Now the difference

\[
\left( \int_0^{T_2} \int_0^{T_1} - \int_0^{T_1} \int_0^{T_2} \right)
\]

of integrals of Gauss type can be estimated by the sum of at most \( \tau \left( \int_0^{T_1} + \int_0^{T_2} \right) + 3 \) terms, none of which exceeds \( c \).

Consequently,

\[
\lambda(x, y) - \lambda(x, y) = \lim_{t, r \to \infty} \frac{1}{T_1 T_2} \left( \int_0^{T_1} \int_0^{T_2} - \int_0^{T_1} \int_0^{T_2} \right)
\]
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(\text{where } T_1 \text{ and } T_2 \text{ tend to infinity on any sequence for which the curves } \Gamma_{T_1} \text{ and } \Gamma_{T_2} \text{ do not intersect}).

4.4. Average linking number

\textbf{Theorem.} The mean value of the asymptotic linking number of a pair of trajectories,

\[ \int \int_{M \times M} \lambda(x_1, x_2) \, dv_1 \, dv_2 / \int \int_{M \times M} dv_1 \, dv_2, \]

is equal to the asymptotic Hopf invariant of the phase velocity field,

\[ I(\xi) = (\text{curl}^{-1} \xi, \xi). \]

\textbf{Proof.} Consider the Biot-Savart integral

\[ \eta(x_1) = -\frac{1}{4\pi} \int_{M} \frac{[\xi(x_1), x_1 - x_2]}{||x_1 - x_2||^3} \, dv(x_2), \]

where \([ , ]\) denotes the vector product. Then \(\text{curl} \, \eta = \xi\) and therefore

\[ \langle \eta, \xi \rangle - (\text{curl}^{-1} \xi, \xi) = -\frac{1}{4\pi} \int \int_{M \times M} \frac{[\xi(x_1), \xi(x_2), (x_1 - x_2)]}{||x_1 - x_2||^3} \, dv(x_1) \, dv(x_2), \]

as was to be proved.

\textbf{Remark.} There is a similar result for any compact simply connected three-dimensional Riemannian manifold \(M\), but the Gauss integral has to be replaced by the integral of de Rham's "double form;" this form cannot be written as explicitly, but has similar properties.

\textbf{Remark.} The question of whether the asymptotic and mean linking numbers are invariant for a pair of trajectories under homeomorphisms that preserve the volume element remains open, as does the closely related question of whether one can define asymptotic and mean linking numbers for trajectories of one-parameter groups of volume-preserving homeomorphisms.

5. Applications to the variational problem

From the existence of the Hopf invariant there follow some lower bounds for the energy of any field obtained from a given field by a volume-preserving diffeomorphism. In particular, on any three-dimensional Riemannian manifold one can find a field that is minimal in its class. In particular, certain special force-free fields have this property.
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5.1. Minimal force-free fields

Let \( M \) be a three-dimensional closed Riemannian manifold. We consider
the operator \( \text{curl} \) on divergence-free fields that are homologous to zero (i.e.,
have a single-valued divergence-free potential). By Weyl's lemma on
orthogonal projections, we can define a single-valued inverse of the operator
\( \text{curl} \) on our space, so that there is an inverse (integral) operator \( \text{curl}^{-1} \)
from the space of divergence-free fields that are homologous to zero, onto itself.
This operator is symmetric, and its spectrum accumulates at zero on both
sides.

**Theorem.** The eigenfield of \( \text{curl}^{-1} \) corresponding to the eigenvalue \( \nu \) of largest
modulus has minimum energy in the class of divergence-free fields obtained
from the eigenfield under the action of volume-preserving diffeomorphisms.

**Proof.** Let \( \nu_\text{min} \) and \( \nu_\text{max} \) be the smallest and largest eigenvalues of the operator
\( \text{curl}^{-1} \). Then for every field \( \xi \) that is homologous to zero we have
\[
0 \leq (\xi, \xi) \leq (\text{curl}^{-1} \xi, \xi) \leq \nu_\text{max} (\xi, \xi), \quad 0 < \nu < \nu_\text{max}.
\]
Consequently, we have the following bound for the energy in terms of the
Hopf invariant:
\[
(\xi, \xi) \leq (\text{curl}^{-1} \xi, \xi)/\nu
\]
where \( \nu \) denotes the value \( \nu_\text{min} \) or \( \nu_\text{max} \) of larger modulus.

The inequality becomes an equality for the eigenfield with the eigenvalue
\( \nu \). The right-hand side of the inequality is invariant under volume-preserving
diffeomorphisms (see Section 2). Consequently, under the action of such a
diffeomorphism on the eigenfield with eigenvalue \( \nu \), the field energy can
only increase. This completes the proof of the theorem.

5.2. Examples

Let us take \( M \) to be the three-sphere with the usual Riemannian metric.
The eigenfield of the operator \( \text{curl}^{-1} \) can be calculated explicitly. The
eigenfields with largest and smallest eigenvalues are the Hopf field and its
symmetric field (corresponding to Hopf invariant \(-1\)). The moduli of these
eigenvalues are equal.

**Corollary.** The Hopf field on the three-sphere has minimum energy among
all fields obtained from it by the action of a volume-preserving diffeomorphism.

(The field lines of the Hopf field are circles, and the linking coefficient of
any two of them is 1.)
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As another example, we consider the three-dimensional torus with the usual Riemannian metric. The eigenfields of the operator curl with largest and smallest eigenvalues can be written explicitly in terms of sines and cosines. We obtain the following corollary:

**Corollary**. Each of the fields

\[
\begin{align*}
\xi_x &= A \sin x + C \cos y, \\
\xi_y &= B \sin x + A \cos y, \\
\xi_z &= C \sin y + B \cos z
\end{align*}
\]

on the three-dimensional torus has minimum energy among all fields obtained from it under volume-preserving diffeomorphisms.

Consequently, a minimal force-free field can have a complicated topology for its field lines, as is the case for generic countable systems (some field lines cover two-dimensional tori densely, others do not lie on any two-dimensional surfaces: see the experiment of Heisen mentioned in Section 1.2).

In conclusion, we remark that we can extract from the asymptotic linking number \(\lambda(x,y)\) more invariants than the mean linking number \(\lambda\); for example, the measure \(\mu(A)\) of the set \(\{x,y \in M \times M : \lambda(x,y) < \lambda_\delta\}\), or the value of the Hopf invariant for various regions that are invariant under the flow of a given field \(\xi\). By using such invariants one can sometimes give lower bounds for the energy of a field obtained from a given field by the action of diffeomorphisms, more precisely than those found by using only the Hopf invariant.

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Note added June 6, 1985. A survey of modern generalizations of the asymptotic Hopf invariant is given in [6]. In the simplest generalization one begins with two closed 2-forms \(a, b\) on \(S^2\) such that \(a \wedge b - b \wedge a = 0\) and considers \(I(a, b) = \int a \wedge d^{-1} b \wedge d^{-1} b + q \int b \wedge d^{-1} a \wedge d^{-1} b\). Such forms define two foliations of \(S^2\) into surfaces intersecting along lines and the functional \(I\) probably has an asymptotic ergodic description similar to that given here for the Hopf invariant.

References


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