THE DIRICHLET PROBLEM
FOR THE LAPLACE OPERATOR

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Part I

The Dirichlet problem via Perron’s method
Chapter 1

The classical Dirichlet problem

1.1 Introduction

Suppose that $\Omega$ is a bounded open set in $\mathbb{R}^n$. We denote by $\Delta$ the Laplace operator, which acts on a function $f$ in $C^2(\Omega)$ by

$$\Delta f(x) = \sum_{j=1}^{n} \partial^2_j f(x) \quad \forall x \in \Omega.$$  

The classical Dirichlet problem is the following: given a continuous function $g$ on $\partial \Omega$, find a function $u$ in $C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} 
\Delta u = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = g.
\end{cases}$$

This is a very challenging problem, which has been considered by many outstanding mathematicians in the past two centuries. The reader is referred to the interesting article [G] of L. Garding for an historical account of the research on the Dirichlet problem until the first half of the twentieth century.

As we shall see, it is comparatively easy to prove that if a solution to the Dirichlet problem exists, then it is unique. By contrast, it is entirely nontrivial to prove that, under suitable assumptions on the domain $\Omega$, the Dirichlet problem is solvable.

In some textbooks, mainly bounded domains with $C^2$ boundary are considered. From the point of view of applications, this assumption is far inadequate. Indeed, Dirichlet problems in a square, or in domains of the plane with polygonal boundary are quite common, if not paradigmatic. Thus, it seems
reasonable to focus on theories which at least cover domains with Lipschitz boundary.

In this chapter, we take up the classical theory of harmonic functions on a very general class of domains, which include Lipschitz domains. Then we will illustrate Perron’s method of solving the Dirichlet problem and give a characterisation of the domains in \( \mathbb{R}^n \) where the Dirichlet problem is solvable. We mainly follow [GT, Chapter 2].

There are many problems in Mathematics and in Physics that lead to consider the Dirichlet problem. Here we recall two of them.

The first is concerned with the heat diffusion in a body \( \Omega \subset \mathbb{R}^n \) (\( n = 2, 3 \) are the most important cases). Assume that a fixed temperature distribution at the boundary is maintained by a heating and refrigeration system. By suitably normalising the physical constants involved, we may assume that the temperature \( u(x, t) \) of the point \( x \in \Omega \) at time \( t \) satisfies the following equation (known as the heat equation)

\[
\partial_t u(x, t) - \Delta u(x, t) = 0 \quad \forall x \in \Omega \quad \forall t > 0,
\]

where the Laplacian \( \Delta \) acts on the \( x \) variable. Denote by \( g(x) \) the temperature at which the heating and refrigeration system keeps the point \( x \in \partial \Omega \). Then we must have

\[
u(x, t) = g(x) \quad \forall x \in \partial \Omega \quad \forall t > 0. \tag{1.1.2}
\]

Of course, the body has an initial temperature \( u(x, 0) \) at each point \( x \in \partial \Omega \).

Given that the boundary temperature is kept at a steady state, it is plausible that the system will evolve towards an equilibrium state \( u(x) \), that is

\[
u(x) = \lim_{t \to \infty} u(x, t).
\]

Since for all \( T > 0 \) the function \( u_T(x, t) := u(x, T + t) \) satisfies the boundary value problem (1.1.1)–(1.1.2), it is reasonable to expect that the same will happen to \( u \). Since \( u \) does not depend on \( t \), \( u \) will satisfy

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad \Omega \\
u|_{\partial \Omega} &= g.
\end{aligned}
\]

The second problem leading to the Dirichlet boundary value problem is internal to Mathematics. Riemann tackled the problem of constructing a conformal mapping \( \varphi \) between a simply connected domain \( \Omega \) in the
complex plane and the unit disc \( D := \{ z \in \mathbb{C} : |z| < 1 \} \). By this we mean that \( \varphi \) is a holomorphic bijection between \( \Omega \) and \( D \). Suppose that \( \Omega \) has smooth boundary. We look for a homeomorphism \( \varphi : \overline{\Omega} \to \overline{D} \), which, restricted to \( \Omega \), is a biholomorphism between \( \Omega \) and \( D \). We may assume that \( 0 \in \Omega \). By possibly composing \( \varphi \) with a Möbius transformation, we may assume that \( \varphi(0) = 0 \). If such a conformal mapping \( \varphi \) exists, then \( z \mapsto \varphi(z)/z \) is nonvanishing (recall that \( \varphi'(z) \neq 0 \) for all \( z \in \Omega \), for \( \varphi \) is conformal), whence there exists a holomorphic function \( f \) on \( \Omega \) such that

\[
\frac{\varphi(z)}{z} = e^{f(z)} \quad \forall z \in \Omega.
\]

Therefore

\[
\log |\varphi(z)| = \log |z| + \text{Re} f(z) \quad \forall z \in \Omega.
\]

Clearly \( \text{Re} f \) is harmonic in \( \Omega \), for it is the real part of a holomorphic function, and it is a solution to the boundary value problem

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = -\log |z|.
\end{cases}
\]

Before closing this section, we illustrate an important method, due to Daniel Bernoulli and known as the \textbf{method of separation of variables}, which allows us to construct a solution to the Dirichlet problem in some cases where \( \Omega \) has suitable geometric features.

Denote by \( R \) the strip

\[
\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \pi, \ y \geq 0\}.
\]

Assume that the stationary temperature \( u(x, y) \) is continuous and bounded and satisfies the following boundary conditions

\[
u(0, y) = 0 = u(\pi, y) \quad \forall y > 0, \quad u(x, 0) = f(x) \quad \forall x \in [0, \pi], \quad (1.1.3)
\]

where \( f \) is an assigned continuous function. We seek functions of the form \( u(x, y) = X(x)Y(y) \) that satisfy the Laplace equation and the boundary conditions on the vertical edges of the strip. We substitute \( u \) in the Laplace equation, and obtain

\[
\frac{X''}{X} = -\frac{Y''}{Y}.
\]

Since the left hand side depends only on \( x \) and the right hand side depends only on \( y \), neither can depend on either. Hence both are equal to a constant,
which we write as $-c$, where $c$ is, for the time being, a complex number. Then, we are led to solve the ordinary differential equations

$$X'' + cX = 0 \quad \text{and} \quad Y'' - cY = 0.$$ 

Furthermore, $X$ must be a solution to the following boundary value problem

$$X'' + cX = 0 \quad \text{and} \quad X(0) = 0 = X(\pi). \quad (1.1.4)$$

It is straightforward to check that this problem has a nontrivial solution only if $c$ is of the form $k^2$, where $k$ is a positive integer (prove this!). The solutions of (1.1.4) are then all the multiples of $\sin(kx)$. The function $Y$ must be a solution to the following problem

$$Y'' - k^2 Y = 0 \quad \text{and} \quad Y \text{ bounded on } [0, \infty). \quad (1.1.5)$$

All the solutions to this problem are multiples of $e^{-ky}$. Thus, we are led to consider the functions

$$u_k(x, y) := e^{-ky} \sin(kx).$$

They satisfy

$$\Delta u_k = 0, \quad u_k(0, y) = 0 = u_k(\pi, y) \quad \forall y > 0.$$ 

Unless the datum $f$ is one of the functions $\sin(kx)$, none of the functions $u_k$ will match the boundary condition $u_k(x, 0) = f(x)$. The idea to circumvent this difficulty is to consider superpositions of the functions $u_k$, i.e., to see whether functions of the form

$$\sum_{k=1}^{\infty} c_k e^{-ky} \sin(kx) \quad (1.1.6)$$

satisfy the given Dirichlet problem. Of course, this idea is suggested by the fact that the Laplace operator $v \mapsto \Delta v$ is linear. At least formally, if the function above satisfies the Dirichlet problem, then its value at $(x, 0)$ must be equal to $f(x)$. In other words, $f$ should have the following Fourier sine series expansion

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx).$$

It is a nontrivial fact, proved long after the appearance of Fourier’s original paper (1822), that the sequence $\{(2/\pi)^{1/2} \sin(kx)\}$ is a complete orthonormal system in $L^2((0, \pi))$. We shall come back to this fact later. This will force $c_k$ to be the $k^{\text{th}}$ Fourier coefficient of the sine expansion of the function $f$. 

Note that if \( f \) is the restriction to \([0, \pi]\) of a \( \pi \) periodic function in \( C^2(\mathbb{R}) \) such that \( f(0) = 0 \), then
\[
c_k = O(k^{-2}),
\]
and the series in (1.1.6) converges uniformly on \([0, \pi] \times [0, \infty)\). Therefore its sum is a continuous function on \(\mathbb{R}\) that matches the boundary values. Furthermore, the sum of the series is infinitely many times differentiable at all the points in \(\mathbb{R}\) with \(y > 0\), and satisfies the Laplace equation therein.

**Exercise 1.1.1** Prove all the assertions above on Fourier sine series, except for the completeness of the system \( \{ (2/\pi)^{1/2} \sin(kx) \} \).

**Exercise 1.1.2** Show that the method of separation of variables for the Dirichlet problem discussed above leads, in the case where \( f(x) = 1 \), to the solution
\[
u(x, y) := 4 \pi \left[ e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin(3x) + \frac{1}{5} e^{-5y} \sin(5x) + \cdots \right]
= \frac{2}{\pi} \arctan \frac{\sin x}{\sinh y}.
\]
Show that \( \nu \) satisfies the boundary conditions except at the corners. Draw the isothermals for small \( x \) and \( y \). *Hint.* To find the sum of the series within square brackets, write \( z = x + iy \) and observe that the series is the imaginary part of a power series in the variable \( e^{iz} \), whose sum is \((1/2) \log[(1+e^{iz})/(1-e^{iz})] \).

### 1.2 Background and preliminary results

We need the following definitions and results. A subset of \( \mathbb{R}^n \) is a **domain** if it is an open connected set.

Now we recall some basic facts from advanced calculus. A subset \( S \) of \( \mathbb{R}^n \) is a **hypersurface of class** \( C^k \) if for every \( x_0 \in S \) there exists an open subset \( V \) of \( \mathbb{R}^n \) containing \( x_0 \) and a real valued function \( \phi \in C^k(V) \) such that \( \nabla \phi \) does not vanish on \( S \cap V \) and
\[
S \cap V = \{ x \in V : \phi(x) = 0 \}.
\]
For the sake of definiteness, suppose that \( \partial_n \phi \) does not vanish on \( S \cap V \). Then, by the implicit function theorem, there exists a \( C^k \) function \( \psi \) such that
\[
x_n = \psi(x_1, \ldots, x_{n-1})
\]
for all \((x_1, \ldots, x_{n-1}, x_n)\) in \(S \cap V\). For convenience, denote by \(x'\) the point \((x_1, \ldots, x_{n-1})\) in \(\mathbb{R}^{n-1}\). Note that the map
\[(x', x_n) \mapsto (x', x_n - \psi(x'))\]
maps \(V \cap S\) onto a neighbourhood of the point \(x'_0\) of the hyperplane \(x_n = 0\).

The inverse of the map
\[x' \mapsto (x', \psi(x'))\]
which is defined in a suitable neighbourhood of \(x'_0\), gives a local chart of \(S\) around \(x_0\).

Now, for every \(x \in S\), the vector \(\nabla \phi(x)\) is orthogonal to \(S\) (i.e., it is orthogonal to the tangent plane to \(S\) at \(x_0\)). We shall assume that \(S\) is oriented, i.e. there is a choice of a unit normal vector \(\nu(x)\) at each point \(x\) of \(S\) that varies continuously with \(x\). In particular
\[
\nu(x) = \pm \frac{\nabla \phi(x)}{\|\nabla \phi(x)\|}.
\]
This formula shows that the normal field \(x \mapsto \nu(x)\) is of class \(C^{k-1}\) on \(S\).

We say that a domain \(\Omega\) has \(C^k\) boundary if \(\partial \Omega\) is a hypersurface of class \(C^k\). We recall the classical divergence theorem.

**Theorem 1.2.1** Suppose that \(\Omega\) is a bounded domain with \(C^1\) boundary, and denote by \(\nu\) the unit outward normal to \(\Omega\). Suppose that \(w \in C^1(\Omega)\). Then
\[
\int_{\Omega} \text{div} \, w \, dV = \int_{\partial \Omega} w \cdot \nu \, d\sigma,
\]
where \(\sigma\) denotes the surface measure of \(\partial \Omega\).

In the applications, the boundary does not always satisfy the assumptions of Theorem 1.2.1. There is a generalisation of the divergence theorem which covers domains with a very general boundary. It is beyond the scope of these notes to deal with such generalisations, for which the reader is referred to the book of Evans and Gariepy [EG]. We just make a few comments concerning domains with Lipschitz boundary.

Recall that a map \(\varphi : \mathbb{R}^{n-1} \to \mathbb{R}\) is **Lipschitz** if there exists a constant \(L\) such that
\[
|\varphi(x) - \varphi(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^{n-1}.
\]
The infimum of all the constants \( L \) such that the above inequality holds is called the **Lipschitz constant** of \( \varphi \).

Lipschitz functions are possibly not differentiable at some points (for instance, \( x \mapsto |x| \) is Lipschitz with constant 1, but it is not differentiable at the origin), but the set where it is not differentiable is small in the measure theoretic sense. In fact, the following nontrivial result holds. We refer the reader to [EG] for the proof of this result.

**Theorem 1.2.2 (Rademacher)** Suppose that \( \varphi \) is Lipschitz. Then the set of points where \( \varphi \) is not differentiable is of null measure.

**Definition 1.2.3** A **Lipschitz domain** is a domain \( \Omega \) with the following property:

for each \( y \in \partial \Omega \) there exists a system of coordinates \((x', x_n) (x' \text{ is an } (n-1)-\text{dimensional vector}) \) centred at \( y \), a ball \( B_r(y) \), a neighbourhood \( U \) of \( x' = 0 \), and a Lipschitz map \( \varphi : U \to \mathbb{R} \), with \( \varphi(0) = 0 \), such that

\[
\begin{align*}
(i) & \quad \partial \Omega \cap B_r(y) = \{(x', x_n) \in B_r(y) : x_n = \varphi(x') \text{ for all } x' \text{ in } U \} \\
(ii) & \quad \Omega \cap B_r(y) = \{(x', x_n) \in B_r(y) : x_n > \varphi(x') \text{ for all } x' \text{ in } U \}.
\end{align*}
\]

A consequence of the definition and of Rademacher’s theorem is that a Lipschitz domain \( \Omega \) admits a well defined outward unit normal \( \nu \) at almost every point of \( \partial \Omega \) (with respect to the surface measure). The surface measure of \( \partial \Omega \) has the following expression in terms of the local coordinates \((x', \varphi(x'))\) in a neighbourhood of the point \( y \) (see Definition 1.2.3 for the notation)

\[
d\sigma(x', \varphi(x')) = \sqrt{1 + |\nabla \varphi(x')|^2} \, dx'.
\]

The following **generalised divergence theorem** holds.

**Theorem 1.2.4** Suppose that \( \Omega \) is a bounded domain with Lipschitz boundary, and denote by \( \nu \) the unit outward normal to \( \Omega \), which is defined at almost every point of the boundary. Suppose that \( w \in C^1(\overline{\Omega}) \). Then

\[
\int_{\Omega} \text{div} \, w \, dV = \int_{\partial \Omega} w \cdot \nu \, d\sigma,
\]

where \( \sigma \) denotes the surface element of \( \partial \Omega \).

**Corollary 1.2.5** Suppose that \( \Omega \) is a bounded Lipschitz domain, and that \( u \) and \( v \) are in \( C^2(\overline{\Omega}) \). The following hold:
(i) \[ \int_{\Omega} \Delta u \, dV = \int_{\partial \Omega} \partial_{\nu} u \, d\sigma; \]

(ii) (first Green’s identity) \[ \int_{\Omega} v \Delta u \, dV + \int_{\Omega} \nabla v \cdot \nabla u \, dV = \int_{\partial \Omega} v \partial_{\nu} u \, d\sigma; \]

(iii) (second Green’s identity) \[ \int_{\Omega} (v \Delta u - u \Delta v) \, dV = \int_{\partial \Omega} (v \partial_{\nu} u - u \partial_{\nu} v) \, d\sigma. \]

(iv) (integration by parts) \[ \int_{\Omega} v \partial_j u \, dV + \int_{\Omega} u \partial_j v \, dV = \int_{\partial \Omega} u v \nu_j \, d\sigma. \]

**Proof.** To prove (i), just take \( w = \nabla u \) in the divergence theorem.

To prove (ii), just take \( w = v \nabla u \) in the divergence theorem.

By interchanging the role of \( u \) and \( v \) in (ii), and subtracting the resulting equalities, we get (iii).

Finally, (iv) follows from the divergence theorem by taking \( w \) to be a vector field, all of whose components vanish but the \( j \)th, which is equal to \( uv \).

Denote by \( \omega_n \) the **surface measure of the sphere** \( \partial B_1(0) \) in \( \mathbb{R}^n \).

**Exercise 1.2.6** Prove that the measure of the unit ball in \( \mathbb{R}^n \) is \( \omega_n/n \).

**Hint:** use the divergence theorem for an appropriate vector field.

**Exercise 1.2.7** By following the steps below, prove that the measure of the unit ball in \( \mathbb{R}^n \) is \( \pi^{n/2}/\Gamma(n/2+1) \) (here \( \Gamma \) denotes Euler’s Gamma function):

(i) show that if \( a > 0 \), then \[ \int_{\mathbb{R}^n} \exp \left( -a|x|^2 \right) \, dx = \left( \pi/a \right)^{n/2}; \]
1.2. BACKGROUND AND PRELIMINARY RESULTS

(ii) compute

\[ I := \int_R p(\theta_1, \ldots, \theta_{n-1}) \, d\theta_1 \cdots d\theta_{n-1}, \]

by computing

\[ \int_{\mathbb{R}^n} \exp \left( -a|x|^2 \right) \, dx \]

with (i), and using polar coordinates;

(iii) compute \( V(B_1(0)) \), by integrating the characteristic function of \( B_1(0) \) in polar coordinates.

Exercise 1.2.8 Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n \). Prove that the divergence operator is the “adjoint” of the gradient, in the sense that for every smooth vector field \( w \) and every smooth function \( \varphi \) with compact support contained in \( \Omega \)

\[ \int_{\Omega} \varphi \, \text{div} \, w \, dV = -\int_{\Omega} \nabla \varphi \cdot w \, dV. \]

Exercise 1.2.9 Suppose that \( \phi \) is a \( C^2 \) diffeomorphism between the domains \( \Omega \) and \( \phi(\Omega) \) and that \( f \) is a function in \( C^2(\phi(\Omega)) \). We denote by \( x \) the variable in \( \Omega \), by \( y \) that in \( \phi(\Omega) \), and by \( J(x) \) the determinant of the differential map \( \phi'(x) \). Prove the following change of variables formulae (in these formulae \( y = \phi(x) \), and the gradient of a scalar function is a row vector):

(i) \( \nabla_y f(y) = \nabla_x (f \circ \phi)(x) \phi'(x)^{-1}; \)

(ii) \( \text{div}_y F(y) = \frac{1}{J(x)} \, \text{div}_x \left[ J(x \circ \phi) \left[ (\phi')^t \right]^{-1} \right](x); \)

(iii) \( \Delta_y f(y) = \frac{1}{J(x)} \, \text{div}_x \left[ J \nabla_x (f \circ \phi) \left[ (\phi')^t \phi' \right]^{-1} \right](x); \)

(iv) if \( \phi \) is an orthogonal transformation, then the columns of \( \phi'(x) \) are of the form \( a_j \, u_j \), where \( u_1, \ldots, u_n \) are orthonormal vectors. Then \( \phi'(x)^t \phi'(x) = \text{diag}(a_1^2, \ldots, a_n^2) \) and \( J(x) = a_1 \cdots a_n \). Compute \( \Delta_y f(y) \);

(v) prove that polar coordinates are associated to an orthogonal transformation, and compute the Laplacian in polar coordinates by using (iii) above, or by specialising the formula found in (iv);
(vi) suppose that $\phi$ is a conformal mapping, i.e., that $\phi'(x) = \varrho(x) U(x)$, where $\varrho(x)$ is a scalar factor and $U(x)$ is an orthogonal matrix. Prove that
\[
\Delta_y f(y) = \frac{1}{\varrho(x)^n} \text{div}_x \left[ \varrho^{n-2} \nabla x (f \circ \phi) \right](x);
\]
(vii) prove that the inversion map $y = x/|x|^2$ is a conformal mapping between appropriate domains, with conformal factor $|x|^{-2}$. Compute the Laplace operator with respect to the new coordinates;
(viii) suppose that $\Omega$ is a domain in $\mathbb{C}$ and that $\phi : \Omega \to f(\Omega)$ is a one-to-one holomorphic function. Prove that
\[
\Delta f(w) = |\phi'(z)|^2 \left( \Delta (f \circ \phi^{-1}) \left( \phi^{-1}(w) \right) \right),
\]
where $w = \phi(z)$.

Hint: Exercise 1.2.8 may be helpful to prove (ii).

**Exercise 1.2.10** Suppose that $u$ is a smooth function on $\mathbb{R}^2 \setminus \{0\}$. Prove that
\[
\Delta u(r, \theta) = \partial^2_r u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial^2_{\theta} u.
\]
Then find all radial solutions to the equation $\Delta u = 0$ in $\mathbb{R}^2 \setminus \{0\}$.

**Exercise 1.2.11** Suppose that $u$ is a smooth function on $\mathbb{R}^3 \setminus \{0\}$. Prove that
\[
\Delta u(r, \theta, \psi) = \partial^2_r u + \frac{2}{r} \partial_r u + \frac{1}{r^2} \left[ \frac{1}{(\sin \psi)^2} \partial^2_{\theta} u + \partial^2_{\psi} u + \cot \psi \partial_{\psi} u \right].
\]
Then find all radial solutions to the equation $\Delta u = 0$ in $\mathbb{R}^3 \setminus \{0\}$. Prove also that the differential operator within square brackets in the formula above is the Laplace–Beltrami operator on the sphere $S^2$ with respect to the Riemannian metric induced by the standard Euclidean metric of $\mathbb{R}^3$.

### 1.3 Subharmonic, superharmonic and harmonic functions

**Definition 1.3.1** Suppose that $\Omega$ is a domain in $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is subharmonic (superharmonic, harmonic) if $\Delta u \geq 0$ ($\leq 0$, $= 0$) in $\Omega$. 


1.3. SUBHARMONIC FUNCTIONS

Exercise 1.3.2 Prove that the real and the imaginary parts of a holomorphic function in a domain $\Omega$ are harmonic in $\Omega$. Prove that the functions $x^2 - y^2$ and $xy$ are harmonic in $\mathbb{R}^2$, and so are the functions $r^j \cos(j \theta)$ and $r^j \sin(j \theta)$ for each nonnegative integer $j$.

Exercise 1.3.3 Suppose that $u$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Prove that the function

$$u^*(x) = |x|^{2-n} u(x/|x|^2)$$

is harmonic in $\mathbb{R}^n \setminus \{0\}$. Hint: Exercise 1.2.9 (vii) may be useful.

Exercise 1.3.4 Suppose that $\Omega$ is a domain in the complex plane, that $\phi : \Omega \rightarrow \phi(\Omega)$ is a one-to-one holomorphic mapping, and that $f$ is harmonic in $\phi(\Omega)$. Prove that the function $f \circ \phi$ is is harmonic in $\phi(\Omega)$. Hint: Exercise 1.2.9 (viii) may be useful.

Exercise 1.3.5 Prove that if $u$ is harmonic in $\Omega$, and $\Omega' \subset \subset \Omega$, then

$$\int_{\partial \Omega} \partial_n u \, d\sigma = 0.$$ 

Conversely, show that if $u \in C^2(\Omega)$ satisfies

$$\int_{\partial B} \partial_n u \, d\sigma = 0$$

for every ball $B \subset \subset \Omega$, then $u$ is harmonic in $\Omega$.

Exercise 1.3.6 For which values of $\alpha$ is the function $x \mapsto |x|^\alpha$ subharmonic? Answer: $\alpha + n - 2 \geq 0$. This shows that if $n \geq 2$, then a convex function need not be subharmonic.

The next result relates subharmonic, superharmonic and harmonic functions with their spherical and solid means over balls. In fact, these kind of functions may be characterised by the behaviour of their means, as we shall see later.

For a ball $B$, we denote by $\sigma(\partial B)$ the surface measure of $\partial B$ and by $V(B)$ the Lebesgue measure of $B$.

**Theorem 1.3.7 (Mean value inequalities)** Suppose that $\Omega$ is a domain. The following hold:
(i) for every subharmonic function $u$ and every ball $B \subset \subset \Omega$

$$u(y) \leq \frac{1}{\sigma(\partial B)} \int_{\partial B} u \, d\sigma \quad \text{and} \quad u(y) \leq \frac{1}{V(B)} \int_B u \, dV;$$

(ii) for every superharmonic function $u$ and every ball $B \subset \subset \Omega$

$$u(y) \geq \frac{1}{\sigma(\partial B)} \int_{\partial B} u \, d\sigma \quad \text{and} \quad u(y) \geq \frac{1}{V(B)} \int_B u \, dV;$$

(iii) for every harmonic function $u$ and every ball $B \subset \subset \Omega$

$$u(y) = \frac{1}{\sigma(\partial B)} \int_{\partial B} u \, d\sigma \quad \text{and} \quad u(y) = \frac{1}{V(B)} \int_B u \, dV.$$

**Proof.** We shall prove (i). The proof of (ii) is similar and is omitted. Part (iii) is a straightforward consequence of (i) and (ii).

To prove (i), suppose that $B = B_R(y)$, that $0 < \rho < R$, and observe that, by Corollary 1.2.5 (i),

$$0 \leq \int_{\partial B_\rho(y)} \partial_\nu u \, d\sigma$$

$$= \int_{\partial B_\rho(0)} \nabla u(y + \omega') \cdot \frac{\omega'}{|\omega'|} \, d\sigma_\rho(\omega')$$

$$= \rho^{n-1} \int_{\partial B_1(0)} \nabla u(y + \rho \omega) \cdot \omega \, d\sigma_1(\omega)$$

$$= \rho^{n-1} \int_{\partial B_1(0)} \frac{d}{d\rho} \left[ u(y + \rho \omega) \right] \, d\sigma_1(\omega)$$

$$= \rho^{n-1} \int_{\partial B_1(0)} u(y + \rho \omega) \, d\sigma_1(\omega)$$

$$= \rho^{n-1} \left[ \frac{1}{\rho^{n-1}} \int_{\partial B_\rho(y)} u \, d\sigma \right].$$

Now we integrate both sides with respect to $\rho$ between $\varepsilon$ and $R$ and obtain

$$0 \leq \frac{1}{R^{n-1}} \int_{\partial B_R(y)} u \, d\sigma - \frac{1}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(y)} u \, d\sigma.$$

Since $u$ is continuous in $y$,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(y)} u \, d\sigma = \omega_n \, u(y).$$
By combining the last two formulae, we obtain that
\[ u(y) \leq \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(y)} u \, d\sigma, \]
which is equivalent to the first formula in (i).

To prove the second formula in (i), we multiply by \( R^{n-1} \) both sides of the inequality above and integrate with respect to \( R \) between 0 and \( r \). We obtain
\[ \frac{1}{n} u(y) \leq \frac{1}{\omega_n} \int_0^r \, dR \int_{\partial B_R(y)} u \, d\sigma, \]
which is equivalent to the required inequality.

**Exercise 1.3.8** Find all harmonic functions on an interval \( I \) of \( \mathbb{R} \). Give a characterisation of subharmonic functions on \( I \). Conclude that a continuous function \( u \) on \( I \) is subharmonic if and only if for every interval \([a,b] \subset I\)
\[ u(x) \leq H_{a,b}(x) \quad \forall x \in [a,b] \]
where \( H_{a,b} \) is the harmonic function on \([a,b] \) such that \( H_{a,b}(a) = u(a) \) and \( H_{a,b}(b) = u(b) \).

It is an important fact that a converse of Theorem 1.3.7 holds.

**Theorem 1.3.9** Suppose that \( u \) is a continuous function in the domain \( \Omega \) and that for every ball \( B \subset \subset \Omega \) the following holds:
\[ u(c_B) = \frac{1}{\sigma(\partial B)} \int_{\partial B} u \, d\sigma, \]
where \( c_B \) denotes the centre of \( B \). Then \( u \) is harmonic in \( \Omega \).

**Proof.** Denote by \( \phi \) a smooth radial function with support contained in the ball \( B_1(0) \) such that \( \int \phi \, dV = 1 \), and denote by \( \psi \) its profile, i.e., \( \psi(|x|) = \phi(x) \). For every \( \varepsilon > 0 \), set \( \phi_{\varepsilon}(x) = \varepsilon^{-n} \phi(x/\varepsilon) \). Clearly the support of \( \phi_{\varepsilon} \) is contained in \( B_{\varepsilon}(0) \). If \( x \) belongs to
\[ \Omega_{\varepsilon} := \{ x : \overline{B}_\varepsilon(x) \subset \Omega \}, \]
then the support of \( x \mapsto \phi_\varepsilon(x - y) \) is contained in \( \Omega \). Observe that
\[
\int_{\mathbb{R}^n} u(y) \phi_\varepsilon(x - y) \, dy = \int_{|y|<1} u(x - \varepsilon y) \phi(y) \, dy
\]
\[
= \int_0^1 \int_{|\gamma'|=1} u(x - \varepsilon \gamma') \psi(r) \, r^{n-1} \, d\sigma(\gamma') \, dr
\]
\[
= \omega_n u(x) \int_0^1 \psi(r) \, r^{n-1} \, dr
\]
\[
= u(x) \int_{\mathbb{R}^n} \phi \, dV
\]
\[
= u(x).
\]
Now, the left hand side, being the convolution of \( u \) with the smooth function \( \phi_\varepsilon \), can be differentiated infinitely many times, whence \( u \) is in \( C^\infty(\Omega_\varepsilon) \). Since \( \varepsilon \) is arbitrary, \( u \) is in \( C^\infty(\Omega) \).

It remains to show that \( u \) is harmonic. Suppose that \( \overline{B}_r(x) \subset \Omega \). By Corollary I.2.5 (i),
\[
\int_{B_r(x)} \Delta u \, dV = \int_{\partial B_r(x)} \partial_\nu u \, d\sigma
\]
\[
= \int_{|\omega|=r} \text{grad} \, u(x + \omega) \cdot \frac{\omega}{|\omega|} \, d\sigma(\omega)
\]
\[
= r^{n-1} \int_{|\omega'|=1} \text{grad} \, u(x + r \omega') \cdot \omega' \, d\sigma(\omega')
\]
\[
= r^{n-1} \frac{d}{dr} \int_{|\omega'|=1} u(x + r \omega') \, d\sigma(\omega').
\]
The last integral is equal to
\[
\omega_n \frac{1}{\sigma(\partial B_r(x))} \int_{\partial B_r(x)} u \, d\sigma,
\]
which, in turn is equal to \( \omega_n \, u(x) \), for \( u \) possesses the mean value property by assumption. Thus, we have proved that
\[
\int_{B_r(x)} \Delta u \, dV = \omega_n \, r^{n-1} \frac{d}{dr} u(x) = 0
\]
for every ball \( B_r(x) \) such that \( \overline{B}_r(x) \subset \Omega \). Since \( \Delta u \) is continuous, \( \Delta u = 0 \), as required. \( \square \)
Exercise 1.3.10 Show that if \( u \) is a function in \( C^2(\Omega) \) and \( x_0 \in \Omega \), then

\[
-\Delta u(x_0) = \lim_{r \to 0} \frac{2n}{r^2} \left[ u(x_0) - \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} u(x_0 + r \omega) \, d\sigma(\omega) \right].
\]

Deduce that if \( u \) satisfies the mean value inequality of Theorem 1.3.7(i), then \( u \) is subharmonic.

1.4 Maximum principle and uniqueness for the Dirichlet problem

In this section we derive some consequences of the mean value inequalities proved in the previous section. In particular, we shall prove the maximum and the minimum principles for harmonic functions. As a corollary we shall obtain that the classical Dirichlet problem on a domain \( \Omega \) has, at most, one solution. The problem of the existence of solutions to the Dirichlet problem will be addressed later.

Theorem 1.4.1 Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n \), and that \( u \in C^2(\Omega) \). The following hold:

(i) (**strong maximum principle**) if \( u \) is subharmonic and there exists a point \( y \in \Omega \) such that \( u(y) = \sup_{\Omega} u \), then \( u \) is constant;

(ii) (**strong minimum principle**) if \( u \) is superharmonic and there exists a point \( y \in \Omega \) such that \( u(y) = \inf_{\Omega} u \), then \( u \) is constant;

(iii) if \( u \) is harmonic, then \( u \) cannot assume an interior maximum or minimum, unless it is constant.

Proof. We prove (i). The proof of (ii) is similar and is omitted. Part (iii) is a direct consequence of (i) and (ii).

Set \( M := \sup_{\Omega} u \), and
\[
\Omega_M := \{ x \in \Omega : u(x) = M \}.
\]

Clearly \( \Omega_M \) contains \( y \), and it is closed, for \( u \) is continuous. We shall prove that \( \Omega_M \) is open. It will follow that \( \Omega_M = \Omega \), for \( \Omega \) is connected by assumption, i.e. \( u = M \) in \( \Omega \), as required.
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To prove that $\Omega_M$ is open, choose $z$ in $\Omega_M$. Observe that $u - M$ is subharmonic ($\Delta(u - M) = \Delta u$), so that, by the mean value inequality,

$$0 = u(z) - M \leq \frac{1}{V(B)} \int_B (u - M) \, dV \leq 0$$

for every ball $B$ with radius small enough. Therefore

$$\int_B (u - M) \, dV = 0.$$ 

If $u$ were strictly less than $M$ on an open subset of $B$, then the integral above would be strictly negative. Therefore $u = M$ on $B$, and $\Omega_M$ is open, as required.

**Corollary 1.4.2** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, and that $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The following hold:

(i) **(weak maximum principle)** if $u$ is subharmonic, then

$$\sup_\Omega u = \sup_{\partial\Omega} u;$$

(ii) **(weak minimum principle)** if $u$ is superharmonic, then

$$\inf_\Omega u = \inf_{\partial\Omega} u;$$

(iii) if $u$ is harmonic, then

$$\inf_{\partial\Omega} u \leq u \leq \sup_{\partial\Omega} u.$$ 

**Proof.** To prove (i), observe that if $\sup_\Omega u > \sup_{\partial\Omega} u$, then there exists a point $y$ in $\Omega$ for which $u(y) = \sup_\Omega u$. Consequently $u$ is constant in $\Omega$ by Theorem 1.4.1 (i). Since $u$ is continuous on $\overline{\Omega}$, $\sup_\Omega u = \sup_{\partial\Omega} u$, thereby contradicting the assumption.

The proof of (ii) is almost verbatim the same as the proof of (i), and is omitted.

Part (iii) follows directly from (i) and (ii).
Exercise 1.4.3 Suppose that $u$ and $v$ are in $C^2(\Omega) \cap C(\overline{\Omega})$ and that $u|_{\partial\Omega} = v|_{\partial\Omega}$. Prove that if $u$ is harmonic and $v$ is subharmonic, then $v \leq u$ on $\overline{\Omega}$. This result justifies the term subharmonic. State and prove a corresponding result for superharmonic functions.

A noteworthy consequence of Corollary 1.4.2 is the following uniqueness result for the Dirichlet problem.

**Theorem 1.4.4** Suppose that $u$ and $v$ are functions in $C^2(\Omega) \cap C(\overline{\Omega})$ that solve the Dirichlet problem

$$ \begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases} $$

Then $u = v$.

**Proof.** Set $w = u - v$. Clearly $w$ solves

$$ \begin{cases} \Delta w = 0 & \text{in } \Omega \\ w|_{\partial\Omega} = 0 \end{cases} $$

In particular, $w$ is a harmonic function in $C^2(\Omega) \cap C(\overline{\Omega})$ which vanishes on $\partial\Omega$. By Corollary 1.4.2 (iii), $w = 0$ on $\overline{\Omega}$, i.e. $u = v$, as required.

We wish to emphasize the fact that no claims are made concerning the existence of a solution to the Dirichlet problem.

### 1.5 Green’s function

In this section we establish an important representation formula for solutions of the Dirichlet problem. We emphasize that we do not prove the existence of a solution, rather we give a formula for the solution under the assumption that a solution exists.

**Definition 1.5.1** The **Newtonian potential** in $\mathbb{R}^n$ is the function $N : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, defined by

$$ N(x) = \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{1}{(2 - n) \omega_n} |x|^{2-n} & \text{if } n \geq 3 \end{cases} $$

(1.5.1)
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Given a point \( x \) in \( \mathbb{R}^n \), the Newtonian potential with pole \( x \) is the function \( N(x - \cdot) \).

Note that if \( n \geq 3 \), then \( N \) is a negative function. By contrast, if \( n = 2 \), then \( N \) has not a definite sign. This simple fact will have far reaching consequences. The reason for the normalisation of \( N \) will be clarified below (see Definition [5.4.2]). It will be proved later that \( N \) is a fundamental solution of the Laplace operator: it will play an important role in finding distributional solutions to the Poisson equation

\[
\Delta u = f,
\]

where \( f \) is a given datum. All this will be discussed in Section [5.4].

**Exercise 1.5.2** Prove, by direct calculation, that \( N \) is harmonic in \( \mathbb{R}^n \setminus \{0\} \).

**Exercise 1.5.3** Prove that

\[
\int_{\partial B_r(x)} \partial_r N(x - y) \, d\sigma(y) = 1,
\]

**Exercise 1.5.4** Recall that the Laplacian in \( \mathbb{R}^n \) may be written in polar coordinates as follows

\[
\Delta = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{n-1}},
\]

where \( \Delta_{S^{n-1}} \) is the Laplace–Beltrami operator on the sphere \( S^{n-1} \) (it is a second order differential operator in the angular variables only). Find all radial harmonic functions in \( \mathbb{R}^n \).

We now establish Green’s representation formula.

**Theorem 1.5.5 (Green’s representation formula)** Suppose that \( \Omega \) is a bounded Lipschitz domain and that \( u \in C^2(\overline{\Omega}) \). The following hold:

(i) for every \( x \) in \( \Omega \)

\[
\begin{align*}
    u(x) &= \int_{\partial\Omega} \left[ u(y) \partial_r N(x - y) - \partial_r u(y) N(x - y) \right] \, d\sigma(y) \\
    &\quad + \int_{\Omega} N(x - y) \Delta u(y) \, dV(y);
\end{align*}
\]

(1.5.2)
(ii) if the support of $u$ is a compact set in $\Omega$, then

$$u(x) = \int_{\Omega} N(x - y) \Delta u(y) \, dV(y) \quad \forall x \in \Omega;$$

(iii) if $u$ is harmonic in $\Omega$, then

$$u(x) = \int_{\partial \Omega} \left[ u(y) \partial_{\nu} N(x - y) - \partial_{\nu} u(y) N(x - y) \right] \, d\sigma(y) \quad \forall x \in \Omega;$$

(iv) suppose that $x \in \Omega$ and that there exists a function $h_x \in C^1(\overline{\Omega})$ which is harmonic on $\Omega$ and satisfies $(h_x)_{|\partial \Omega} = -N(x - \cdot)_{|\partial \Omega}$. Denote by $G(x, \cdot)$ the function $h_x + N(x - \cdot)$. Then

$$u(x) = \int_{\partial \Omega} u(y) \partial_{\nu} G(x, y) \, d\sigma(y) + \int_{\Omega} G(x, y) \Delta u(y) \, dV(y).$$

The normal derivative of $G$ in the integral above is taken with respect to the variable $y$.

**Proof.** We prove the theorem in the case where $n \geq 3$. The case $n = 2$ is left to the reader.

First we prove (i). Consider the domain $\Omega \setminus \overline{B}_r(x)$, for $r$ small. Apply the second Green’s identity (see Corollary 1.2.5 (iii)) with $N(x - \cdot)$ in place of $u$ and $\Omega \setminus \overline{B}_r(x)$ in place of $\Omega$. We obtain

$$\int_{\Omega \setminus \overline{B}_r(x)} N(x - y) \Delta u(y) \, dV(y)$$

$$= \int_{\partial \Omega} \left[ N(x - y) \partial_{\nu} u(y) - u(y) \partial_{\nu} N(x - y) \right] \, d\sigma(y)$$

$$+ \int_{\partial B_r(x)} \left[ N(x - y) \partial_{\nu} u(y) - u(y) \partial_{\nu} N(x - y) \right] \, d\sigma(y).$$

The required formula will follow from this and the dominated convergence theorem once we prove that

$$\lim_{r \downarrow 0} \int_{\partial B_r(x)} N(x - y) \partial_{\nu} u(y) \, d\sigma(y) = 0 \quad (1.5.3)$$

and that

$$\lim_{r \downarrow 0} \int_{\partial B_r(x)} u(y) \partial_{\nu} N(x - y) \, d\sigma(y) = u(x). \quad (1.5.4)$$
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To prove (1.5.3), note that
\[ |N(x - y)| \leq C r^{2-n} \quad \forall y \in \partial B_r(x) \]
and that
\[ |\partial_x u(y)| \leq \max_{y \in \Omega} |\nabla u(y)|, \]
which is finite, because \( u \in C^2(\Omega) \). Thus,
\[
\left| \int_{\partial B_r(x)} N(x - y) \partial_x u(y) \, d\sigma(y) \right| \leq C \max_{y \in \Omega} |\nabla u(y)| \, r^{2-n} \, \sigma(\partial B_r(x)),
\]
which tends to 0 as \( r \) tends to 0, for \( \sigma(\partial B_r(x)) \approx r^{n-1} \), as required.

To prove (1.5.4), write the integral in (1.5.4) as
\[
\int_{\partial B_r(x)} \left[ u(y) - u(x) \right] \partial_x N(x - y) \, d\sigma(y) + u(x) \int_{\partial B_r(x)} \partial_x N(x - y) \, d\sigma(y).
\]
Since \( u \) is smooth,
\[ |u(y) - u(x)| \leq r \max_{\Omega} |\nabla u| \quad \forall y \in \partial B_r(x). \]
Recall that \( N \) is homogeneous of degree \( 2 - n \), hence its partial derivatives are homogeneous of degree \( 1 - n \). Thus,
\[ |\partial_x N(x - y)| \leq C r^{1-n} \quad \forall y \in \partial B_r(x). \]
Therefore
\[
\left| \int_{\partial B_r(x)} \left[ u(y) - u(x) \right] \partial_x N(x - y) \, d\sigma(y) \right| \leq C r r^{1-n} \sigma(\partial B_r(y)),
\]
which tends to 0 as \( r \) tends to 0. Finally, by Exercise 1.5.3
\[
\int_{\partial B_r(x)} \partial_x N(x - y) \, d\sigma(y) = 1,
\]
which completes the proof of (i).

Note that (ii) and (iii) are direct consequences of the representation formula established in (i).

It remains to prove (iv). Write \( N(x - y) = G(x, y) - h_x(y) \) in (1.5.2). We obtain
\[
u(x) = \int_{\partial \Omega} \left[ u \partial_x G(x, \cdot) - \partial_x u \, G(x, \cdot) \right] \, d\sigma + \int_{\Omega} G(x, y) \, \Delta u(y) \, dV(y) \]
\[ + \int_{\partial \Omega} \partial_x u \, h_x \, d\sigma - \int_{\partial \Omega} u \, \partial_x h_x \, d\sigma - \int_{\Omega} h_x \, \Delta u \, dV.\]
Note that, by second Green’s identity and the fact that $h_x$ is harmonic,

$$
\int_{\partial\Omega} \partial_{\nu} u h_x \, d\sigma = \int_{\partial\Omega} u \partial_{\nu} h_x \, d\sigma + \int_{\Omega} h_x \Delta u \, dV,
$$

and the required formula follows.

The last part of the theorem above suggest that the function $G$ defined therein may play an important role in the theory. This justifies the following definition.

**Definition 1.5.6** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$. Assume that for every $x \in \Omega$ the Dirichlet problem

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
u|_{\partial\Omega} &= -N(x - \cdot)
\end{aligned}
\]

is solvable, and that the solution $h_x$ is in $C^1(\overline{\Omega})$. The function $G : \Omega \times \Omega \setminus \{(z, z) : z \in \Omega\}$, defined by

$$
G(x, y) := h_x(y) + N(x - y),
$$

is called the **Green function** for the domain $\Omega$.

Note that we do not assert the *existence* of a Green’s function for a generic domain $\Omega$. In fact, there are domains which do not admit a Green’s function.

**Exercise 1.5.7** Prove that the Green’s function for $\Omega$, if it exists, is unique.

We explicitly state a straightforward but important consequence of Theorem 1.5.5 (iv): a representation formula for the solution to the Dirichlet problem

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
u|_{\partial\Omega} &= g,
\end{aligned}
\]

under the assumption that a solution $u$ exists and it is of class $C^1(\overline{\Omega})$.

**Corollary 1.5.8** Suppose that the domain $\Omega$ admits a Green’s function $G$. If $u \in C^1(\overline{\Omega})$ is a solution to the Dirichlet problem above, then

$$
u u(x) = \int_{\partial\Omega} g(y) \partial_{\nu} G(x - y) \, d\sigma(y) \quad \forall x \in \Omega.$$
**Definition 1.5.9** Suppose that $G$ is the Green function for the domain $\Omega$. The function $P : \Omega \times \partial \Omega \to \mathbb{R}$, defined by

$$P(x, Y) := \partial_{\nu(Y)} G(x, Y)$$

is called the **Poisson kernel** for the domain $\Omega$.

Corollary 1.5.8 indicates that it is reasonable to produce efforts to determine the Poisson kernel of a given domain $\Omega$. However, the Poisson kernel may be explicitly found only in a few, albeit important, cases, where the domain $\Omega$ has a suitable shape. The next section is devoted to the case of the upper half space in $\mathbb{R}^n$.

**Exercise 1.5.10** Suppose that $G$ is the Green’s function of the bounded domain $\Omega$. Prove the following:

(i) $G(x, y) = G(y, x)$ for every $x$, $y$ in $\Omega$, $x \neq y$;

(ii) $G(x, y) < 0$ for every $x$, $y$ in $\Omega$, $x \neq y$;

(iii) prove that $|G(x, y)| \leq |N(x - y)|$ for every $x$ and $y$ in $\Omega$;

(iv) if $f$ is bounded in $\Omega$, then $\int_{\Omega} G(x, y) f(y) \, dV(y) \to 0$ as $x \to \partial \Omega$.

**Hints:** (i) for every $z$ in $\Omega$ and every (small) $\varepsilon > 0$, denote by $\Omega_\varepsilon(z)$ the set $\Omega \setminus B_\varepsilon(z)$. Then write the harmonic function $G(x, \cdot)$ by using the representation formula in Theorem 1.5.5 (iii) with $\Omega_\varepsilon(x)$ in place of $\Omega$ and similarly write the harmonic function $G(y, \cdot)$ by using the representation formula with $\Omega_\varepsilon(y)$ in place of $\Omega$. Then use second Green’s identity and the fact that $G(x, \cdot) = N(x - \cdot) + h_x(\cdot)$;

(ii) note that $G(x, y)$ tends to $-\infty$ if $y$ tends to $x$. Therefore, $G < 0$ near $x$. Apply the maximum principle to the domain $\Omega \setminus B_\varepsilon(x)$;

(iii) observe that, by the minimum principle, $h_x > 0$ in $\Omega$ and use (ii) and the definition of $G$;

(iv) use (iii) and the Lebesgue dominated convergence theorem.

**1.6 The Poisson kernel for the half space**

Our aim is to compute the Poisson kernel for the upper half space $\mathbb{R}_+^n$ in $\mathbb{R}^n$, defined as follows

$$\mathbb{R}_+^n := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}.$$
1.6. THE POISSON KERNEL FOR THE HALF SPACE

First we need to compute the Green’s function for \( \mathbb{R}^n_+ \). Fix a point \( x = (x', x_n) \) in the upper half space, and consider the Newtonian potential \( N(x-\cdot) \) with pole \( x \). We must find a function \( h_x \) in \( C^1(\mathbb{R}^n_+) \) that is harmonic on \( \mathbb{R}^n_+ \) and such that \( N(x-\cdot) + h_x \) vanishes identically when \( x_n = 0 \). Observe that the Newtonian potential is, up to a constant, the potential generated by a unit negative charge placed at the point \( x^* := (x', -x_n) \). By symmetry, its values on the hyperplane \( x_n = 0 \) are the same of those of a Newtonian potential generated by a unit negative charge placed at the point \( x^* \). Thus, the Green’s function \( G \) of \( \mathbb{R}^n_+ \) is given by

\[
G(x,y) = \begin{cases} 
\frac{1}{2\pi} \log \frac{|x-y|}{|x^*-y|} & \text{if } n = 2 \\
\frac{1}{(2-n)\omega_n} \left(|x-y|^{2-n} - |x^*-y|^{2-n}\right) & \text{if } n \geq 3.
\end{cases}
\]

(1.6.1)

**Exercise 1.6.1** Check that the function \( G \) defined above is the Green’s function for the upper half space.

First we consider the case where \( n \geq 3 \). By definition, for every \( x \in \mathbb{R}^n_+ \) and \( y' \in \mathbb{R}^{n-1} \) the Poisson kernel \( P(x,y') \) is then given by

\[
P(x,y') = \partial_{y_n} G(x,y') \\
= -\frac{1}{\omega_n} \left[ \frac{y_n - x_n}{|x-y|^n} - \frac{y_n + x_n}{|x^*-y|^n} \right]_{y_n=0}.
\]

Observe that \( |x-y| = |x^*-y| \) when \( y_n = 0 \). Hence

\[
P(x,y') = \frac{2}{\omega_n} \frac{x_n}{|x-y|^n} = \frac{2}{\omega_n} \frac{x_n}{\left(|x' - y'|^2 + x_n^2\right)^{n/2}} \quad \forall x_n > 0, x', y' \in \mathbb{R}^{n-1}.
\]

(1.6.2)

A similar computation shows that the formula above holds also in the case where \( n = 2 \).

**Exercise 1.6.2** Compute the Poisson kernel for a quadrant of the plane. 
*Hint:* refine the method of images illustrated above for the half plane.

Note that, given a function \( g \) on \( \mathbb{R}^{n-1} \) the Dirichlet problem

\[
\begin{cases} 
\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\
u|_{\partial\mathbb{R}^n_+} = g 
\end{cases}
\]
may have more than one solution. In particular, if \( g \) vanishes identically, then the null function and the function \( u(x', x_n) = x_n \) both solve the Dirichlet problem above. Thus, if we want to recover uniqueness, we must impose further restriction on the solution. A typical statement which holds in this case is the following.

**Theorem 1.6.3** Suppose that \( g \) is continuous and bounded on \( \mathbb{R}^{n-1} \). Then there is a unique bounded continuous function \( u \) on \( \mathbb{R}^n_+ \) which solves the Dirichlet problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n_+ \\
\left. u \right|_{\partial \mathbb{R}^n_+} &= g.
\end{align*}
\]

Furthermore

\[
u(x) = \int_{\mathbb{R}^{n-1}} g(y') P(x, y') \, dy' = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} g(y') \frac{x_n}{(|x' - y'|^2 + x_n^2)^{n/2}} \, dy' \quad \forall x \in \mathbb{R}^n_+,
\]

and \( u(x', 0) = g(x') \) for every \( x' \in \mathbb{R}^{n-1} \).

**Remark 1.6.4** Set

\[
p(x') := \frac{2}{\omega_n} \frac{1}{(|x'|^2 + 1)^{n/2}} \quad \forall x' \in \mathbb{R}^{n-1}.
\]

Note that \( p \) is in \( L^1(\mathbb{R}^{n-1}) \) and that

\[P(x, y') = p_{x_n}(x' - y'),\]

where, as customary, we write \( p_{x_n} \) to be the **dilated and normalised version** of \( p \) in \( L^1(\mathbb{R}^{n-1}) \). Explicitly,

\[p_{x_n}(w') = x_n^{-n} p\left( w'/x_n \right) \quad \forall w' \in \mathbb{R}^{n-1} \quad \forall x_n > 0.
\]

Observe that \( \| p_{x_n} \|_1 = \| p \|_1 \), thereby justifying the terminology. With this notation, formula (1.6.3) may be rewritten as follows

\[u(x) = g * p_{x_n}(x') \quad \forall x \in \mathbb{R}^n_+,
\]

where \( * \) stands for convolution on \( \mathbb{R}^{n-1} \).

**Exercise 1.6.5** Prove that (1.6.3) gives, indeed, a solution to the Dirichlet problem on the upper half space with boundary datum \( g \).

By using a celebrated theorem of Liouville and the Schwartz reflection principle, we may prove that if \( g \) is continuous and bounded, then (1.6.4) gives the unique bounded solution to the Dirichlet problem in the upper half space.
1.7 The Poisson kernel for the ball

The aim of this section is to provide the solution to the classical Dirichlet problem in the ball $B_R(0)$. The theory developed here is important at least for two reasons: (i) it leads to an explicit formula for the solution; (ii) the fact that the classical Dirichlet problem in every ball is solvable is a key step towards the solution of the classical Dirichlet problem in more general domains (see Definition 1.8.3 and the proof of Theorem 1.8.11).

1.7.1 The two dimensional case

We first look at the two dimensional case, where the theory of functions of a complex variable is of valuable help. Suppose that $\varphi$ is a conformal map between the domains $\Omega \subset \mathbb{C}$ and $\varphi(\Omega)$. In particular, $\varphi$ is a bijection between $\Omega$ and $\varphi(\Omega)$ and $\varphi'(z) \neq 0$ for every $z$ in $\Omega$.

**Proposition 1.7.1** Suppose that $\Omega$ and $\varphi$ are as above, that $\varphi$ extends to a biholomorphic map between $\Omega$ and $\varphi(\Omega)$ and assume that $\varphi(\Omega)$ admits a Green’s function $G^{\varphi(\Omega)}$. Then $\Omega$ admits a Green’s function $G^{\Omega}$ and

$$G^{\Omega}(z, \zeta) = G^{\varphi(\Omega)}(\varphi(z), \varphi(\zeta)) \quad \forall (z, \zeta) \in \Omega \times \Omega \setminus \{(\omega, \omega) : \omega \in \Omega\}.$$

**Proof.** For each $\zeta \in \partial \Omega$ the point $\varphi(\zeta)$ is in $\partial(\varphi(\Omega))$. Therefore for each $z \in \Omega$

$$G^{\varphi(\Omega)}(\varphi(z), \varphi(\zeta)) = 0,$$

because $G^{\varphi(\Omega)}$ is the Green’s function for $\varphi(\Omega)$. For the same reason

$$\Delta_w G^{\varphi(\Omega)}(w, \varphi(\zeta)) = 0.$$

Since $\varphi$ is conformal,

$$\Delta_w G^{\varphi(\Omega)}(w, \varphi(\zeta)) = |\varphi'(z)|^2 \Delta_z G^{\varphi(\Omega)}(\varphi(z), \varphi(\zeta)),$$

by (1.2.9) (viii), whence $G^{\Omega}(\cdot, \zeta)$ is harmonic for every $\zeta$ in $\Omega$.

Finally,

$$G^{\Omega}(z, \zeta) = \frac{1}{2\pi} \log |z - \zeta|$$

$$= G^{\varphi(\Omega)}(\varphi(z), \varphi(\zeta)) - \frac{1}{2\pi} \log |z - \zeta|$$

$$= \frac{1}{2\pi} \log |\varphi(z) - \varphi(\zeta)| + h_{\varphi(z)}(\varphi(\zeta)) - \frac{1}{2\pi} \log |z - \zeta|$$

$$= \frac{1}{2\pi} \log \left| \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta} \right| + h_{\varphi(z)}(\varphi(\zeta))$$
Observe that $\varphi(z) - \varphi(\zeta)/z - \zeta$ tends to $\varphi'(z)$ as $\zeta$ tends to $z$. Therefore the function

$$
\zeta \mapsto \frac{1}{2\pi} \log \left| \frac{\varphi(z) - \varphi(\zeta)}{(z - \zeta)} \right| + h_{\varphi(z)}(\varphi(\zeta))
$$

is harmonic in $\Omega \setminus \{z\}$ and bounded in a neighbourhood of $z$, hence it is harmonic in $\Omega$. We have proved that $G^\Pi(z, \cdot)$ may be written as the sum of the Newtonian potential with pole $z$ and a function harmonic in $\Omega$.

This completes the proof of the proposition. □

We apply this result to the case where the domain $\Omega$ is just the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ and 

$$
\varphi(z) = i \frac{1 + z}{1 - z}, \quad \forall z \in D.
$$

It is not hard to check that $\varphi$ maps $D$ conformally onto the upper half plane $\Pi$, and $\partial D$ onto $\mathbb{R} \cup \infty$. In the last section we have proved that the Green’s function for the upper half plane is given by 

$$
G^\Pi(w, \omega) := \frac{1}{2\pi} \log \frac{|w - \omega|}{|\overline{w} - \omega|}.
$$

Denote by $z$ and $\zeta$ the points in $D$ such that 

$$
\varphi(z) = w \quad \text{and} \quad \varphi(\zeta) = \omega.
$$

It is straightforward to check that $\overline{w} = \varphi(1/\overline{z})$, i.e., $\overline{w}$ is the image under $\varphi$ of the point obtained from $z$ by the mapping 

$$
z \mapsto \frac{1}{\overline{z}},
$$

which is the inversion with respect to the circle $\{|z| = 1\}$. Observe that 

$$
G^D(z, \zeta) = G^\Pi(\varphi(z), \varphi(\zeta))
= \frac{1}{2\pi} \log \frac{|\varphi(z) - \varphi(\zeta)|}{|\varphi(1/\overline{z}) - \varphi(\zeta)|}
= \frac{1}{2\pi} \log \frac{|1 + z - 1 + \zeta|}{|1 + (1/\overline{z}) - 1 + \zeta|}
= \frac{1}{2\pi} \log \frac{|z - \zeta|}{|1 - \overline{z}\zeta|}.
$$
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We write \( z = re^{i\phi} \) and \( \zeta = se^{i\theta} \), and obtain

\[
GD(r e^{i\phi}, s e^{i\theta}) = \frac{1}{2\pi} \log \frac{|r - se^{i(\theta - \phi)}|}{|1 - rs e^{i(\theta - \phi)}|}
\]

\[
= \frac{1}{4\pi} \log \frac{r^2 - 2rs \cos(\theta - \phi) + s^2}{1 - 2rs \cos(\theta - \phi) + r^2 s^2}
\]

The Poisson kernel is then

\[
\partial_s GD(r e^{i\phi}, s e^{i\theta})|_{s=1} = \frac{1}{4\pi} \left[ \frac{-2r \cos(\theta - \phi) + 2}{r^2 - 2r \cos(\theta - \phi) + 1} - \frac{-2r \cos(\theta - \phi) + 2r^2}{1 - 2r \cos(\theta - \phi) + r^2} \right]
\]

\[
= \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2}.
\]

Therefore, if a solution \( u \in C^1(D) \) to the Dirichlet problem

\[
\begin{cases}
\Delta u = 0 & \text{in } D \\
u|_{\partial D} = g,
\end{cases}
\]

exists, then

\[
u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} g(e^{i\theta}) \, d\theta. \quad (1.7.1)
\]

The aim of the following series of exercises is to determine the Green’s function, hence the Poisson kernel, of certain domains in the complex plane that are conformally equivalent to the unit disc.

**Exercise 1.7.2** Compute the Green’s function and the Poisson kernel of the half disc

\[
F := \{ z \in \mathbb{C} : |z| < 1, \ \text{Im}(z) > 0 \}.
\]

**Exercise 1.7.3** Prove that the map \( z \mapsto e^z \) is a conformal map of the strip

\[
S := \{ z \in \mathbb{C} : 0 < \text{Im}(z) < \pi \}
\]

onto the upper half plane. Then compute the Green’s function and the Poisson kernel of the strip \( S \).

**Exercise 1.7.4** Prove that the map \( z \mapsto z^{\alpha/\pi} \) is a conformal map of the upper half plane onto the sector

\[
\Gamma := \{ z \in \mathbb{C} : 0 < \text{arg}(z) < \alpha \}.
\]

Then compute the Green’s function and the Poisson kernel of the sector \( \Gamma \).
CHAPTER 1. THE CLASSICAL DIRICHLET PROBLEM

1.7.2 The higher dimensional case

We now consider the case where \( n \geq 3 \). First, we deal with the case where \( R = 1 \). The idea is that the potential on the unit sphere \( \partial B_1(0) \) induced by a unit charge at \( x \neq 0 \) is equal to the potential on \( \partial B_1(0) \) induced by a charge of magnitude \( |x|^{2-n} \) at the point \( x/|x|^2 \), obtained by inversion with respect to \( \partial B_1(0) \) of \( x \).

**Lemma 1.7.5** Suppose that \( Y \in \partial B_1(0) \) and that \( x \in B_1(0) \setminus \{0\} \). Then

\[
|Y - x| = \frac{|x|}{|x|^2} Y - \frac{x}{|x|},
\]

**Proof.** This lemma is geometrically obvious (just draw a picture). For an analytic proof, just square both sides of the formula above and make the required computation. \( \square \)

**Proposition 1.7.6** Suppose that \( R > 0 \). The following hold:

(i) the function, defined for every \( x \in B_1(0) \) and for every \( y \in \overline{B_1(0)} \) by

\[
G_1(x, y) = N(x - y) - |x|^{2-n} N(x/|x|^2 - y) = \frac{1}{(2 - n) \omega_n} \left[ |x - y|^{2-n} - |x|^{-1} x - |x| |y|^{2-n} \right],
\]

is the Green’s function of \( B_1(0) \);

(ii) the Poisson kernel for the ball \( B_R(0) \) is given by

\[
P_R(x, Y) = \frac{R^2 - |x|^2}{\omega_n R |x - Y|^n} \quad \forall x \in B_R(0) \quad \forall Y \in \partial B_R(0).
\]

**Proof.** To prove (i) observe that the first equation shows that \( G_1(x, \cdot) \) is harmonic in \( B_1(0) \), and, by Lemma 1.7.5, the second proves that \( G_1(x, y) = N(x - y) + h_x(y) \), where \( h_x \) is harmonic in \( B_1(0) \) and \( h_x(Y) = -N(x - Y) \) when \( Y \in \partial B_1(0) \). Since the Green’s function, if it exists, is unique, \( G_1 \) is the Green’s function of \( B_1 \).

Next we prove (ii). It is straightforward to check that the function

\[
G_R(x, y) = \frac{1}{R^{n-2}} G_1(x/R, y/R)
\]
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is the Green’s function for \( B_R(0) \). Indeed, by Lemma 1.7.5, if \( Y \in \partial B_R(0) \) and \( x \in B_R(0) \), then

\[
|Y - x| = \left| Y \frac{|x|}{R} - R \frac{x}{|x|} \right|.
\]

(1.7.3)

Clearly \( G_R(x, \cdot) \) is harmonic in \( B_R(0) \). Furthermore, if \( Y \in \partial B_R(0) \), then

\[
G_R(x, Y) = \frac{1}{(2 - n) \omega_n} R^{2-n} \left[ \frac{|x|}{R} - Y \right]^{2-n} - \left| \frac{R}{|x|} x - \frac{|x|}{R} Y \right|^{2-n}
\]

\[
= 0,
\]

where the last equality follows from (1.7.3). Denote by \( \nu \) the outward normal of \( \partial B_R(0) \) at the point \( Y \). By the definition of Poisson kernel,

\[
P_R(x, Y) = \partial_\nu G_R(x, Y)
\]

\[
= \frac{1}{\omega_n} \left[ \frac{Y - x}{|x - Y|^n} \cdot \frac{Y}{|Y|} - \frac{|x|}{R} \frac{Y - R x}{|x|^n} \cdot \frac{Y}{|Y|} \right]
\]

\[
= \frac{R^2 - |x|^2}{\omega_n R |x - Y|^n},
\]

as required. \( \square \)

A direct consequence of Corollary 1.5.8 and Definition 1.5.9 is the following representation formula for solutions to the Dirichlet problem on \( B_R(0) \).

**Corollary 1.7.7** Suppose that \( u \in C^1(\overline{B_R(0)}) \) solves the Dirichlet problem

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad \Omega \\
u|_{\partial \Omega} &= g.
\end{aligned}
\]

Then

\[
u(y) = \frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(X)}{|y - X|^n} \, d\sigma(X) \quad \forall y \in \Omega.
\]

In addition, we can prove that a solution of the Dirichlet problem above with continuous boundary data exists. This is the main result of this section.
Theorem 1.7.8 Suppose that \( g \in C(\partial B_R(0)) \). Then the Dirichlet problem

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = g
\end{cases}
\]

has a unique solution \( u \in C(\overline{B}_R(0)) \), given by

\[
u(y) = \begin{cases}
R^2 - |y|^2 \\
g(y)
\end{cases} \frac{\int_{\partial B_R(0)} g(X) \, d\sigma(X)}{\omega_n R |y - X|^n} \quad \forall y \in \Omega
\]

\[
u(y) = \begin{cases}
R^2 - |y|^2 \\
g(y)
\end{cases} \frac{\int_{\partial B_R(0)} g(X) \, d\sigma(X) - g(X_0) |y - X|^n}{\omega_n R} + 1 \quad \forall y \in \partial B_R(0).
\]

Proof. By differentiating under the integral sign, it is straightforward to check that the function \( u \) defined above is harmonic in \( B_R(0) \). Thus, it remains to prove that for every \( X_0 \in \partial B_R(0) \)

\[
\lim_{y \to X_0} u(y) = g(X_0),
\]

as \( y \) tends to \( X_0 \) within \( B_R(0) \). For \( y \in B_R(0) \) we write

\[
u(y) = \frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X)
\]

\[
+ g(X_0) \frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{d\sigma(X)}{|y - X|^n}.
\]

We shall prove that

\[
\lim_{y \to X_0} \frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X) = 0 \quad (1.7.5)
\]

and that

\[
\frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{d\sigma(X)}{|y - X|^n} = 1 \quad \forall y \in B_R(0). \quad (1.7.6)
\]

These formulae will imply the required conclusion \((1.7.4)\). The second formula is a direct consequence of the representation formula of Corollary 1.7.7 (with the constant function 1 in place of \( g \) and \( u \)).

To prove \((1.7.5)\), we proceed as follows. Since \( g \) is continuous in \( X_0 \), for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|g(X) - g(X_0)| < \varepsilon \quad \forall X \in B_{2\delta}(X_0) \cap \partial B_R(0).
\]
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We split the integral over $\partial B_R(0)$ as the sum of the integral over $\partial B_R(0) \cap B_{2\delta}(X_0)$ and the integral over $\partial B_R(0) \setminus B_{2\delta}(X_0)$, and estimate them separately.

To estimate the first, observe that

$$\left| \int_{\partial B_R(0) \cap B_{2\delta}(X_0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X) \right| \leq \varepsilon \int_{\partial B_R(0) \cap B_{2\delta}(X_0)} |y - X|^{-n} \, d\sigma(X) \leq \varepsilon \frac{\omega_n R}{R^2 - |y|^2} \varepsilon \quad \forall y \in B_R(0).$$

Moreover

$$\left| \int_{\partial B_R(0) \setminus B_{2\delta}(X_0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X) \right| \leq 2\|g\|_{\infty} \int_{\partial B_R(0) \setminus B_{2\delta}(X_0)} |y - X|^{-n} \, d\sigma(X) \delta^{-n} \sigma(\partial B_R(0)) \quad \forall y \in B_\delta(X_0).$$

We observe that $|y - X| \geq \delta$ for all $y$ in $B_R(0) \cap B_\delta(X_0)$. Therefore the integral on the right hand side is estimated from above by $\delta^{-n} \sigma(\partial B_R(0) \cap B_{2\delta}(X_0))$,

and

$$\left| \int_{\partial B_R(0) \setminus B_{2\delta}(X_0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X) \right| \leq \frac{2\|g\|_{\infty}}{\omega_n} R^2 - |y|^2 \varepsilon \quad \forall y \in B_\delta(X_0).$$

By combining the estimates above we see that given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y$ in $B_R(0) \cap B_\delta(X_0)$

$$\frac{R^2 - |y|^2}{\omega_n R} \left| \int_{\partial B_R(0) \setminus B_{2\delta}(X_0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X) \right| \leq \varepsilon + C \delta^{-n} (R - |y|).$$

Thus, for every $\varepsilon > 0$

$$\lim_{y \to X_0} \frac{R^2 - |y|^2}{\omega_n R} \left| \int_{\partial B_R(0)} \frac{g(X) - g(X_0)}{|y - X|^n} \, d\sigma(X) \right| \leq \varepsilon,$$

which clearly implies (1.7.5).

The proof of the theorem is complete. \qed
1.8 Perron’s method

In this section we characterise all domains in $\mathbb{R}^n$, $n \geq 2$, for which the Dirichlet problem with continuous boundary data is solvable. The method we employ is due to Perron. One of its merits lies in the fact that the theory separates in a natural way the existence of a solution in $\Omega$ from the boundary behaviour of the solution.

We need to enlarge the definition of subharmonic and superharmonic functions to functions in $C(\Omega)$. Recall that so far we have given the definition of $C^2(\Omega)$ subharmonic functions.

**Definition 1.8.1** Suppose that $\Omega$ is a domain in $\mathbb{R}^n$. A function $u \in C(\Omega)$ is subharmonic in $\Omega$ if for every ball $B \subset \subset \Omega$ and for every function $h$, which is continuous on $B$, harmonic in $B$, and satisfies $u \leq h$ on $\partial B$, we have that $u \leq h$ in $B$.

The definition of $C(\Omega)$ superharmonic function is similar (just revert all $\leq$ signs into $\geq$). Henceforth by subharmonic function in $\Omega$ we mean a $C(\Omega)$ subharmonic function in the sense of the above definition.

**Exercise 1.8.2** Prove that a function $u \in C^2(\Omega)$ such that $\Delta u \geq 0$ in $\Omega$ is subharmonic in the sense of the definition above.

Perron’s method hinges on the notion of harmonic lifting, which we now define.

**Definition 1.8.3** Suppose that $\Omega$ is a domain in $\mathbb{R}^n$, that $u$ is subharmonic in $\Omega$, and that $B \subset \subset \Omega$. We denote by $H_B(u)$ the harmonic lifting of $u$ in $B$, defined as the function on $\Omega$ that agrees with $u$ on $\Omega \setminus B$, and with the solution to the Dirichlet problem

$$\begin{cases}
\Delta h = 0 & \text{in } B \\
h|_{\partial B} = u
\end{cases}$$

on $\overline{B}$. Clearly $H_B(u)$ is given by the Poisson integral in $B$ of the restriction of $u$ to $\partial B$.

The following lemma summarizes some useful properties of subharmonic functions.
Lemma 1.8.4 Suppose that $\Omega$ is a domain in $\mathbb{R}^n$. The following hold:

(i) if $u$ is subharmonic in $\Omega$, then it satisfies the strong maximum principle in $\Omega$;

(ii) if $\Omega$ is bounded, $u, v$ are in $C(\overline{\Omega})$, $u$ is subharmonic in $\Omega$, $v$ is superharmonic in $\Omega$ and $u \leq v$ on $\partial \Omega$, then either $u < v$ in $\Omega$, or $u = v$;

(iii) if $u$ is subharmonic in $\Omega$ and $B \subset \subset \Omega$, then the harmonic lifting $H_B(u)$ is subharmonic in $\Omega$;

(iv) if $u_1, \ldots, u_N$ are subharmonic functions in $\Omega$, then

$$\max(u_1, \ldots, u_N)$$

is subharmonic in $\Omega$.

We leave the reader the task of stating and proving the corresponding properties for superharmonic functions.

Proof. We first prove (i). Suppose that $y$ is a point in $\Omega$ such that

$$u(y) = \sup_{\Omega} u.$$

We need to prove that $u$ is constant. Clearly the set $\Omega_0 := \{ x \in \Omega : u(x) = u(y) \}$ is closed, because $u$ is continuous in $\Omega$. We prove that $\Omega_0$ is open. Suppose that $x \in \Omega_0$ and that $B_R(x) \subset \subset \Omega$. Denote by $h$ the solution to the Dirichlet problem on $B_R(x)$ with boundary values $u|_{\partial B_R(x)}$. Since $u$ is subharmonic, $u \leq h$ on $\overline{B_R(x)}$. Notice that

$$\sup_{\partial B_R(x)} h = \sup_{\partial B_R(x)} u \leq u(x) \leq h(x).$$

By the strong maximum principle, $h$ is constant on $\overline{B_R(y)}$ and the constant must be $h(y)$. Thus,

$$u|_{\partial B_R(x)} = u(x).$$

By repeating the same reasoning for all $r < R$, we find that $u = u(x)$ on $B_R(x)$. Thus $\Omega_0$ is open. Since $\Omega$ is connected, and $\Omega_0$ is nonempty, $\Omega_0 = \Omega$, i.e., $u$ is constant on $\Omega$, as required.

To prove (ii), we observe preliminarily that $-v$ is a subharmonic function, and that the sum of two subharmonic functions is subharmonic. Indeed, if $B \subset \subset \Omega$ and $h$ is continuous on $\overline{B}$, harmonic in $\Omega$, and satisfies $-v|_{\partial B} \leq h|_{\partial B}$, then $v|_{\partial B} \geq -h|_{\partial B}$. Since $v$ is superharmonic and $-h$ is harmonic, we
may conclude that \( v \geq -h \) in \( B \), equivalently that \(-v \leq h\). Hence \(-v\) is subharmonic. Suppose that \( u_1 \) and \( u_2 \) are subharmonic functions in \( \Omega \) and fix \( B \subset \subset \Omega \). Denote by \( U_j \) the unique solution to the Dirichlet problem

\[
\begin{align*}
\Delta w &= 0 \quad \text{in } B \\
 w|_{\partial B} &= (u_j)|_{\partial B}.
\end{align*}
\]

Clearly \( u_j \leq U_j \) in \( B \), because \( u_j \) is subharmonic. Assume that \( h \) is continuous on \( B \), harmonic in \( B \), and satisfies \((u_1 + u_2)|_{\partial B} \leq h|_{\partial B}\). Then \( U_1 + U_2 \leq h \) in \( \overline{B} \) by the maximum principle. Therefore \( u_1 + u_2 \leq h \) on \( \overline{B} \), as required.

As a consequence, \( u - v \) is subharmonic in \( \Omega \). Clearly \( u - v \) belongs to \( C(\overline{\Omega}) \), and, by assumption, \( u - v \leq 0 \) on \( \partial \Omega \). Set

\[
M := \sup_{\Omega} (u - v).
\]

We first prove that \( M \leq 0 \). Indeed, if \( M \) were strictly positive, then, by compactness, there would exist a point \( x_0 \in \Omega \) such that

\[
M = (u - v)(x_0).
\]

Then, by (i) (i.e., by the strong maximum principle), \( u - v \) would be constant (equal to \( M > 0 \)) in \( \Omega \). This, together with the continuity of \( u - v \) on \( C(\overline{\Omega}) \), would then contradict the fact that \( u - v \leq 0 \) on \( \partial \Omega \).

Thus, \( M \leq 0 \). Clearly, if \( M < 0 \), then \( u < v \) on \( \Omega \). To conclude the proof of (ii), assume that \( M = 0 \). Again, either \( u < v \) in \( \Omega \), or the inequality \( u < v \) fails at some point of \( \Omega \). In the latter case there exists \( x_0 \in \Omega \) such that

\[
(u - v)(x_0) = 0.
\]

Then \( u = v \) by the strong maximum principle (i).

Next we prove (iii). Suppose that \( B' \) is a ball, which is relatively compact in \( \Omega \). We need to show that for every function \( h \), which is continuous on \( \overline{B'} \), harmonic in \( B' \), and satisfies \( H_B(u)|_{\partial B'} \leq h|_{\partial B'} \), we have \( H_B(u) \leq h \) in \( B' \). This is clear if \( B' \subset B \) or \( B' \subset (\Omega \setminus B) \). It remains to consider the case where \( B' \cap (\partial B) \neq \emptyset \).

Since \( h \geq H_B(u) \) on \( \partial B' \) and \( H_B(u) \geq u \) on \( \Omega \), \( h \geq u \) on \( \partial B' \). Then \( h \geq u \) on \( \overline{B'} \), because \( u \) is subharmonic. Recall that \( H_B(u) = u \) on \( \Omega \setminus B \), whence \( h \geq H_B(u) \) on \( \partial (B \cap B') \). Since both \( h \) and \( H_B(u) \) are harmonic on \( B \cap B' \), \( h \geq H_B(u) \) on \( \overline{(B \cap B')} \) by the maximum principle. This concludes the proof of (iii).
Finally we prove (iv). Suppose that $B$ is a relatively compact ball in $\Omega$, and that $h$ is continuous on $\overline{B}$, harmonic in $B$, and satisfies $\max(u_1, \ldots, u_N)_{|\partial B} \leq h_{|\partial B}$. Then $(u_j)_{|\partial B} \leq h_{|\partial B}$ for each $j$. By assumption $u_j$ is subharmonic, so that $u_j \leq h$ in $\overline{B}$ for every $j$, whence $\max(u_1, \ldots, u_N) \leq h$ on $\overline{B}$. Therefore $\max(u_1, \ldots, u_N)$ is subharmonic, as required.

To proceed further, we need a compactness criterion for harmonic functions. First we recall the classical compactness result of Ascoli and Arzelà.

Recall that a sequence $\{f_j\}$ of continuous functions on a set $E \subset \mathbb{R}^n$ is **uniformly bounded** if there exists a constant $C$ such that

$$\sup_E |f_j| \leq C \quad \forall j.$$  

The sequence $\{f_j\}$ is **equicontinuous** if for every $\varepsilon > 0$ there exists a constant $\delta$ such that

$$x, y \in E, \ |x - y| < \delta \quad \implies \quad |f_j(x) - f_j(y)| < \varepsilon \quad \forall j.$$  

Suppose that $E$ is an interval of the real line, that $f_j$ is differentiable, and that there exists a constant $C$ such that

$$\sup_E |f_j'| \leq C \quad \forall j.$$  

Then $\{f_j\}$ is equicontinuous.

**Exercise 1.8.5** Prove the assertion above.

**Theorem 1.8.6** Suppose that $\{f_j\}$ is a sequence of uniformly bounded and equicontinuous functions on the compact set $K \subset \mathbb{R}^n$. Then there exists a subsequence $\{f_{j_k}\}$ such that $\{f_{j_k}\}$ is uniformly convergent on $K$ to a continuous function $f$.

We refer the reader to [Forl Thm 4.43, p. 137] for a proof of this result.

We shall apply Ascoli–Arzelà’s theorem to sequences of harmonic functions on domains of $\mathbb{R}^n$. The main point is that the equicontinuity of such a sequence is a consequence of its uniform boundedness. A key step towards a compactness criterion for harmonic functions is the following result, of independent interest, which gives estimates for derivatives of harmonic functions in terms of bounds of the functions themselves.
Proposition 1.8.7 Suppose that \( u \) is harmonic in the domain \( \Omega \), and that \( K \) is a compact subset of \( \Omega \). Then, for every multiindex \( \alpha \)

\[
\sup_K |D^\alpha u| \leq \left( \frac{n |\alpha|}{\text{dist}(K, \partial \Omega)} \right)^{|\alpha|} \sup_\Omega |u|.
\]

**Proof.** We prove the required estimate in the case where \( |\alpha| = 1 \). Since \( \partial_j u \) is harmonic in \( \Omega \) (because \( \partial_j \) and \( \Delta \) commute), we may use the mean value property for harmonic functions and write

\[
\partial_j u(x) = \frac{1}{V(B_R(x))} \int_{B_R(x)} \partial_j u \, dV = \frac{1}{V(B_R(x))} \int_{\partial B_R(x)} u \nu_j \, d\sigma.
\]

We have used the formula of integration by parts in the last equality. Here \( R < \text{dist}(K, \partial \Omega) \), so that \( B_R(x) \subset \Omega \). Therefore

\[
|\partial_j u(x)| \leq \frac{1}{V(B_R(x))} \sigma_R(\partial B_R(x)) \sup_\Omega |u| = \frac{n}{R} \sup_\Omega |u|
\]

The required estimate follows by taking the supremum of both sides with respect to \( R < \text{dist}(K, \partial \Omega) \). \( \square \)

**Exercise 1.8.8** Prove that every bounded harmonic function on \( \mathbb{R}^n \) is constant.

**Corollary 1.8.9** Every bounded sequence of harmonic functions on a domain \( \Omega \) contains a subsequence converging uniformly on compact subdomains of \( \Omega \) to a harmonic function.

**Proof.** Suppose that \( \{u_j\} \) is a bounded sequence of harmonic functions on \( \Omega \), and that \( K \) is a compact subdomain of \( \Omega \). By Proposition 1.8.7,

\[
\sup_K |\nabla u_j| \leq \frac{n}{\text{dist}(K, \partial \Omega)} \sup_\Omega |u_j| \leq C_{n,K}.
\]

Therefore \( \{u_j\} \) is an equicontinuous sequence of functions on \( K \). By Ascoli–Arzelà’s theorem, there exists a subsequence \( \{u_{j_k}\} \) such that \( \{u_{j_k}\} \) converges...
uniformly on $K$ to a continuous function $u$. It remains to prove that $u$ is harmonic in the interior of $K$. It suffices to show that $u$ satisfies the mean value theorem therein. Suppose that $x$ is in the interior of $K$. Then $B_R(x)$ is contained in the interior of $K$ for small $R$. Since $u_{j\ell}$ is harmonic in $\Omega$,

$$u_{j\ell}(x) = \frac{1}{\sigma(\partial B_R(x))} \int_{\partial B_R(x)} u_{j\ell} \, d\sigma.$$  

The uniform convergence of $\{u_{j\ell}\}$ to $u$ implies that

$$u(x) = \frac{1}{\sigma(\partial B_R(x))} \int_{\partial B_R(x)} u \, d\sigma.$$  

Hence $u$ is a continuous function on the interior of $K$ that satisfies the mean value property. The required conclusion then follows from Theorem 1.3.9.

### Definition 1.8.10
Suppose that $\Omega$ is a bounded domain and that $\varphi$ is a bounded function defined on $\partial \Omega$. We say that a function $v \in C(\overline{\Omega})$ is a subfunction relative to $\varphi$ if $v$ is subharmonic in $\Omega$ and $v \leq \varphi$ on $\partial \Omega$. The set of all subfunctions relative to $\varphi$ is denote by $S_{\varphi}$.

### Theorem 1.8.11
Suppose that $\Omega$ is a bounded domain and that $\varphi$ is a bounded function defined on $\partial \Omega$. The function $u$, defined by

$$u(x) := \sup_{v \in S_{\varphi}} v(x) \quad \forall x \in \Omega,$$

is harmonic in $\Omega$.

**Proof.** Note that every subfunction relative to $\varphi$ is bounded above by $\sup \varphi$, so that $u$ is well defined and bounded above by $\sup \varphi$. Note that $\inf \varphi$ is a subfunction relative to $\varphi$. Hence $u \geq \inf \varphi$ in $\Omega$.

Fix $y \in \Omega$. We shall show that $u$ is harmonic in a neighbourhood of $y$. By letting $y$ vary in $\Omega$ we may then conclude that $u$ is harmonic in $\Omega$, as required.

By the definition of $u$, there exists a sequence $\{v_\ell\}$ of subfunctions relative to $\varphi$ such that

$$u(y) = \lim_{\ell \to \infty} v_\ell(y).$$

It may very well happen that $\{v_\ell\}$ is unbounded from below. However, the sequence $\{\max(v_n, \inf \varphi)\}$ is bounded from below, and still

$$u(y) = \lim_{\ell \to \infty} \max(v_\ell(y), \inf \varphi),$$
CHAPTER 1. THE CLASSICAL DIRICHLET PROBLEM

so that we may assume that \( \{v_\ell\} \) is bounded. Now, we choose \( R \) such that \( B_R(y) \subset \Omega \). We consider the harmonic lifting \( H_{B_R(y)}(v_\ell) \) of \( v_\ell \). Recall that \( H_{B_R(y)}(v_\ell) \) is subharmonic in \( \Omega \), and harmonic in \( B_R(y) \).

Clearly \( H_{B_R(y)}(v_\ell) \) is a subfunction relative to \( \varphi \) and

\[
  u(y) = \lim_{\ell \to \infty} H_{B_R(y)}(v_\ell)(y).
\]

Observe that \( \{H_{B_R(y)}(v_\ell)\} \) is a sequence of uniformly bounded harmonic functions on \( B_R(y) \). By Corollary [1.8.9] there exists a subsequence \( \{\ell_j\} \) such that \( \{H_{B_R(y)}(v_{\ell_j})\} \) is uniformly convergent on \( \overline{B_{R/2}(y)} \) to a function \( v \), which is harmonic in \( B_{R/2}(y) \).

Obviously \( v \leq u \) in \( \overline{B_{R/2}(y)} \). We claim that \( v = u \) in \( B_{R/2}(y) \).

We argue by contradiction. Suppose that there exists \( z \in B_{R/2}(y) \) such that

\[
  v(z) < u(z).
\]

By the definition of the function \( u \), there exists a subfunction \( \tilde{v} \) relative to \( \varphi \) in \( \Omega \) such that

\[
  v(z) < \tilde{v}(z) \leq u(z). \tag{1.8.1}
\]

Consider the sequence

\[
  w_j := \max(\tilde{v}, H_{B_R(y)}(v_{\ell_j})).
\]

Clearly \( w_j \) is a subfunction relative to \( \varphi \). Its harmonic lifting \( H_{B_R(y)}(w_j) \) is harmonic in \( B_R(y) \),

\[
  u(y) = \lim_{j \to \infty} H_{B_R(y)}(w_j)(y),
\]

and, possibly passing to a subsequence, \( \{H_{B_R(y)}(w_j)\} \) is uniformly convergent on \( \overline{B_{R/2}(y)} \) to a function \( w \), which is harmonic in \( B_{R/2}(y) \). Clearly \( w \) satisfies

\[
  v \leq w \leq u \quad \text{on} \quad \overline{B_{R/2}(y)}, \quad v(z) < \tilde{v}(z) \leq w(z) \quad \text{and} \quad v(y) = w(y) = u(y).
\]

Observe that \( v - w \) is harmonic in \( B_{R/2}(y) \) and that \( (v - w)|_{\partial B_{R/2}(y)} \leq 0 \). Since \( (v - w)(y) = 0 \), \( v - w = 0 \) in \( B_R(y) \) by the maximum principle. This contradicts (1.8.1). Therefore \( v = u \) on \( B_{R/2}(y) \). This proves the claim.

The proof of the theorem is complete. \( \square \)

**Definition 1.8.12** The function

\[
  u(x) := \sup_{v \in S_\varphi} v(x) \quad \forall x \in \Omega,
\]

is called the **Perron solution** with boundary datum \( \varphi \).
Corollary 1.8.13  Suppose that Ω is a bounded domain and that φ is a continuous function on ∂Ω. If the classical Dirichlet problem

\[
\begin{aligned}
\Delta w &= 0 \quad \text{in } \Omega \\
\left. w \right|_{\partial \Omega} &= \varphi
\end{aligned}
\]

has a solution \( w \in C^2(\Omega) \cap C(\overline{\Omega}) \), then \( w \) agrees on \( \Omega \) with the Perron solution with boundary datum \( \varphi \).

Proof.  On the one hand, observe that \( w \) is in \( S_{\varphi} \). Hence \( u = \sup_{v \in S_{\varphi}} v \geq w \) in \( \Omega \).

On the other hand, if \( v \in S_{\varphi} \), then \( v \leq w \) on \( \partial \Omega \), whence \( v \leq w \) on \( \overline{\Omega} \), because \( v \) is subharmonic. Then

\[
u = \sup_{v \in S_{\varphi}} v \leq w \quad \text{in } \Omega.
\]

Thus, \( w = u \) in \( \Omega \), as required. \( \square \)

Now, we examine the boundary behaviour of the Perron solution to the Dirichlet boundary value problem. We need the following definition.

Definition 1.8.14  Suppose that \( \Omega \) is a domain and that \( X \in \partial \Omega \). A function \( w \), which is continuous on \( \overline{\Omega} \) is a barrier function at \( X \) relative to \( \Omega \) if the following hold

(i) \( w \) is superharmonic in \( \Omega \);

(ii) \( w > 0 \) in \( \overline{\Omega} \setminus \{X\} \) and \( w(X) = 0 \).

Suppose that \( \Omega \) is the upper half space \( \{x \in \mathbb{R}^n : x_n > 0\} \) in \( \mathbb{R}^n \), \( n \geq 3 \). Choose \( y \in \Omega \) and \( R > 0 \). Set \( \tilde{y} = (y', -R) \). Then the function

\[
w(x) := R^{2-n} - |x - \tilde{y}|^{2-n}
\]

is a barrier at the point \((y', 0)\).

Exercise 1.8.15  Prove that the function above is a barrier at \((y', 0)\). Do the same for the function

\[
w(x) := \log \frac{|x - \tilde{y}|}{R}
\]

in the case where \( n = 2 \).
**Exercise 1.8.16** Suppose that $\Omega$ is a domain that satisfies the following **uniform exterior sphere condition**: there exists $R > 0$ such that for every $X \in \partial \Omega$ there exists a ball $B_R(y)$ such that $B_R(y) \cap \overline{\Omega} = \{X\}$. Prove that for each $X \in \partial \Omega$ there exists a barrier at $X$ relative to $\Omega$.

Observe that the concept of barrier function is a local property of $\Omega$. Indeed, suppose that $w$ is a **local barrier** at $X$ relative to $\Omega$, i.e., there exists $R > 0$ such that $w$ is continuous on $\Omega \cap B_R(X)$ and

(i) $w$ is superharmonic in $\Omega \cap B_R(X)$;

(ii) $w > 0$ in $(\overline{\Omega} \cap B_R(X)) \setminus \{X\}$ and $w(X) = 0$.

Choose a ball $B \subset B_R(X)$ with centre $X$, and such that

$$m := \inf_{B_R(X) \setminus B} w > 0.$$  

Then the function

$$\tilde{w}(x) = \begin{cases} 
\min\{m, w(x)\} & \forall x \in \overline{\Omega} \cap B \\
\quad m & \forall x \in \overline{\Omega} \setminus B,
\end{cases}$$

is a barrier at $X$ relative to $\Omega$.

Indeed, $\tilde{w}$ is clearly continuous on $\overline{\Omega}$, and is superharmonic in $\Omega$ by Lemma 1.8.4 (iv). Furthermore, $\tilde{w} > 0$ on $\overline{\Omega} \setminus \{X\}$ and $w(X) = 0$.

**Definition 1.8.17** A point $X \in \partial \Omega$ is **regular** if there exists a barrier at that point.

**Lemma 1.8.18** Denote by $u$ the Perron solution in $\Omega$ relative to the datum $\varphi$. If $X \in \partial \Omega$ is a regular point and $\varphi$ is a function on $\partial \Omega$ that is continuous at $X$, then

$$\lim_{x \to X} u(x) = \varphi(X);$$

here $x$ tends to the boundary point $X$ within $\Omega$.

**Proof.** Set $M := \sup |\varphi|$ and choose $\varepsilon > 0$. Since $X$ is a regular point, there exists a barrier $w$ at $X$ relative to $\Omega$. Since $\varphi$ is continuous at $X$, there exists $\delta > 0$ such that

$$|\varphi(Y) - \varphi(X)| < \varepsilon$$
for all $Y \in \partial \Omega$ such that $|X - Y| < \delta$. Furthermore, there exists a constant $k$ such that

$$k \, w(x) \geq 2M \quad \forall x \text{ such that } |x - X| \geq \delta. \tag{1.8.2}$$

It is straightforward to check that the functions $\varphi(X) + \varepsilon + kw$ and $\varphi(X) - \varepsilon - kw$ are superfunction and subfunction relative to $X$, respectively. By the definition of $u$ and the fact that every superfunction dominates every subfunction, we have that

$$\varphi(X) - \varepsilon - kw(x) \leq u(x) \leq \varphi(X) - \varepsilon - kw(x) \quad \forall x \in \Omega.$$ 

This is clearly equivalent to

$$|u(x) - \varphi(X)| \leq \varepsilon + kw(x) \quad \forall x \in \Omega.$$

Since $w$ is a barrier at $X$, $w(x) \to 0$ as $x$ tends to $X$, so that for every $\varepsilon > 0$

$$\lim_{x \to X} |u(x) - \varphi(X)| \leq \varepsilon,$$

which implies the required conclusion. \hfill \Box

**Corollary 1.8.19** Suppose that $\Omega$ is a bounded domain. Then the classical Dirichlet problem is uniquely solvable in $\Omega$ if and only if every point of $\partial \Omega$ is a regular point.

**Proof.** If every point of $\partial \Omega$ is regular, then, by Lemma [1.8.18] the classical Dirichlet problem is solvable. Uniqueness then follows from the maximum principle.

Conversely, if the classical Dirichlet problem is solvable, and $X$ is a point of $\partial \Omega$, then the solution to the problem

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = g,
\end{cases}$$

where $g(Y) := |X - Y|$ is a barrier function at $X$, hence $X$ is a regular point. Consequently, every point of $\partial \Omega$ is regular, as required. \hfill \Box

Now, the problem becomes to give criteria which imply that every point of the boundary of a given domain are regular. A simple criterion is discussed in the following exercise.
Exercise 1.8.20 Suppose that \( \Omega \) is a bounded \( C^2 \) domain. Prove that the classical Dirichlet problem is uniquely solvable in \( \Omega \).

We conclude this section with a criterion which ensures the solvability of the classical Dirichlet problem in Lipschitz domains. We need the following definition.

Definition 1.8.21 A bounded domain \( \Omega \) satisfies the uniform exterior cone condition if there exist constants \( h \) and \( \alpha \) such that for each point \( X \in \partial \Omega \) there exists a cone \( \Gamma(X, \alpha, h) \) with vertex \( X \), aperture \( \alpha \) and height \( h \) such that \( \Gamma(X, \alpha, h) \cap \overline{\Omega} = \{X\} \).

Exercise 1.8.22 Prove that every bounded Lipschitz domain satisfies the exterior cone condition.

Proposition 1.8.23 Suppose that \( \Omega \) is a bounded domain, which satisfies the uniform exterior cone condition. Then the classical Dirichlet problem is uniquely solvable in \( \Omega \).

Proof. Fix a point \( X \in \partial \Omega \). Since \( \Omega \) satisfies the uniform exterior cone condition, there exist constants \( \alpha \) and \( h \) such that the cone \( \Gamma(X, \alpha, h) \) is such that \( \Gamma(X, \alpha, h) \cap \overline{\Omega} = \{X\} \). We normalise our setting by assuming that \( \Gamma(X, \alpha, h) \cap (X + \mathbb{S}^{n-1}) \neq \emptyset \). To construct a barrier at \( X \) we argue as follows. Consider all functions \( v \) of the form \( w(x) = r^{\lambda} \varphi_0(\omega) \), where \( x = r \omega \) (polar coordinates centred at \( X \)), \( \lambda > 0 \), and \( \varphi_0 \) is the eigenfunction associated to the lowest eigenvalue \( \mu \) of the eigenvalue problem

\[
\Delta_{\mathbb{S}^{n-1}} U = -\mu U
\]
on \( \mathbb{S}^{n-1} \setminus \Gamma(0, \alpha, h) \) with Dirichlet boundary conditions. A well known result in spectral theory, which we do not prove in these notes, asserts that \( \mu > 0 \) and that \( \varphi_0 \) is a smooth function, which is strictly positive in \( \mathbb{S}^{n-1} \setminus \Gamma(0, \alpha, h) \). By using the formula for the Laplacian \( \Delta \) in polar coordinates, we find that

\[
\Delta w(x) = r^{\lambda - 2} \varphi_0 \left[ \alpha^2 + (n - 2) \alpha - \mu \right].
\]

It is straightforward to check that if \( \alpha = (1/2)\left[\sqrt{(n - 2)^2 + 4\mu} - (n - 2)\right] \), then \( w \) is harmonic in \( \Omega \). Furthermore, \( w(X) = 0 \), and \( w > 0 \) in \( \Omega \setminus \{X\} \). Thus, \( w \) is a barrier at \( X \). Hence \( X \) is a regular point relative to \( \Omega \).

Since this holds for every point in \( \partial \Omega \), the classical Dirichlet problem is solvable by Lemma 1.8.18. The uniqueness follows from the maximum principle.
1.9 The Lebesgue spine

The aim of this section is to provide a counterexample to the solvability of the classical Dirichlet problem on bounded domains. Specifically, we shall prove that there exists a bounded domain in $\mathbb{R}^3$, rotationally invariant with respect to the $z$ axis, for which the classical Dirichlet problem has no solutions. This domain has a sharp inward cusp, which is responsible for the nonsolvability of the Dirichlet problem. The counterexample is due to Lebesgue and dates back to 1912.

We shall denote by $(x, y, z)$ the coordinates in $\mathbb{R}^3$, and set $r^2 = x^2 + y^2$. We consider the domain $\Omega$, defined by

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 : r^2 + z^2 < 1, \ r > e^{-1/(2z)} \text{ if } z > 0\}.$$  

We call the inward cusp at $(0, 0, 0)$ a Lebesgue spine. We shall prove that $(0, 0, 0)$ is not a regular point relative to $\Omega$.

Indeed, consider the positive Radon measure $\mu$, defined as the continuous linear functional on $C_c(\mathbb{R}^3)$, defined by

$$\mu(\varphi) = \int_0^1 \varphi(0, 0, \tau) \tau \, d\tau$$  

(see Section 5.1 and, in particular, Theorem 5.1.2 for more on the connection between measures and continuous linear functionals on $C_c(\mathbb{R}^n)$). Note that the support of $\mu$ is contained in the segment

$$\Sigma := \{(0, 0, z) \in \mathbb{R}^3 : z \in [0, 1]\}.$$  

We consider the potential associated to the measure $\mu$, i.e., the function $u$ given by

$$u = N * \mu.$$  

Since $\mu$ has compact support, the convolution on the right hand side of the formula above makes perfect sense (see Definition 5.4.9 below). We shall compute $u$ explicitly. For every $\varphi \in C_c^\infty(\mathbb{R}^3)$

$$\langle \varphi, N * \mu \rangle = \langle \varphi * \tilde{N}, \mu \rangle$$

$$= \int_0^1 \varphi * \tilde{N}(0, 0, \tau) \tau \, d\tau$$

$$= \int_0^1 \tau \, d\tau \int_{\mathbb{R}^3} \varphi(x, y, z) \, N(x, y, z - \tau) \, dx \, dy \, dz$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^3} \varphi(x, y, z) \, dx \, dy \, dz \int_0^1 \frac{\tau}{\sqrt{x^2 + y^2 + (z - \tau)^2}} \, d\tau.$$
Therefore

\[ N * \mu(x, y, z) = \frac{1}{4\pi} \int_0^1 \frac{\tau}{\sqrt{x^2 + y^2 + (z - \tau)^2}} \, d\tau. \]

The right hand side can be computed explicitly. We find that, up to a constant factor,

\[ N * \mu(x, y, z) = \sqrt{r^2 + (z - 1)^2 - \sqrt{r^2 + z^2}} \]

\[ + \log \left[ \sqrt{r^2 + z^2 + z} \left| \sqrt{r^2 + (z - 1)^2 + 1 - z} \right| \right] \]

\[ - 2z \log r \quad \forall (x, y, z) \in \Omega \text{ such that } z > 0. \]

**Exercise 1.9.1** Compute \( N * \mu(x, y, z) \) for \( z < 0 \).

It is straightward to check that the restriction of \( u \) to \( \partial \Omega \) is a continuous function. Clearly \( u \) is harmonic in \( \Omega \). However, \( u \) does not admit a limit as \((x, y, z)\) tends to \((0, 0, 0)\) within \( \Omega \).

Indeed, suppose that \( \alpha \in (0, 1) \), and consider the surfaces

\[ r_\alpha(z) := e^{-\alpha/(2z)}, \quad z > 0, \]

which, for \( z \in (0, 1) \) is contained in \( \Omega \). It is straightward to check that

\[ \lim_{z \to 0^+} u(r_\alpha(z) \cos \theta, r_\alpha(z) \sin \theta, z) = \alpha, \]

as required. We have used cylindrical coordinates in the formula above.

We are now ready to prove that \((0, 0, 0)\) is not a regular point relative to \( \Omega \), i.e., the Dirichlet problem with datum \( u|_{\partial \Omega} \) does not admit a classical solution.

We argue by contradiction. If \( v \) were a classical solution to the Dirichlet problem with datum \( u|_{\partial \Omega} \), then its restriction to the subdomain \( \Omega_\varepsilon \), defined by

\[ \Omega_\varepsilon := \{ (x, y, z) \in \Omega : r^2 > \varepsilon^2 \}, \]

solves the Dirichlet problem

\[
\begin{cases}
\Delta w = 0 & \text{in } \Omega_\varepsilon \\
 \left. w \right|_{\partial \Omega_\varepsilon} = u|_{\partial \Omega_\varepsilon}.
\end{cases}
\]

Since every point of \( \partial \Omega_\varepsilon \) is regular relative to \( \Omega_\varepsilon \) and \( u \) is clearly a solution of the problem above, \( u = v \) on \( \Omega_\varepsilon \) for every \( \varepsilon > 0 \). Then \( v = u \) on \( \Omega \setminus (0, 0, 0) \), a fact which contradicts the continuity of \( v \) at the point \((0, 0, 0)\).
1.9. THE LEBESGUE SPINE

It may be worth noticing that there are domains with inward cusps for which the classical Dirichlet problem is solvable. For instance, it may be shown that this is the case for the domain

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 : r^2 + z^2 < 1, \ r > z^{2k} \text{ if } z > 0\}.$$ 

for every positive integer $k$. The proof of this fact uses the theory of capacity, which we do not tackle in this notes. The interested reader is referred to [K]. Similar results hold in $\mathbb{R}^n$ for $n \geq 3$. However, the case of bounded domains in the plane is quite different. Indeed, it is straightforward to show that if $\Omega$ is a planar domain and $z_0$ is a boundary point for which the following is true: there exists a straight line segment $[z_0, z_1]$ entirely contained in $\mathbb{R}^2 \setminus \Omega$, then $z_0$ is a regular point relative to $\Omega$.

**Exercise 1.9.2** Prove the assertion above. *Hint:* consider a branch of $\log(z - z_0)$ adapted to the situation and study the function $-\text{Re} \left[1/\log(z - z_0)\right]$.

This criterion implies that the two dimensional analogue of the Lebesgue spine is a regular point relative to $\Omega$. 

Part II

The Dirichlet problem
via integral equations
In this part we discuss a method for solving the Dirichlet problem on a bounded domain $\Omega$ that hinges on certain integral equations on the boundary $\partial \Omega$. The idea behind the method is very simple. We recall that if $u \in C(\Omega) \cap C^2(\Omega)$ is the classical solution of the Dirichlet problem

$$\begin{cases}
\Delta u = 0 \quad \text{in} \quad \Omega \\
u|_{\partial \Omega} = g,
\end{cases}$$

then

$$u(x) = \int_{\partial \Omega} g(Y) \, \partial_{\nu(Y)} G(x,Y) \, d\sigma(Y) \quad \forall x \in \Omega$$

(see Corollary 1.5.8), where $G$ is the Green’s function of $\Omega$, and $\nu(Y)$ denotes the outward unit normal to $\partial \Omega$ at the point $Y$. Furthermore,

$$\lim_{x \to X} u(x) = g(X) \quad \forall X \in \partial \Omega.$$ 

The idea that leads to the method of integral equations is to replace the Green’s function of $\Omega$ with the Newtonian potential $N$ in the integral above, and to consider the operator that associates to each reasonable function $g$ on $\partial \Omega$ the function $Dg$, defined by

$$Dg(x) = \int_{\partial \Omega} g(Y) \, \partial_{\nu(Y)} N(x-Y) \, d\sigma(Y) \quad \forall x \in \Omega.$$ 

Clearly $Dg$ will not be the solution of the Dirichlet problem with datum $g$ for the obvious reason that $N(x-Y)$ does not agree with $G(x,Y)$. However, the operator $D$ is related to the solution of the Dirichlet problem, as we shall explain below. The operator $g \mapsto Dg$ is called the double layer potential and the function $Dg$ is called the double layer potential with density $g$. Its properties will play a key role in the theory.

We shall prove that for $\varphi \in C(\partial \Omega)$, the integral

$$\int_{\partial \Omega} \varphi(Y) \, \partial_{\nu(Y)} N(x-Y) \, d\sigma(Y)$$

makes sense also when $X \in \partial \Omega$. Thus, we may define the operator $K$ on $C(\partial \Omega)$ by

$$K\varphi(X) := \int_{\partial \Omega} \varphi(Y) \, \partial_{\nu(Y)} N(x-Y) \, d\sigma(Y) \quad \forall X \in \partial \Omega.$$
Furthermore, the following formula holds for every $\varphi \in C(\partial \Omega)$:

$$
\lim_{x \to X \atop x \in \Omega} D\varphi(x) = \frac{1}{2} \varphi(X) + K\varphi(X) \quad \forall X \in \partial \Omega.
$$

Thus, the boundary value of the double layer potential $D\varphi$ with density $\varphi$ is the datum $g$ provided that $\varphi$ satisfies the following integral equation

$$
g(X) = \frac{1}{2} \varphi(X) + K\varphi(X) \quad \forall X \in \partial \Omega.
$$

It is a nontrivial fact, which will be proved below, that for every continuous function $g$ on $\partial \Omega$ the integral equation above has a unique solution $\varphi \in C(\partial \Omega)$. In the language of spectral theory, this statement may be reformulated by saying that $-1/2$ is in the resolvent set of the operator $K$.

The operator $K$ is a compact operator. This is the reason why in the following we shall study in some detail the class of compact operators and their spectra.

One of the major drawbacks of the method of integral equation is that it is hard to make it work unless the domain $\Omega$ has a reasonable boundary, say $C^2$. The method can be pushed to cover Lipschitz domains, and $L^p$ boundary data, but this requires a lot of very refined analysis, which is well beyond the scope of these notes. The interested reader is referred to the paper of B. Dahlberg and C. Kenig [DK] for a detailed exposition of this comparatively recent piece of research.
Chapter 2

Linear operators on Banach spaces

Throughout this chapter $B$ will denote a Banach space over $\mathbb{C}$.

2.1 Basic definitions

**Definition 2.1.1** A linear operator on $B$ is a pair $(X, A)$, where $X$ is a (possibly not closed) linear subspace of $B$ and $A : X \to B$ is a linear operator: $X$ is called the domain of $A$ and will be denoted by $\text{Dom}(A)$.

Note that we do not assume that $\text{Dom}(A)$ is dense in $B$, unless explicitly stated.

**Definition 2.1.2** Given two linear operators $A$ and $B$ on $B$, we say that $B$ is an extension of $A$, and we write $B \supset A$, if $\text{Dom}(B) \supset \text{Dom}(A)$ and the restriction of $B$ to $\text{Dom}(A)$ agrees with $A$.

Consider the operators $A$ and $B$ on $L^2(\mathbb{R})$ defined by

\[
\begin{align*}
\text{Dom}(A) &= C^\infty_c(\mathbb{R}) & Af &= i f' & \forall f \in \text{Dom}(A) \\
\text{Dom}(B) &= C^1_c(\mathbb{R}) & Bf &= i f' & \forall f \in \text{Dom}(B).
\end{align*}
\]

Clearly $B \supset A$.

We consider two more examples of linear operators, which are one dimensional models of the **Dirichlet** and **Neumann Laplacian**. Suppose that
Let $-\infty < a < b < \infty$. Let $H_D$ and $H_N$ be the operators on $L^2((a,b))$ defined by

$\text{Dom}(H_D) = \{ f \in C^2([a,b]) : f(a) = 0 = f(b) \}$ \quad $H_D f = -f''$

$\text{Dom}(H_N) = \{ f \in C^2([a,b]) : f'(a) = 0 = f'(b) \}$ \quad $H_N f = -f''$.

It is not hard to check that the domains of $H_D$ and $H_N$ are dense in $L^2((a,b))$.

**Exercise 2.1.3** Suppose that $A$ is a densely defined linear operator in a Hilbert space $H$ such that $(Af,f) = 0$ for all $f$ in $\text{Dom}(A)$. Prove that $A = 0$. Note that this fails on real vector spaces (take, e.g., a rotation of $\pi/2$ on $\mathbb{R}^2$).

### 2.2 Bounded operators

**Definition 2.2.1** A linear operator on $B$ is **bounded** if $\text{Dom}(A) = B$ and

$$\sup_{f \neq 0} \frac{\|Af\|}{\|f\|} < \infty.$$ 

The supremum above is called the **operator norm** of $A$ and it is denoted by $\|A\|$.

**Exercise 2.2.2** Suppose that $B$ is a finite dimensional (Banach) space. Prove that every linear operator with domain $B$ is bounded. (Hint: It suffices to “control” the operator on a basis of $B$.)

**Exercise 2.2.3** Prove that for a linear operator $A$ on $B$ the following are equivalent:

(i) $A$ is a bounded operator on $B$;

(ii) $A$ is continuous at every point of $B$;

(iii) $A$ is continuous at 0.

**Exercise 2.2.4** Suppose that $A$ is a linear operator on $B$ with dense domain. Assume that

$$\sup_{f \in \text{Dom}(A) \setminus \{0\}} \frac{\|Af\|}{\|f\|} < \infty.$$

Prove that there exists a unique bounded linear operator $\tilde{A}$ that extends $A$. 
2.2. BOUNDED OPERATORS

An interesting application of the exercise above is the following special case of Poincaré’s inequality, whose general form will be considered in Section ?? below.

**Definition 2.2.5** Suppose that $I$ is an open interval of the real line. We consider the following norm on $C_c^\infty(I)$:

$$\|\varphi\|_{H_0^1(I)} := \|\varphi'\|_{L^2(I)}.$$  

Denote by $H_0^1(I)$ the completion of $C_c^\infty(I)$ with respect to the norm above.

The following proposition illustrates the role of Exercise 2.2.4 in proving the boundedness of certain operators.

**Proposition 2.2.6** Suppose that $I$ is a bounded interval of the real line. The identity map $j : \varphi \mapsto \varphi$, initially defined on $C_c^\infty(I)$, extends to a bounded linear operator from $H_0^1(I)$ to $L^2(I)$.

**Proof.** By Exercise 2.2.4 it suffices to show that

$$\|\varphi\|_{L^2(I)} \leq |I| \|\varphi'\|_{L^2(I)} \quad \forall \varphi \in C_c^\infty(I).$$

This is elementary. Indeed, denote by $p$ the left endpoint of $I$ and write

$$|\varphi(x)| = \left| \int_p^x \varphi'(s) \, ds \right|$$

$$\leq \int_p^x |\varphi'(s)| \, ds$$

$$\leq \|\varphi'\|_1$$

(by Hölder’s inequality)

$$\leq |I|^{1/2} \|\varphi'\|_2 \quad \forall x \in I,$$

and the required estimate follows by integrating the squares of both sides on $I$. \(\square\)

It is worth observing that the space $H_0^1(I)$ may be realised as a (nonclosed) subspace of $L^2(I)$.

Indeed, to prove that the map $j$ in Proposition 2.2.6 is an immersion, we have to show that if $f \in H_0^1(I)$ satisfies $j(f) = 0$, then $f = 0$. Note that this is trivial if $f \in C_c^\infty(I)$, because $j(f) = f$ as functions. However, we do not know *a priori* how the map $j$ acts on elements of $H_0^1(I)$, so we need to prove that $j(f) = 0$ implies $f = 0$. Since $H_0^1(I)$ is the completion of $C_c^\infty(I)$,
given $f \in H^1_0(I)$, there exists a sequence $\{f_n\}$ of functions in $C^\infty_c(I)$ such that $\|f_n - f\|_{H^1_0(I)} \to 0$ as $n$ tends to $\infty$. Note that $\|f_n\|_2 \to 0$, because $j$ is continuous by Proposition [2.2.6] and $f_n = j(f_n) \to j(f) = 0$.

Furthermore, $\{f'_n\}$ is a Cauchy sequence in $L^2(I)$ by the definition of the norm in $H^1_0(I)$. Since $L^2(I)$ is complete, there exists a function $g \in L^2(I)$ such that $\|f'_n - g\|_2 \to 0$ as $n$ tends to $\infty$. By possibly passing to a subsequence, we deduce that $\{f_n\}$ is convergent to $0$ a.e. By the fundamental theorem of calculus

$$f_n(x) = \int_0^x f'_n(s) \, ds,$$

so that, by taking the limit of both sides,

$$0 = \int_0^x g(s) \, ds$$

for all $x \in I$. Therefore $g = 0$ a.e. Thus,

$$\|f_n\|_{H^1_0(I)} = \|f'_n\|_2 = \|f'_n - g\|_2 \to 0$$

as $n$ tends to $\infty$. By the uniqueness of the limit in $H^1_0(I)$, we may conclude that $f = 0$, as required.

Thus, elements of $H^1_0(I)$ may be interpreted as equivalence classes of square integrable functions.

**Exercise 2.2.7** Prove that the operators $A$ and $B$ defined just below Definition [2.1.2] do not admit bounded extensions to $L^2(\mathbb{R})$.

It is straightforward to check that the set of bounded operators on $\mathcal{B}$ is a vector space, which we denote by $\mathcal{L}(\mathcal{B})$, and that the operator norm is a norm on $\mathcal{L}(\mathcal{B})$.

**Proposition 2.2.8** The space $\mathcal{L}(\mathcal{B})$ is a Banach space with respect to the operator norm.

**Proof.** Suppose that $\{A_n\}$ is a Cauchy sequence of operators in $\mathcal{L}(\mathcal{B})$. Then for every $\varepsilon > 0$ there exists an integer $\nu$ such that

$$\|A_m - A_n\| \leq \varepsilon \quad \forall m, n \geq \nu.$$

Consequently,

$$\|A_m f - A_n f\| \leq \varepsilon \|f\| \quad \forall m, n \geq \nu \quad \forall f \in \mathcal{B}. \quad (2.2.1)$$
Thus, for every \( f \) in \( \mathcal{B} \), \( \{ A_n f \} \) is a Cauchy sequence in \( \mathcal{B} \). Since \( \mathcal{B} \) is complete, there exists an element in \( \mathcal{B} \) that we denote \( A f \) such that

\[
\lim_{n \to \infty} A_n f = A f,
\]

where the convergence is in the norm of \( \mathcal{B} \). It is straightforward to check that the operator \( A \) thus defined is linear. By letting \( m \) tend to \( \infty \) in (2.2.1), we obtain

\[
\| A f - A_n f \| \leq \varepsilon \| f \| \quad \forall n \geq \nu \quad \forall f \in \mathcal{B}.
\]

This implies that \( A - A_n \) is a bounded operator on \( \mathcal{B} \), and that

\[
\| A - A_n \| \leq \varepsilon \quad \forall n \geq \nu.
\]

Thus, \( A \) is bounded (because it is the sum of the bounded operators \( A_n \) and \( A - A_n \)), and \( \lim_{n \to \infty} \| A_n - A \| = 0 \), as required.

2.3 The spectrum of a linear operator

Definition 2.3.1 Suppose that \( A \) is a linear operator on \( \mathcal{B} \). We say that the complex number \( \zeta \) is in the resolvent set of \( A \) if \( \zeta I - A \) maps \( \text{Dom}(A) \) in a one-to-one fashion onto \( \mathcal{B} \) and the resolvent operator

\[
R(\zeta; A) := (\zeta I - A)^{-1}
\]

is a bounded operator on \( \mathcal{B} \). The resolvent set of \( A \) will be denoted by \( \varrho(A) \). The spectrum \( \sigma(A) \) of \( A \) is defined to be \( \mathbb{C} \setminus \varrho(A) \).

If \( \mathcal{B} \) is finite dimensional, then the spectrum of a linear operator \( A \) is just the set of its eigenvalues.

One of the reasons for which spectral theory on infinite dimensional Banach spaces is far more difficult than in the finite dimensional case is that the following result, whose proof we leave as an exercise, is false in the infinite dimensional case.

Proposition 2.3.2 Suppose that \( \mathcal{B} \) is finite dimensional, and that \( A \) is a linear operator defined on \( \mathcal{B} \). The following hold:

(i) \( A \) is bounded;

(ii) \( A \) is invertible if and only if \( A \) is injective if and only if \( A \) is surjective;
(iii) \( \text{Ran}(A) \) is dense if and only if \( \text{Ran}(A) = \mathcal{B} \);
(iv) if \( A \) is invertible, then \( A^{-1} \) is bounded.

**Proof.** Exercise. \( \square \)

Consider the operator \( A \) defined just below Definition 2.1.2. There exists a (purely algebraic) extension of \( A \) to a linear operator on \( \mathcal{B} \). Note that this extension cannot be bounded for we have proved in Exercise 2.2.7 that \( A \) does not admit a bounded extension to all of \( \mathcal{B} \).

This extension provides an example of an unbounded linear operator which is defined everywhere.

**Exercise 2.3.3** Find counterexamples to statements (ii)-(iv) in the case of operators on infinite dimensional Banach spaces, by looking at the operators \( L \) and \( R \), defined on \( \ell^2 \) by
\[
L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots) \quad R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),
\]
and \( C \), defined on \( \ell^1 \) by
\[
C(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).
\]

**Definition 2.3.4** Suppose that \( A \) is a closed operator on \( \mathcal{B} \). We say that a point \( \zeta \) in \( \sigma(A) \) is in the **point spectrum** of \( A \) if \( \zeta \) is an eigenvalue of \( A \).

We say that a point \( \zeta \) in \( \sigma(A) \) is in the **discrete spectrum** of \( A \) if it is an eigenvalue and it is an isolated point in \( \sigma(A) \).

The main reason for introducing the residual spectrum of an operator is that self adjoint operators have empty residual spectrum.

## 2.4 The adjoint of a bounded operator

In this section we assume that \( A \) is a bounded linear operator on the Hilbert space \( \mathcal{H} \).

In the case where \( \mathcal{H} \) is a finite dimensional Hilbert space with inner product \( (\cdot, \cdot) \), once we fix a orthonormal basis \( \{v_1, \ldots, v_n\} \) of \( \mathcal{H} \), each linear

operator $A$ may be represented by a unique matrix $M_A = \{m_{j,k}\}$, where $m_{j,k} := (Av_j, v_k)$, and

$$(A(x_1v_1 + \ldots + x_nv_n), y_1v_1 + \ldots + y_nv_n) = (M_Ax, y),$$

where $x$ is the column vector $(x_1, \ldots, x_n)^t$, and similarly for $y$, and the inner product on the right hand side is just the Hermitian inner product on $\mathbb{C}^n$, defined by

$$(X, Y) = \sum_{j=1}^{n} X_j Y_j.$$

Denote by $M_A^*$ the matrix $M_A^t$ (conjugate transpose of $M_A$). Then

$$(M_Ax, y) = (x, M_A^*y) \quad \forall x, y \in \mathbb{C}^n.$$ 

Denote by $A^*$ the operator on $\mathcal{H}$ associated to the matrix $M_A^*$. Then, clearly

$$(A(x_1v_1 + \ldots + x_nv_n), y_1v_1 + \ldots + y_nv_n) = (x_1v_1 + \ldots + x_nv_n, A^*(y_1v_1 + \ldots + y_nv_n)),$$

The operator $A^*$ is called the adjoint of $A$.

This notion may be easily generalised to all bounded operators on infinite dimensional Hilbert spaces. Suppose that $A$ is a bounded operator on the Hilbert space $\mathcal{H}$. For each $g$ in $\mathcal{H}$, the functional

$$f \mapsto (Af, g)$$

is bounded on $\mathcal{H}$, because, by Schwarz’s inequality,

$$| (Af, g) | \leq \|Af\| \|g\|$$

(since $A$ is bounded)

$$\leq \|A\| \|f\| \|g\|. \quad (2.4.1)$$

Therefore, by the Riesz representation theorem, there exists a unique element $\gamma_g \in \mathcal{H}$ such that

$$(Af, g) = (f, \gamma_g)$$

It is straightforward to check that the assignment $g \mapsto \gamma_g$ is linear, whence it defines a linear operator, which we call $A^*$. Thus,

$$(Af, g) = (f, A^*g) \quad \forall f, g \in \mathcal{H}.$$ 

Observe that $A^*$ is bounded on $\mathcal{H}$, for

$$\|A^*g\| = \sup_{\|f\| \leq 1} \left| (f, A^*g) \right|$$

$$= \sup_{\|f\| \leq 1} \left| (Af, g) \right|$$

(by $\|A\| \|f\| \|g\|$, $2.4.1$)

$$\leq \|A\| \|f\| \|g\|$$

$$= \|A\| \|g\|,$$
and that $\|A^*\| \leq \|A\|$. By reversing the role of $A$ and $A^*$, we find that $\|A\| \leq \|A^*\|$, whence $\|A\| = \|A^*\|$.

**Definition 2.4.1** Suppose that $A$ is a bounded operator on the Hilbert space $\mathcal{H}$. The operator $A^*$ defined above is called the (Hilbert space) adjoint of $A$.

Before looking at examples of adjoints, we define an important class of operators, acting on $L^2(\mathbb{R}^n)$.

**Definition 2.4.2** An operator $A$, acting on $L^2(\mathbb{R}^n)$, is a Hilbert–Schmidt operator if it is of the following form

$$Af(x) := \int_{\mathbb{R}^n} K(x, y) f(y) \, dy \quad \forall f \in L^2(\mathbb{R}^n),$$

where $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. The function $K$ is called the integral kernel (or simply the kernel) of $A$.

**Exercise 2.4.3** Prove that a Hilbert–Schmidt operator $A$ with kernel $K$ is bounded on $L^2(\mathbb{R}^n)$ with norm $\leq \|K\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}$, and that the adjoint operator $A^*$ is the Hilbert–Schmidt operator with kernel $(x, y) \mapsto \overline{K(y, x)}$.

**Exercise 2.4.4** Consider the convolution operator $T$ on $L^2(\mathbb{R})$, defined by

$$Tf = f * \varphi \quad \forall f \in L^2(\mathbb{R}),$$

where $\varphi$ is in $L^1(\mathbb{R})$. Prove that $T$ is bounded on $L^2(\mathbb{R})$ and find $T^*$.

**Exercise 2.4.5** Consider the left shift operator $L$ on $\ell^2$, defined by

$$L(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

for every sequence $\{x_n\}$ in $\ell^2$. Compute $L^*$. Do the same for the right shift operator $R$, defined by

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

The following proposition summarises the main properties of the $*$ operator.

**Proposition 2.4.6** The following hold:
2.4. THE ADJOINT OF A BOUNDED OPERATOR

(i) $\ast$ is a conjugate linear isometric isomorphism of $L(\mathcal{H})$;

(ii) $(A\ast)^\ast = A$, i.e., $\ast$ is involutive;

(iii) $(ST)^\ast = T^\ast S^\ast$ for all $S$ and $T$ in $L(\mathcal{H})$;

(iv) if $A$ and $A^{-1}$ are in $L(\mathcal{H})$, then $A^\ast$ is invertible in $L(\mathcal{H})$ and $(A^\ast)^{-1} = (A^{-1})^\ast$;

(v) $\|A^\ast A\| = \|A\|^2$ for every $A$ in $L(\mathcal{H})$.

Proof. The proof of (i)-(iv) is left to the reader.

To prove (v) observe that, on the one hand,

$$\|A^\ast A\| \leq \|A^\ast\| \|A\| = \|A\|^2,$$

where we have used the formula $\|A^\ast\| = \|A\|$. On the other hand

$$\|A\|^2 = \sup_{\|f\|=1} \|Af\|^2$$

$$= \sup_{\|f\|=1} |(Af, Af)|$$

$$= \sup_{\|f\|=1} |(f, A^\ast Af)|$$

$$\leq \|A^\ast A\|,$$

and the required equality follows.

The properties of the involution $\ast$ contained in the proposition above and the fact that $L(\mathcal{H})$ is complete are usually summarised by saying that $L(\mathcal{H})$ is a $C^\ast$-algebra (with involution $\ast$).

Exercise 2.4.7 Prove statements (i)-(iv) in Proposition 2.4.6.

We now introduce a very important class of operators in Hilbert spaces.

Definition 2.4.8 A bounded operator $A$ on $\mathcal{H}$ is self adjoint if $A^\ast = A$.

Bounded self adjoint operators on Hilbert spaces generalise Hermitian matrices on $\mathbb{C}^n$, and play a distinguished role in Analysis. In particular, every bounded self adjoint operator may be “diagonalised”. In other words, for bounded self adjoint operator a generalisation of the Spectral Theorem for Hermitian matrices holds. This result will be proved in the next section for a noteworthy subclass of bounded operators, namely the compact operators. One of the step that will be used in its proof is the following proposition, which is of independent interest.
Proposition 2.4.9 Suppose that $A$ is a bounded self adjoint operator on the Hilbert space $\mathcal{H}$. Then

$$
\|A\| = \sup_{\|f\|=1} |(Af,f)|.
$$

Proof. Denote by $\eta_A$ the right hand side of the equality above. The inequality $\eta_A \leq \|A\|$ is trivial and its proof is left to the reader.

It remains to prove that $\|A\| \leq \eta_A$. We use the following polarisation identity: for every $f, g$ in $\mathcal{H}$

\begin{equation}
4\langle Af, g \rangle = \langle A(f + g), f + g \rangle - \langle A(f - g), f - g \rangle + i\langle A(f + ig), f + ig \rangle - i\langle A(f - ig), f - ig \rangle,
\end{equation}

which is easily verified. Observe that for each $\varphi$ in $\mathcal{H}$ the number $\langle A\varphi, \varphi \rangle$ is real, for

$$
\langle A\varphi, \varphi \rangle = (\varphi, A^*\varphi) = (\varphi, A\varphi) = (A\varphi, \varphi).
$$

Here we have used the assumption that $A$ is self adjoint. From (2.4.2) we then deduce that

$$
4 \Re \langle Af, g \rangle = \langle A(f + g), f + g \rangle - \langle A(f - g), f - g \rangle,
$$

whence

\begin{equation}
4|\Re \langle Af, g \rangle| \leq \eta_A \left[\|f + g\|^2 + \|f - g\|^2\right] = 2\eta_A \left[\|f\|^2 + \|g\|^2\right].
\end{equation}

Now, we may choose a real number $\theta$ so that

$$
|\langle Af, g \rangle| = |\Re \left[e^{i\theta} \langle Af, g \rangle\right]| = |\Re \langle Af, e^{-i\theta}g \rangle|.
$$

We now use (2.4.3) with $e^{-i\theta}g$ in place of $g$, and obtain

\begin{equation}
|\Re \langle Af, g \rangle| \leq \frac{\eta_A}{2} \left[\|f\|^2 + \|e^{-i\theta}g\|^2\right].
\end{equation}

By taking the supremum over all $f$ and $g$ with $\|f\| = \|g\| = 1$, we see that $\|A\| \leq \eta_A$, as required to conclude the proof of the proposition.

Next, we illustrate a connection between bounded bilinear forms on $\mathcal{H} \times \mathcal{H}$ and bounded operators on $\mathcal{H}$. Unbounded operators and unbounded bilinear forms may also be considered.

Definition 2.4.10 A bounded bilinear form $B$ on $\mathcal{H}$ is a complex valued map on $\mathcal{H} \times \mathcal{H}$ so that
2.4. THE ADJOINT OF A BOUNDED OPERATOR

(i) $f \mapsto B(f, g)$ is a linear functional on $\mathcal{H}$ for all $g \in \mathcal{H}$;

(ii) $g \mapsto B(f, g)$ is a conjugate linear functional on $\mathcal{H}$ for all $f \in \mathcal{H}$;

(iii) there exists a constant $M$ such that

$$|B(f, g)| \leq M \|f\| \|g\| \quad \forall f, g \in \mathcal{H}.$$ 

The norm of $B$ is the infimum of all $M$ for which the above inequality holds.

Suppose that $A$ is a bounded operator on $\mathcal{H}$. We define the bilinear form $B$ associated to $A$ as follows

$$B(f, g) := (Af, g) \quad \forall f, g \in \mathcal{H}.$$ 

It is straightforward to check that $B$ is a bounded bilinear form, with norm $\|A\|$.

Conversely, suppose that $B$ is a bounded bilinear form on $\mathcal{H}$ and that $f \in \mathcal{H}$. Then, by (iii) above, the functional $\lambda_f : g \mapsto B(f, g)$ is bounded on $\mathcal{H}$. By the Riesz representation theorem there exists a unique element of $\mathcal{H}$ that represents $\lambda_f$. We call it $Af$. Then $B(f, g) = (g, Af)$, whence

$$B(f, g) = (Af, g) \quad \forall f, g \in \mathcal{H}.$$ 

It is straightforward to check that $A$ is a linear operator on $\mathcal{H}$. Finally,

$$\|Af\| = \sup_{\|g\|=1} |(Af, g)|$$

$$= \sup_{\|g\|=1} |B(f, g)|$$

$$\leq M \|f\|,$$

where we have used the boundedness of the form $B$ in the last inequality. Therefore the operator $A$ is bounded: $A$ will be called the bounded linear operator associated to the form $B$.

Thus, bounded bilinear forms and bounded operators on $\mathcal{H}$ are somewhat “interchangeable”. Furthermore, if $A$ is self adjoint, then the associated bilinear form is symmetric, i.e.

$$B(f, g) = (Af, g) = (f, Ag) = (Ag, f) = B(g, f) \quad \forall f, g \in \mathcal{H}.$$ 

Conversely, given a bounded bilinear form satisfying

$$B(f, g) = \overline{B(g, f)} \quad \forall f, g \in \mathcal{H},$$

the bounded linear operator $A$ associated to $B$ is self adjoint.
Definition 2.4.11  A linear operator on $\mathcal{H}$ is said to be **unitary** if 

$$U^*U = UU^* = I.$$  

Exercise 2.4.12  Prove that the following are equivalent:

(i) $U$ is unitary;

(ii) $\text{Ran}(U) = \mathcal{H}$ and $(Uf, Ug) = (f, g)$ for every $f$ and $g$ in $\mathcal{H}$.

(iii) $\text{Ran}(U) = \mathcal{H}$ and $\|Uf\| = \|f\|$ for every $f$ in $\mathcal{H}$;

Show that if $\mathcal{H}$ is finite dimensional, we may omit the requirement $\text{Ran}(U) = \mathcal{H}$ in (ii) and (iii) above. An operator $U$ such that $\|Uf\| = \|f\|$ for every $f$ in $\mathcal{H}$ is called an **isometry**. Construct an isometry on $\ell^2$ which is not onto.

The next proposition relates the kernel and the range of the bounded operators $A$ and $A^*$.

**Proposition 2.4.13** Suppose that $A$ is a bounded linear operator on $\mathcal{H}$. Then

$$\text{Ker}(A^*) = \text{Ran}(A)^\perp \quad \text{and} \quad \text{Ker}(A) = \text{Ran}(A^*)^\perp.$$  

**Proof.** Note that $A^*y = 0$ if and only if 

$$(x, A^*y) = 0 \quad \forall x \in \mathcal{H},$$

which is equivalent to 

$$(Ax, y) = 0 \quad \forall x \in \mathcal{H},$$

which, of course, says that $y$ is in the orthogonal complement of $\text{Ran}(A)^\perp$. This proves the first assertion.

The second follows from the first applied to $A^*$ and the elementary fact that $A^{**} = A$ (see Proposition 2.4.6 (ii)).  

Note that $\text{Ker}(A)$ and $\text{Ker}(A^*)$ are always closed subspaces of $\mathcal{H}$, but this is, in general, no longer true for $\text{Ran}(A)$ and $\text{Ran}(A^*)$. Therefore the statements

$$\text{Ker}(A^*)^\perp = \text{Ran}(A) \quad \text{and} \quad \text{Ker}(A)^\perp = \text{Ran}(A^*)$$

are, in general, false.
Exercise 2.4.14  Construct examples of bounded operators for which the statements above fail.

Definition 2.4.15  A bounded linear operator $E$ on $\mathcal{H}$ is a projection if $E^2 = E$. A projection $E$ is said to be orthogonal when it is self adjoint.

Exercise 2.4.16  Suppose that $E$ is a projection on $\mathcal{H}$. The following are equivalent:

(i) $E$ is self adjoint;

(ii) $\operatorname{Ran}(E) = \operatorname{Ker}(E)^\perp$;

(iii) $(Ev, v) = \|Ev\|^2$ for every $v$ in $\mathcal{H}$.

2.5 Compact operators

In this section we begin the study of compact operators, which will play a key role in the solution to the Dirichlet problem via integral equations. In a sense that we shall make precise later, compact operators on infinite dimensional Banach spaces have spectral properties not dissimilar from those of linear operators on finite dimensional Banach spaces.

Definition 2.5.1  A bounded operator $T$ on a Banach space $\mathcal{B}$ is said to be compact if $TB$ is relatively compact (with respect to the norm topology): here $B$ denotes the unit ball in $\mathcal{B}$. The vector space of compact operators on $\mathcal{B}$ will be denoted by $\mathcal{K}(\mathcal{B})$.

It is a well known fact (that we shall not prove here) that the closed unit ball in a Banach space $\mathcal{B}$ is compact if and only if $\mathcal{B}$ is finite dimensional. As a consequence, the identity operator $I$ on a Banach space $\mathcal{B}$ is compact if and only if $\mathcal{B}$ is finite dimensional.

Exercise 2.5.2  Suppose that $T$ is in $\mathcal{L}(\mathcal{B})$. Prove that the following are equivalent:

(i) $T$ is a compact operator;

(ii) $T$ transforms bounded sets into relatively compact sets;
(iii) for every bounded sequence \( \{ f_k \} \) in \( \mathcal{B} \), there exists a subsequence \( \{ f_{n_k} \} \) such that \( T f_{n_k} \) is convergent in \( \mathcal{B} \).

**Definition 2.5.3** A bounded linear operator \( T \) on \( \mathcal{B} \) is said to have **finite rank** provided \( \text{Ran}(T) \) is finite-dimensional.

Clearly, finite rank operators are compact, because the image of the unit ball \( B \) in \( \mathcal{B} \) is a bounded set in the finite-dimensional Banach space \( \text{Ran}(T) \), and therefore it is relatively compact.

Compact operators need not have finite rank. Indeed, consider the interval \( I = [0, 1] \) and the operator \( T : C(I) \to C(I) \), defined by

\[
T f(x) = \int_0^x f(s) \, ds \quad \forall f \in C(I) \quad \forall x \in I.
\]

Observe that \( T \) is compact. Indeed, suppose that \( \{ f_k \} \) is a bounded sequence in \( C(I) \). Then \( \{ T f_k \} \) is a bounded and equicontinuous sequence of continuous functions in \( I \). By the Ascoli–Arzelà theorem, \( \{ T f_k \} \) is a relatively compact sequence in \( C(I) \). Therefore there exists a subsequence \( \{ n_k \} \) such that \( \{ T f_{n_k} \} \) is convergent in the uniform norm on \( I \), and hence in \( C(I) \).

However, \( \text{Ran}(T) = C^1(I) \), which is infinite dimensional, whence \( T \) does not have finite rank.

By using a celebrated compactness criterion of M. Riesz, Fréchet and Kolmogorov, it is not hard to show that the operator \( T \) above is compact from \( L^p(I) \) to \( L^p(I) \) for all \( p \in [1, \infty) \).

It is a celebrated result of P. Enflo that there are Banach spaces \( \mathcal{B} \) such that the vector space of all finite rank operators is not dense in \( \mathcal{K}(\mathcal{B}) \) (with respect to the operator norm). Fortunately, finite rank operators in Hilbert spaces are dense in the space of compact operators, as the following result shows.

**Exercise 2.5.4** Prove that the vector space \( \mathcal{K}(\mathcal{B}) \) of all compact linear operators on \( \mathcal{B} \) is a two sided closed ideal in \( \mathcal{L}(\mathcal{B}) \).

**Exercise 2.5.5** Suppose that \( K \) is a continuous function on \( C([0,1]^2) \). Prove that the operator

\[
Af(s) := \int_0^1 K(s, t) f(t) \, dt \quad \forall f \in C([0,1])
\]
is compact. Prove that if \( K \) is of the following form
\[
K(s, t) := \sum_{j=1}^{J} \phi_j(s) \psi_j(t) \quad \forall s, t \in [0, 1],
\]
then \( A \) has finite rank.

**Exercise 2.5.6** Suppose that \( A \) is as in the previous exercise, except that \( K \) is a function on \([0, 1]^2\) such that its points of discontinuity are contained in the graphs of a finite number \( \varphi_1, \ldots, \varphi_N \) of continuous functions of the form \( t = \varphi_j(s) \) (\( s \) is the second variable!). Prove that \( A \) is a compact operator.

**Proposition 2.5.7** Suppose that \( \mathcal{H} \) is a Hilbert space. Then the space of finite rank operators is dense in \( \mathcal{K}(\mathcal{H}) \) with respect to the operator norm.

**Proof.** Pick \( C \) in \( \mathcal{K}(\mathcal{H}) \). Denote by \( \{P_n\} \) a sequence of orthogonal projections such that \( P_n \uparrow I \) (in the strong operator topology, i.e., \( \lim_{n \to \infty} \|P_n f - f\| = 0 \) for every \( f \in \mathcal{H} \)). Then \( P_n C \) is a finite rank operator. We prove that
\[
\lim_{n \to \infty} \|C - P_n C\| = 0.
\]
Indeed,
\[
\|C - P_n C\| = \sup_{x \in \overline{B}} \|(I - P_n)Cx\| \leq \sup_{y \in \overline{K}} \|(I - P_n)y\|
\]
where \( \overline{K} = \overline{C \mathcal{B}} \). Since \( C \) is compact, so is \( \overline{K} \). Now, set
\[
f_n(y) = \|(I - P_n)y\| \quad \forall y \in \overline{K}.
\]
Then \( f_n \downarrow 0 \). Since \( \overline{K} \) is compact and \( f_n \) continuous, \( f_n \downarrow 0 \) uniformly, so that
\[
\|C - P_n C\| \leq \sup_{y \in \overline{K}} \|(I - P_n)y\| \to 0,
\]
as required. \( \square \)

**Exercise 2.5.8** By using Proposition 2.5.7 prove that a bounded operator \( A \) on \( \mathcal{H} \) is compact if and only if \( A^* \) is compact.

**Exercise 2.5.9** By using Proposition 2.5.7 prove that Hilbert–Schmidt operators are compact. It may be useful to show preliminarily that if \( \{\varphi_j\} \) is an orthonormal basis of \( L^2(\mathbb{R}^n) \), then the set of functions \( \psi_{j,k} \), defined by
\[
\psi_{j,k}(x, y) := \varphi_j(x) \varphi_k(y)
\]
is an orthonormal basis of \( L^2(\mathbb{R}^n \times \mathbb{R}^n) \).
Exercise 2.5.10 Suppose that $\mathcal{H}$ is a Hilbert space and that $\mathcal{J}$ is a nonnull norm closed ideal in $L(\mathcal{H})$. Prove that $K(\mathcal{H})$ is contained in $\mathcal{J}$, by showing that any finite rank operator is in $\mathcal{J}$. Hint: follow the following steps: First step: Given $v,w$ in $\mathcal{H}$, define
\[ R_{v,w} f := (f, v) w. \]
Show that $R_{v,w}$ is a rank one operator with $\text{Ran}(R_{v,m}) = \langle w \rangle$, and that the following formulae hold:
\[ R_{v,w} R_{v',w'} = (w', v) R_{v',w} \quad \alpha R_{v,w} = R_{v,\alpha w}, \quad R_{v,v} = \|v\|^2 P_{\langle v \rangle}, \]
where $P_{\langle v \rangle}$ denotes the orthogonal projection onto the subspace generated by $v$. Furthermore, if $T$ is in $L(\mathcal{H})$, then
\[ R_{v,w} T = R_{T^*,v,w} \quad \text{and} \quad T R_{v,w} = R_{v,Tw}. \]
Finally, if $T \neq 0$ and $Th \neq 0$, then
\[ \frac{1}{\|Th\|^2} R_{Th,w} TR_{w,h} = R_{w,w}. \]
Second step: Show that $\mathcal{J}$ contains all finite rank operators. Third step: use Proposition 2.5.7

2.6 The spectra of compact operators

In this section we discuss some properties of the spectrum of a compact operator acting on a Hilbert space $\mathcal{H}$. The spectrum of a bounded operator on $\mathcal{H}$ may be any closed bounded subset of the complex plane. By contrast, the spectrum of a compact operator is a bounded discrete set of the complex plane. If the spectrum is not a finite set, then it has 0 as the unique accumulation point.

Exercise 2.6.1 Suppose that $A$ is a compact operator on an infinite dimensional Hilbert space. Prove that 0 belongs to the spectrum of $A$.

As a preliminary step towards a better understanding of the spectral properties of compact operators, we consider operators of a very special form. Suppose that $\{\varphi_k\}$ is an orthonormal basis of $\mathcal{H}$, and that $A$ acts diagonally on $\{\varphi_k\}$, i.e., there exist constants $\lambda_k$ such that
\[ A \varphi_k = \lambda_k \varphi_k \quad k = 1, 2, 3, \ldots \quad (2.6.1) \]
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We claim that $A$ is compact if and only if $\lambda_k \to 0$ as $k$ tends to infinity.

Indeed, denote by $P_N$ the orthogonal projection onto the subspace of $\mathcal{H}$ generated by $\varphi_1, \ldots, \varphi_N$. Clearly $P_N$ is compact, for it has finite rank.

Now, if $\lambda_k \to 0$, then

$$\|A - P_N A\| = \sup\{\|Af\| : \|f\| = 1, f \in \text{Ran}(P_N)^\perp\}.\$$

Every $f \in \text{Ran}(P_N)^\perp$ with norm equal to one is of the form $\sum_{j>N} c_j \varphi_j$, where the constants $c_j$ satisfy $\sum_{j>N} |c_j|^2 = 1$. Then

$$Af = \sum_{j>N} c_j \lambda_j \varphi_j.$$

Hence

$$\|Af\|^2 = \sum_{j>N} |c_j|^2 |\lambda_j|^2.$$

For every $\varepsilon > 0$ there exists $\nu$ such that $|\lambda_j| < \varepsilon$ for every $N \geq \nu$. Therefore

$$\|A - P_N A\| \leq \sup_{\|f\|=1, f \in \text{Ran}(P_N)^\perp} \varepsilon^2 \sum_{j>N} |c_j|^2$$

$$= \sup_{\|f\|=1, f \in \text{Ran}(P_N)^\perp} \varepsilon^2 \|f\|^2,$$

so that $P_N A$ tends to $A$ in the uniform operator topology. Since $P_N A$ is compact, $A$ is compact by Proposition 2.5.7, as required.

Conversely, assume that $A$ is compact, and consider the bounded sequence $\{\varphi_k\}$. Then there must exist a subsequence $\{k_j\}$ such that $\{A\varphi_{k_j}\}$ is convergent in $\mathcal{H}$ as $j$ tends to $\infty$. Since $A\varphi_{k_j} = \lambda_{k_j} \varphi_{k_j}$ and $\{A\varphi_{k_j}\}$ is a Cauchy sequence, the difference

$$\|A\varphi_{k_j} - A\varphi_{k_j'}\|^2 = \|\lambda_{k_j} \varphi_{k_j} - \lambda_{k_j'} \varphi_{k_j'}\|^2 = |\lambda_{k_j}|^2 + |\lambda_{k_j'}|^2$$

must converge to 0 as $j$ and $j'$ tend to infinity. Therefore $\lambda_{k_j}$ must converge to 0. By applying a similar argument to every subsequence of the sequence $\{\varphi_k\}$, we see that $\lambda_k$ is convergent to 0.

As a consequence, we see that a compact operator, which acts diagonally on an orthonormal basis, has the following property:

for each $\mu > 0$, the set $\mathcal{E}_\mu$ of all the eigenvalues $\lambda$ of $A$ with $|\lambda| > \mu$ is finite, and for each $\lambda \in \mathcal{E}_\mu$ the nullspace $\text{Ker}(\lambda I - A)$ is finite dimensional.

The next theorem shows that this property holds for all compact operators.
Exercise 2.6.2  Prove that the spectrum of the operator $A$, defined in (2.6.1), is the set
\[ \{ \lambda_k : k = 1, 2, 3, \ldots \} \cup \{ 0 \}, \]
and that 0 may or may not be an eigenvalue.

Theorem 2.6.3  Suppose that $A$ is a compact operator on $\mathcal{H}$. Then for each $\mu > 0$, the set $\mathcal{E}_\mu$ of all the eigenvalues $\lambda$ of $A$ with $|\lambda| > \mu$ is finite, and for each $\lambda \in \mathcal{E}_\mu$ the nullspace $\text{Ker}(\lambda I - A)$ is finite dimensional.

Proof. We argue by contradiction. Suppose that there exists a sequence $\{ \varphi_j \}$ of linearly independent eigenvectors, associated to the eigenvalues $\{ \lambda_j \}$ with $\|\varphi_j\| = 1$ and
\[ \mu \leq |\lambda_j| \leq \|A\|. \]
By (possibly) passing to a subsequence, we may assume that $\{ \lambda_j \}$ is convergent to $\lambda$, say. Obviously
\[ \mu \leq |\lambda| \leq \|A\|. \]
Set
\[ \mathcal{H}_m := \text{span}\{ \varphi_1, \ldots, \varphi_m \}, \]
and denote by $g_m$ and element of $\mathcal{H}_m \cap \mathcal{H}_{m-1}^\perp$ such that $\|g_m\| = 1$.
Since $g_m \in \mathcal{H}_m$, there exist constants $\{ c^m_j : j = 1, \ldots, m \}$ such that
\[ g_m = \sum_{j=1}^m c^m_j \varphi_j. \]
Thus,
\[ c^m_m \varphi_m = g_m - \sum_{j=1}^{m-1} c^m_j \varphi_j. \]
Since $\varphi_j$ is an eigenvector associated to the eigenvalue $\lambda_j$
\[ Ag_m = \sum_{j=1}^m c^m_j \lambda_j \varphi_j \]
\[ = c^m_m \lambda_m \varphi_m + \sum_{j=1}^{m-1} c^m_j \lambda_j \varphi_j \]
\[ = \lambda_m g_m + \sum_{j=1}^{m-1} c^m_j (\lambda_j - \lambda_m) \varphi_j. \]
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Therefore
\[ Ag_m - \lambda_m g_m \in \mathcal{H}_{m-1}. \]
Clearly \( \{g_m\} \) is a bounded sequence in \( \mathcal{H} \). We shall prove that \( \{Ag_m\} \) does not admit a convergent subsequence, thereby contradicting the compactness of \( A \).

Indeed, suppose that \( m < n \). Then
\[
\|Ag_n - Ag_m\|^2 = \|\lambda_n g_n - v\|^2,
\]
where, by (2.6.2), \( v \in \mathcal{H}_{n-1} \). Since \( g_n \in \mathcal{H}_{n-1}^\perp \),
\[
\|Ag_n - Ag_m\|^2 \geq \|\lambda_n g_n\|^2 = |\lambda_n|^2 \|g_n\|^2 \geq \mu^2.
\]
Hence \( \{Ag_m\} \) has no Cauchy subsequences, as required.

Operators of the form \( \lambda I - A \), where \( A \) is compact, arise also in the theory of second order elliptic equations, as we shall see. Fredholm studied some of the properties of these operators, and the resulting theory is now now commonly referred to as Fredholm theory. The following theorem is the main result of Fredholm theory. We point out that Fredholm theory was extended to Banach spaces, mainly by Riesz and Schauder. See [Br] or [Y] for excellent expositions of this generalisation.

**Theorem 2.6.4** Suppose that \( A \) is a compact operator on \( \mathcal{H} \) and that \( \lambda \neq 0 \). Then the following hold:

(i) the range of \( \lambda I - A \) is closed;

(ii) the range of \( \lambda I - A \) is \( \mathcal{H} \) if and only if the kernel of \( \overline{\lambda I - A^*} \) is trivial;

(iii) \( \lambda I - A \) is injective if and only if \( \overline{\lambda I - A^*} \) is injective;

(iv) \( \lambda I - A \) is injective if and only if \( \lambda I - A \) is surjective;

(iv) \( \text{Ker}(\lambda I - A) \) and \( \text{Ker}(\overline{\lambda I - A^*}) \) have the same dimension.

**Proof.** To prove (i), suppose that \( g_j \to g \), where \( g_j = (\lambda I - A)f_j \), for some \( f_j \) in \( \mathcal{H} \). We must show that there exists \( f \in \mathcal{H} \) such that \( g = (\lambda I - A)f \). We have a unique decomposition
\[
f_j = u_j + v_j,
\]
where \( u_j \in \text{Ker}(\lambda I - A) \), and \( v_j \in \text{Ker}(\lambda I - A)^\perp \). Recall that \( \text{Ker}(\lambda I - A) \) is closed, because it is finite dimensional. Thus,

\[
g_j = \lambda v_j - Av_j \quad j = 1, 2, \ldots
\]

(2.6.3)

We claim that \( \{v_j\} \) is bounded. We argue by contradiction. Suppose that \( \{v_j\} \) is not bounded. We may assume that \( \|v_j\| \to \infty \) (otherwise we consider a suitable subsequence of \( \{v_j\} \)). Set

\[
w_j := \frac{v_j}{\|v_j\|} \quad j = 1, 2, \ldots
\]

Since \( A \) is compact, there exists a subsequence, which we still denote by \( \{w_j\} \), such that \( \{Aw_j\} \) is convergent to \( z \), say. From (2.6.3), and the fact that \( \{g_j\} \) is bounded (for it is a convergent sequence) we deduce that

\[
0 = \lambda \lim_j w_j - z.
\]

In particular, \( \|z\| = |\lambda| \). Observe that \( z \) is in \( \text{Ker}(\lambda I - A) \), for

\[
\lambda z - Az = \lambda \lim_j (\lambda w_j - Aw_j) = 0.
\]

But \( v_j \in \text{Ker}(\lambda I - A)^\perp \) by assumption, hence so is \( w_j \). Since \( v_j \in \text{Ker}(\lambda I - A)^\perp \) is closed and \( \lambda \neq 0 \), \( z \) is in \( \text{Ker}(\lambda I - A)^\perp \). Therefore \( z = 0 \), which contradicts the fact that \( \|z\| = |\lambda| \) and \( \lambda \neq 0 \).

The claim is proved.

Since \( \{v_j\} \) is bounded, there exists a subsequence, which we denote still by \( \{v_j\} \), and a point \( \zeta \in H \) such that \( Av_j \to \zeta \). Thus, by (2.6.3),

\[
\lambda \lim_j v_j = g + \zeta,
\]

and

\[
(\lambda I - A)((g + \zeta)/\lambda) = \lim_{j \to \infty} (\lambda I - A)v_j = \lim_{j \to \infty} g_j = g.
\]

This concludes the proof of (i).

Part (ii) follows directly from (i) and Proposition 2.4.13.

Next we prove (iii). We begin by showing that if \( A_\lambda \) is injective, then \( A_\lambda^* \) is injective. We argue by *reductio ad absurdum*. Suppose that \( A_\lambda^* \) is not injective. Observe that, by (i) and Proposition 2.4.13, there is an orthogonal decomposition

\[
H = \text{Ran}(A_\lambda) + \text{Ker}(A_\lambda^*) \quad \text{(orthogonal sum)}.
\]
Since $A_\lambda^*$ is not injective, $\text{Ker}(A_\lambda^*)$ is not trivial, whence $\text{Ran}(A_\lambda)$ is a proper (closed) subspace of $\mathcal{H}$. Set $\mathcal{R}_\lambda^j := \text{Ran}(A_\lambda^j)$.

We claim that
\[ \mathcal{R}_\lambda^j \supset \mathcal{R}_\lambda^{j+1}, \quad j = 1, 2, 3, \ldots \quad (\text{proper inclusion}). \tag{2.6.4} \]

For otherwise there is an index $j$ such that $\mathcal{R}_\lambda^j = \mathcal{R}_\lambda^{j+1}$. Then there are $v \in \mathcal{R}_\lambda^{j-1} \setminus \mathcal{R}_\lambda^j$ and $w \in \mathcal{R}_\lambda^j$ such that $A_\lambda v = A_\lambda w$. But this and the injectivity of $A_\lambda$ imply that $v = w$, which contradicts the fact that $v \notin \mathcal{R}_\lambda^j \setminus \mathcal{R}_\lambda^{j-1}$.

By (i), each $\mathcal{R}_\lambda^j$ is a closed subspace of $\mathcal{H}$, hence a proper closed subspace of $\mathcal{R}_\lambda^{j-1}$. Therefore there exists a unit vector $f_j \in \mathcal{R}_\lambda^j$, orthogonal to $\mathcal{R}_\lambda^{j+1}$.

We claim that $\{Af_j\}$ has no Cauchy subsequences. Indeed, suppose that $j < k$. Then
\[ A f_k - A f_j = \lambda f_k - A_\lambda f_k - \lambda f_j + A_\lambda f_j. \]

Observe that $A_\lambda f_k + \lambda f_j - A_\lambda f_j$ is in $\mathcal{R}_\lambda^{k+1}$, and that $\lambda f_k$ is orthogonal to $\mathcal{R}_\lambda^{k+1}$. Therefore
\[ \|A f_k - A f_j\|^2 \geq |\lambda|^2 \|f_k\|^2 = |\lambda|^2, \]

as claimed.

This contradicts the compactness of $A$, and concludes the proof of one implication of (iii).

To prove the reverse implication, we may repeat the same reasoning with $A_\lambda^*$ playing the role of $A_\lambda$, and using the fact that $(A_\lambda^*)^* = A_\lambda$.

The proof of (iii) is complete.

To prove (iv), observe that, by Proposition 2.4.13 and (i), $\lambda I - A$ is injective if and only if $\text{Ran}(\overline{\lambda I - A^*}) = \mathcal{H}$.

Finally we prove (v). For the sake of simplicity, set
\[ \mathcal{N}_\lambda := \text{Ker}(\lambda I - A) \quad \text{and} \quad \mathcal{N}_\lambda^* := \text{Ker}(\overline{\lambda I - A^*}). \]

Denote by $\delta$ and $\delta^*$ the dimensions of $\mathcal{N}_\lambda$ and $\mathcal{N}_\lambda^*$, respectively.

We prove that $\delta^* \leq \delta$. Suppose that $\delta < \delta^*$. Then there exists a linear operator $\Lambda : \mathcal{N}_\lambda \to \mathcal{N}_\lambda^*$, which is injective but not surjective. Denote by $P$ the orthogonal projection onto $\mathcal{N}_\lambda$, and consider the operator
\[ S := A + \Lambda \circ P. \]

Note that $\Lambda \circ P$ is of finite rank, hence compact. Thus, $S$ is compact, for it is the sum of two compact operators.
We claim that $\ker(\lambda I - S) = \{0\}$. Indeed, suppose that $u$ is in $\ker(\lambda I - S)$. Then

$$\lambda u - Au - \Lambda(Pu) = 0.$$ 

Since $\Lambda(Pu)$ is in $N_\lambda^*$, $\lambda u - Au$ is in $\text{Ran}(\lambda I - A)$, and $\text{Ran}(\lambda I - A) = (N_\lambda^*)^\perp$ by Proposition 2.4.13 and (i), we may conclude that

$$\lambda u - Au = 0 \quad \text{and} \quad \Lambda(Pu) = 0.$$ 

These equations imply that $Au = 0$, whence $u = 0$ because $\Lambda$ is injective. The proof of the claim is complete.

By the claim and (iv), $\lambda I - S$ is surjective. But this contradicts the fact that $\Lambda$ is not surjective. Indeed, suppose that $g$ belongs to $N_\lambda^* \setminus \text{Ran}(\Lambda)$, and consider the equation

$$\lambda u - Su = g,$$ 

which is equivalent to

$$\lambda u - Au - \Lambda(Pu) = g.$$ 

Since $g \in N_\lambda^*$ and $\lambda u - Au$ is orthogonal to $N_\lambda^*$, we must have $g = \Lambda(Pu)$, which contradicts the assumption that $g$ be off the range of $\Lambda$.

This concludes the proof that $\delta^* \leq \delta$. The same reasoning, with $\overline{\lambda} - A^*$ in place of $\lambda I - A$, and the fact that

$$\lambda I - A = (\overline{\lambda} - A^*)^*,$$ 

prove that $\delta \leq \delta^*$.

This concludes the proof of (v), and of the theorem. \qed

We conclude this section by deriving some consequences of the result above. Recall the following consequence of the closed graph theorem.

**Proposition 2.6.5** Suppose that $B$ is a Banach space, that $A$ is in $\mathcal{L}(B)$ is a bijective operator (onto $B$). Then $A^{-1}$ (which clearly exists) is in $\mathcal{L}(B)$.

The following corollary describes the spectrum of a compact operator.

**Corollary 2.6.6** Suppose that $A$ is a compact operator on the Hilbert space $H$. If $\lambda$ is in $\sigma(A) \setminus \{0\}$, then $\lambda$ is an isolated eigenvalue of $A$ of finite multiplicity.
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Proof. Suppose that $\lambda \in \sigma(A)$. If $\lambda$ is not an eigenvalue, then $\lambda I - A$ is injective, hence it is surjective by Theorem 2.6.4 (iv). Then $(\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{H})$ by Proposition 2.6.5, so that $\lambda$ is in the resolvent set of $A$.

Thus, $\lambda$ is an eigenvalue. The associated eigenspace is finite dimensional by Theorem 2.6.3 as required.

Thus, the spectrum of a compact operator on an infinite dimensional separable Hilbert space is one of the following

(i) 0 (which may or may not be an eigenvalue) and a finite number (possibly 0) of nonzero eigenvalues with finite multiplicities;

(ii) 0 (which may or may not be an eigenvalue) and a sequence $\{\lambda_k\}$ of nonzero eigenvalues with finite multiplicities that tends to 0 as $k$ tends to $\infty$.

An interesting fact, which we shall use in the sequel without any further reference, is that all the results in this section hold, with different proofs, for compact operators on Banach spaces (see the books [Br, Y] for the corresponding proofs).

2.7 The spectral theorem for compact self adjoint operators

In this section we prove the spectral theorem for compact self adjoint operators on a Hilbert space $\mathcal{H}$ that generalises the spectral theorem for Hermitian matrices on finite dimensional Hilbert spaces. This result may be further generalised to bounded and even to unbounded linear operators on $\mathcal{H}$. The result for bounded operators is due to Hilbert and that for unbounded operators generalisation is a deep result of von Neumann, which we do not discuss here.

Theorem 2.7.1 Suppose that $A$ is a compact self adjoint operator on the Hilbert space $\mathcal{H}$. Then there exists an orthonormal basis $\{\varphi_k\}$, which consists of eigenvectors of $A$. Furthermore, the sequence $\{\lambda_k\}$ of the corresponding eigenvalues is real and is convergent to 0 as $k$ tends to $\infty$.

We have already proved that if $\{\varphi_k\}$ is an orthonormal basis of $\mathcal{H}$, and $A$ acts diagonally on $\{\varphi_k\}$ by

$$A\varphi_k = \lambda_k \varphi_k,$$
then $\lambda_k \to 0$ as $k$ tends to infinity if and only if $A$ is a compact operator. If all the eigenvalues $\lambda_k$ are real, then $A$ is self adjoint.

**Proof.** The proof is quite long. We split it up in several steps.

**Step I:** if $\lambda$ is an eigenvalue of $A$, then $\lambda$ is real.

Suppose that $\lambda$ is a nonzero eigenvalue and that $\varphi$ is a corresponding eigenvector. Then

$$\lambda (\varphi, \varphi) = (A\varphi, \varphi) = (\varphi, A\varphi) = \overline{\lambda} (\varphi, \varphi),$$

which implies the required conclusion.

**Step II:** eigenvectors associated to distinct eigenvalues are orthogonal.

Suppose that $\varphi_1$ and $\varphi_2$ are eigenvectors associated to the eigenvalues $\lambda_1$ and $\lambda_2$, with $\lambda_1 \neq \lambda_2$. Then

$$\lambda_1 (\varphi_1, \varphi_2) = (A\varphi_1, \varphi_2) = (\varphi_1, A\varphi_2) = \lambda_2 (\varphi_1, \varphi_2),$$

so that $(\varphi_1, \varphi_2) = 0$, as required.

**Step III:** for each $\mu > 0$, the set $\mathcal{E}_\mu$ of all the eigenvalues $\lambda$ of $A$ with $|\lambda| > \mu$ is finite, and for each $\lambda \in \mathcal{E}_\mu$ the nullspace $\ker(\lambda I - A)$ is finite dimensional.

This is just Theorem 2.6.3 above, which holds for all compact operators.

**Step IV:** either $\|A\|$ or $-\|A\|$ is an eigenvalue of $A$.

By Proposition 2.4.9, either

$$\|A\| = \sup_{\|f\|=1} (Af, f) \quad \text{or} \quad -\|A\| = \inf_{\|f\|=1} (Af, f).$$

For the sake of definiteness, assume that the first of the two equalities above holds. Clearly, there exists a sequence $\{f_k\} \in \mathcal{H}$ with $\|f_k\| = 1$, such that

$$\|A\| = \lim_{k \to \infty} (Af_k, f_k). \quad (2.7.1)$$

Since $A$ is compact, there exists a subsequence of $\{f_k\}$, which we still denote by $\{f_k\}$, such that $\{Af_k\}$ is convergent in $\mathcal{H}$ to a function, $g$ say. Write $Af_k = Af_k - g + g$ on the right hand side of (2.7.1) above. Observe that

$$|(Af_k - g, f_k)| \leq \|Af_k - g\| \|f_k\| \to 0$$
as $k$ tends to infinity, so that

$$\|A\| = \lim_{k \to \infty} (g, f_k).$$

Note also that for every $k$

$$\|A\| \geq \sup_{\|f\|=1} \|Af\| \geq \|Af_k\|.$$

By taking the limit of both sides as $k$ tends to infinity, we get that

$$\|A\| \geq \|g\|. \quad (2.7.2)$$

Now, notice that

$$\|Af_k - \|A\| f_k\|^2 = \|Af_k\|^2 - 2\|A\| (Af_k, f_k) + \|A\|^2 \|f_k\|^2.$$

Then we take the limit of both sides as $k$ tends to infinity, and obtain

$$\lim_{k \to \infty} \|Af_k - \|A\| f_k\|^2 = \|g\|^2 - \|A\|^2.$$

From (2.7.2) and the fact that the left hand side is nonnegative, we deduce that $\|A\| = \|g\|$ and, consequently, that

$$\lim_{k \to \infty} \|Af_k - \|A\| f_k\|^2 = 0.$$

Therefore $f_k$ tends to $g/\|A\|$, and $Ag = \|A\| g$, so that $g$ is an eigenvector associated to the eigenvalue $\|A\|$, as required to conclude the proof of Step IV.

**Step V: conclusion.**

We argue by *reductio ad absurdum*. Suppose that the closure $S$ of the span of all the eigenvectors of $A$ is not dense in $H$. Then $S^\perp$ is a nontrivial closed subspace of $H$, hence an Hilbert space itself. It is straightforward to check that $AS \subseteq S$ and that $AS^\perp \subseteq S^\perp$. Furthermore, $A$, restricted to $S^\perp$, is clearly a bounded self adjoint operator on $S^\perp$. Therefore, by Step IV (applied to $A_{|S^\perp}$), it has at least one nonzero eigenvalue. This clearly would be also an eigenvalue of the original operator $A$, which contradicts the fact that $S$ is the closure of the span of all the eigenvalues of $A$.

This concludes the proof of Step V, and of the theorem. \qed

Note that 0 may or may not be an eigenvalue of a bounded self adjoint operator. Furthermore, if 0 is an eigenvalue, then the corresponding eigenspace may be finite or infinite dimensional. The following examples should help clarifying the situation.
Example 2.7.2 Suppose that $S$ is a finite dimensional subspace of the infinite dimensional Hilbert space $\mathcal{H}$ and denote by $E$ the orthogonal projection of $\mathcal{H}$ onto $S$. Clearly $E$ is a compact operator which has two eigenvalues, 0 and 1: the eigenspace associated to the eigenvalue 0 is just $S^\perp$, which is infinite dimensional.

Example 2.7.3 Suppose that $\{\varphi_k : k = 1, 2, \ldots\}$ is an orthonormal basis of $\mathcal{H}$. Define the operator $A$ on this basis by

$$A\varphi_k = \frac{1}{k}\varphi_k,$$

and then extend it by linearity on $\mathcal{H}$. Then $A$ is a compact self adjoint operator with eigenvalues $\{1/k\}$: thus, 0 is not an eigenvalue.

Exercise 2.7.4 Give all the details of the example above.

Example 2.7.5 Denote by $I$ the interval $[0, 1]$ and by $K : I \times I \to \mathbb{R}$ the function defined by

$$K(t, x) = \begin{cases} 1 & \text{if } t \leq x \\ 0 & \text{if } t > x. \end{cases}$$

Let $A : L^2(I) \to L^2(I)$ be the Hilbert–Schmidt operator with kernel $K$. By Exercise 2.5.9 the operator $A$ is compact. Since the kernel $K$ is not Hermitian, $A$ is not self adjoint. It may be proved that $A$ has no eigenvalues at all.

Exercise 2.7.6 Give all the details of the example above.

Example 2.7.7 Denote $I$ the interval $[0, 1]$, and by $A : L^2(I) \to L^2(I)$ the operator defined by

$$Af(x) := x^2 f(x) \quad \forall x \in I.$$

It is straightforward to check that $A$ is a bounded self adjoint operator, with $\|A\| = 1$. It may be proved that $A$ is not compact and has no eigenvalues.

Exercise 2.7.8 Give all the details of the example above.
Chapter 3

Dirichlet problem via integral equations

3.1 Kernels of type $\alpha$

In this section we analyse the properties of a class of integral operators on smooth compact hypersurfaces of $\mathbb{R}^n$. In particular, we investigate their compactness. The results we shall obtain will be applied to the double layer potential, and to a suitable derivative of the single layer potential.

Definition 3.1.1 Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$ and that $\alpha$ is a number in $(0, n - 1)$. A measurable function $K : \partial \Omega \times \partial \Omega \to \mathbb{C}$ is a kernel of type $\alpha$ if

$$K(X, Y) = A(X, Y) |X - Y|^{-\alpha} \quad \forall X, Y \in \partial \Omega, \ X \neq Y,$$

where $A$ is a bounded function on $\partial \Omega \times \partial \Omega$. If $K$ is of the form

$$K(X, Y) = A(X, Y) \log |X - Y| + B(X, Y) \quad \forall X, Y \in \partial \Omega, \ X \neq Y,$$

where $A$ and $B$ are bounded functions on $\partial \Omega \times \partial \Omega$, then we say that $K$ is a kernel of type 0. If $K$ is a kernel of type $\alpha \in (0, n - 1)$ and $A$ is a continuous function on $\partial \Omega \times \partial \Omega$, then we say that $K$ is a continuous kernel of type $\alpha$. Similarly, if $K$ is a kernel of type 0, and the functions $A$ and $B$ are continuous on $\partial \Omega \times \partial \Omega$, then we say that $K$ is a continuous kernel of type 0.

If $K$ is a kernel of order $\alpha \in [0, n - 1)$, then we define the operator $T_K$ associated to $K$ formally by

$$T_K f(X) := \int_{\partial \Omega} K(X, Y) f(Y) d\sigma(Y) \quad \forall X \in \partial \Omega. \quad (3.1.1)$$
It is straightforward to check that $T_K f$ makes sense for every bounded function $f$ on $\partial \Omega$. We now investigate possible extensions of $T_K$ to $L^p(\partial \Omega)$. We shall need the following boundedness criterion for operators acting on $L^p(\partial \Omega)$. We state it in the more general setting of a $\sigma$-finite measure space.

**Proposition 3.1.2** Suppose that $M$ is a measure space (with measure $\mu$) and that $K$ is a complex valued measurable function on $M \times M$, which satisfies the following estimates

\[
C_0 := \sup_{x \in M} \int_M |K(x, y)| \, d\mu(y) < \infty
\]

and

\[
C_1 := \sup_{y \in M} \int_M |K(x, y)| \, d\mu(x) < \infty.
\]

Then the operator $T_K$, formally defined by

\[
T_K f(x) = \int_M K(x, y) f(y) \, d\mu(y),
\]

extends to a bounded operator on $L^p(M)$ for all $p \in [1, \infty]$. Furthermore,

\[
\|T_K\| \leq C_0^{1/p'} C_1^{1/p},
\]

where $p'$ denotes the index conjugate to $p$.

**Proof.** If $p = \infty$, the hypothesis $C_1 < \infty$ is superfluous, and the conclusion is obvious. Thus, we may assume that $p < \infty$. By Hölder’s inequality

\[
|T_K f(x)| \leq \left( \int_M |K(x, y)| \, d\mu(y) \right)^{1/p'} \left( \int_M |K(x, y)||f(y)|^p \, d\mu(y) \right)^{1/p}
\]

\[
\leq C_0^{1/p'} \left( \int_M |K(x, y)||f(y)|^p \, d\mu(y) \right)^{1/p}.
\]

We now compute the $p$ norm of both sides (with respect to the variable $x$), and obtain

\[
\|T_K f\|_p \leq C_0^{1/p'} \left( \int_M d\mu(x) \int_M |K(x, y)| |f(y)|^p \, d\mu(y) \right)^{1/p}
\]

\[
\leq C_0^{1/p'} C_1^{1/p} \|f\|_p,
\]

as required. \qed
Some important properties of kernels of type $\alpha$ are established in the next proposition. We shall need two lemmata. Loosely speaking, Lemma 3.1.3 below says that, at a local scale, the distance between two points in $\partial \Omega$ is (uniformly) comparable to that of the corresponding images via a (suitable) coordinate function.

**Lemma 3.1.3** Suppose that $\Omega$ is a bounded domain with $C^2$ boundary. Then there exist local charts $(U_j, \Phi_j)$, $j \in \{1, \ldots, N\}$, such that the following hold:

(i) $\Phi_j$ is an homeomorphism between $U_j$ and $\Phi_j(U_j) =: V_j$, and its inverse $\Psi_j := \Phi_j^{-1} : V_j \to U_j$ is of class $C^2(V_j)$;

(ii) $\Psi_j$ has the following form

$$\Psi_j(\xi) = (\xi, f_j(\xi)) \quad \forall \xi \in V_j,$$

i.e., that $\partial \Omega \cap U_j$ is the graph of the function $f_j : V_j \to \mathbb{R}$ (clearly $f_j \in C^1(V_j)$), because $\partial \Omega$ is assumed to be of class $C^2$;

(iii) $\bigcup_j U_j = \partial \Omega$;

(iv) there exist positive constants $C_0$ and $\delta_0$ such that for each $X \in \partial \Omega$ there exists $j \in \{1, \ldots, N\}$ with $B_{\delta}(X) \cap \partial \Omega \subset U_j$, and

$$\Phi_j(B_{\delta}(X) \cap \partial \Omega) \subseteq B_{\delta}(\Phi_j(X)) \subseteq \Phi_j(B_{C_0 \delta}(X) \cap \partial \Omega) \quad \forall \delta \in (0, \delta_0],$$

equivalently

$$B_{\delta}(X) \cap \partial \Omega \subseteq \Psi_j(B_{\delta}(\Phi_j(X))) \subseteq B_{C_0 \delta}(X) \cap \partial \Omega \quad \forall \delta \in (0, \delta_0].$$

**Proof.** For each point $Z \in \partial \Omega$ there is a local chart $(U_Z, \Phi_Z)$ such that $Z$ is in $U_Z$, the map $\Phi_Z$ is an homeomorphism between $U_Z$ and $\Phi(U_Z) =: V_Z$ and its inverse $\Psi_Z := \Phi_Z^{-1}$ is of class $C^2(V_Z)$. By possibly shrinking the open set $U_Z$, and remaining the variables, we may assume that $\Psi_Z$ is of the form $\xi \mapsto (\xi, f_Z(\xi))$, where $f_Z$ is a function of class $C^2(V_Z)$, and that $V_Z$ is a ball with centre $0 \in \mathbb{R}^{n-1}$. Clearly $\bigcup_{Z \in \partial \Omega} U_Z = \partial \Omega$. Since $\partial \Omega$ is compact, we can choose a finite subcovering $U_{Z_1}, \ldots, U_{Z_N}$, which clearly satisfies (i)-(iii). In the sequel we shall write $U_j$ instead of $U_{Z_j}$.

Now we prove (iv). For each $X \in \partial \Omega$ and $j \in \{1, \ldots, N\}$, set

$$\tau_j(X) := \begin{cases} 0 & \text{if } X \notin U_j \\ \sup \{ r > 0 : B_r(X) \cap \partial \Omega \subseteq U_j \} & \text{if } X \in U_j. \end{cases}$$
Then set
\[ \tau(X) := \max\{\tau_1(X), \ldots, \tau_N(X)\} \]
If \( \tau(X) = \tau_k(X) \), then we say that \( X \) is related to \( U_k \), and write \( X \sim U_k \).
Since \( \{U_1, \ldots, U_N\} \) is a covering of \( \partial \Omega \), each \( X \) belongs to at least to one of the sets \( U_1, \ldots, U_N \), so that \( \tau(X) > 0 \) for every \( X \in \partial \Omega \). Define
\[ \tau := \inf_{X \in \partial \Omega} \tau(X). \tag{3.1.2} \]
We claim that \( \tau > 0 \). We argue by reductio ad absurdum. Suppose that \( \tau = 0 \). Then there exists a sequence \( \{X_k\} \) of points in \( \partial \Omega \) such that
\[ \lim_{k \to \infty} \tau(X_k) = 0. \tag{3.1.3} \]
By compactness, there exists a subsequence \( \{X_{k_\ell}\} \) of \( \{X_k\} \) which is convergent to some point, \( X_* \), say, in \( \partial \Omega \). Suppose that \( \varepsilon = (1/2) \tau(X_*) \), and that \( X_* \) is related to \( U_h \). Then there exists \( \ell_0 \) such that
\[ |X_{k_\ell} - X_*| < \varepsilon \quad \forall \ell \geq \ell_0. \]
Then, by the triangle inequality, \( B_{\varepsilon/2}(X_{k_\ell}) \subset U_h \) for every \( \ell \geq \ell_0 \), whence \( \tau(X_{k_\ell}) > \varepsilon/2 \) for all such \( \ell \), thereby contradicting (3.1.3). Thus, the claim is proved.

Now, we define
\[ K_j := \bigcup_{X \in U_j} (B_{\tau/2}(X) \cap \partial \Omega). \tag{3.1.4} \]
Each point in \( B_{\tau/2}(X) \cap \partial \Omega \) has distance at least \( \tau/2 \) from \( \partial U_j \), for \( X \sim U_j \) and \( \tau < \tau(X) = \tau_j(X) \). Thus, \( K_j \) is a compact subset of \( U_j \). Define
\[ \kappa_j := \text{co}(\Phi_j(K_j)), \tag{3.1.5} \]
the convex hull of \( K_j \). Clearly \( \Phi_j(K_j) \) is compact, hence \( \kappa_j \) is compact (see Exercise 3.1.4 below). Furthermore, \( \Phi_j(K_j) \subset V_{Z_j} \), whence \( \kappa_j \) is a compact subset of \( V_{Z_j} \) (recall that \( V_{Z_j} \) is an open ball in \( \mathbb{R}^{n-1} \)). Set
\[ C_j := \sup_{\xi \in \kappa_j} \|d\Psi_j(\xi)\| \quad \text{and} \quad C_0 := \max_{j=1,\ldots,N} \sqrt{1 + C_j^2}. \tag{3.1.6} \]
Suppose that \( X \) and \( Y \) are in \( U_j \) and set \( \Phi_j(X) = \xi \) and \( \Phi_j(Y) = \eta \). Observe that
\[ |\xi - \eta|^2 \leq |\Psi_j(\xi) - \Psi_j(\eta)|^2 = |X - Y|^2 \]
\[ = |\xi - \eta|^2 + |f_j(\xi) - f_j(\eta)|^2 \]
\[ \leq |\xi - \eta|^2 + |\nabla f_j(\omega) \cdot (\xi - \eta)|^2 \]
\[ \leq |\xi - \eta|^2 \left[ 1 + \sup_{\omega \in [\xi, \eta]} |\nabla f_j(\omega)|^2 \right]. \tag{3.1.7} \]
Choose any $\delta_0 > 0$ such that
\[
\delta_0 < \frac{\tau}{4C_0},
\] (3.1.8)
and suppose that $\delta \leq \delta_0$. Fix a point $X \in \partial \Omega$. For the sake of definiteness, suppose that $X$ is related to $U_j$. Assume that $Y \in B_\delta(X) \cap \partial \Omega$. Then $X$ and $Y$ belong to $K_j$, and estimate (3.1.7) applies. Notice that the segment $[\xi, \eta]$ is contained in $\kappa_j$, so that
\[
|\xi - \eta|^2 \leq |X - Y|^2 \leq C_0^2 |\xi - \eta|^2.
\] (3.1.9)

The left hand inequality in (3.1.9) implies that $\Phi_j(B_\delta(X)) \subset B_\delta(\Phi_j(X))$ for every $\delta \leq \delta_0$. The right hand inequality in (3.1.9) implies that $B_\delta(\Phi_j(X)) \subset \Phi_j(B_{C_0\delta}(X))$ for every $\delta \leq \delta_0$.

This concludes the proof of (iv), and of the lemma.

\section*{Exercise 3.1.4}
Prove that the convex hull of a compact set in $\mathbb{R}^n$ is compact.

The next lemma is a sort of substitute for integration of “radial” functions in polar coordinates on $\partial \Omega$.

\begin{lemma}
Suppose that $\Omega$ is a bounded domain with $C^2$ boundary and that $\alpha \in [0, n-1)$. Then there exists a constant $C_1$ (see (3.1.13) below) such that
\[
\sup_{X \in \partial \Omega} \int_{B_\delta(X) \cap \partial \Omega} |X - Y|^{-\alpha} \, d\sigma(Y) \leq C_1 \delta^{n-1-\alpha} \quad \forall \delta > 0.
\]
\end{lemma}

\section*{Proof.}
Let $\delta_0 > 0$, $U_1, \ldots, U_N$ and $\Phi_j : U_j \rightarrow \mathbb{R}^{n-1}$, $j = 1, \ldots, N$ be as in Lemma 3.1.3. The left hand inequality in (3.1.9) implies that
\[
|X - Y|^{-\alpha} \leq |\xi - \eta|^{-\alpha} \quad \forall \xi, \eta \in \Phi(K_j).
\] (3.1.10)
Furthermore, the left hand inequality in (3.1.9) implies that $\Phi_j(B_\delta(X)) \subset B_\delta(\Phi_j(X))$ for every $\delta \leq \delta_0$. Thus, if $X$ is related to $U_j$,
\[
\int_{B_\delta(X) \cap \partial \Omega} |X - Y|^{-\alpha} \, d\sigma(Y) \leq \int_{B_\delta(\Phi_j(X))} |\xi - \eta|^{-\alpha} \sqrt{1 + \sup_{\Phi_j(K_j)} |\nabla f_j|^2} \, d\eta
\leq C_0 \int_{B_\delta(\Phi_j(X))} |\xi - \eta|^{-\alpha} \, d\eta
= C_0 \omega_{n-1} \int_0^\delta r^{n-2-\alpha} \, dr
= C_0 \frac{\omega_{n-1}}{n - 1 - \alpha} \delta^{n-1-\alpha} \quad \forall \delta \in (0, \delta_0].
\] (3.1.11)
This proves the required result for \( \delta \leq \delta_0 \). Suppose now that \( \delta > \delta_0 \). Then

\[
\int_{B_\delta(X) \cap \partial \Omega} |X - Y|^{-\alpha} \, d\sigma(Y)
\leq \int_{B_{\delta_0}(X) \cap \partial \Omega} |X - Y|^{-\alpha} \, d\sigma(Y) + \delta_0^{-\alpha} \sigma(\partial \Omega)
\leq \frac{\omega_{n-1}}{n - 1 - \alpha} C_0 \delta_0^{n-1-\alpha} + \delta_0^{-\alpha} \sigma(\partial \Omega)
\leq \delta^{n-1-\alpha} \left[ \frac{\omega_{n-1}}{n - 1 - \alpha} C_0 + \delta^{1+\alpha-n} \delta_0^{-\alpha} \sigma(\partial \Omega) \right]
\leq \delta^{n-1-\alpha} \left[ \frac{\omega_{n-1}}{n - 1 - \alpha} C_0 + \delta^{1+\alpha-n} \delta_0^{-\alpha} \sigma(\partial \Omega) \right]
= C_1 \delta^{n-1-\alpha},
\]

where

\[
C_1 := \frac{\omega_{n-1}}{n - 1 - \alpha} C_0 + \delta_0^{1-n} \sigma(\partial \Omega).
\]

This concludes the proof of the lemma. \( \square \)

**Proposition 3.1.6** Suppose that \( K \) is a kernel of type \( \alpha \in [0, n - 1) \) and that \( T_K \) denotes the corresponding integral operator (see (3.1.1)). The following hold:

(i) for every \( p \in [1, \infty) \) the operator \( T_K \) is bounded on \( L^p(\partial \Omega) \);

(ii) if \( \varepsilon > 0 \), and \( K(X, Y) = 0 \) for all \( X, Y \in \partial \Omega \) such that \( |X - Y| \geq \varepsilon \), then

\[
\| T_K \|_p \leq C \varepsilon^{n-1-\alpha} \| A \|_\infty
\]

when \( \alpha > 0 \), and

\[
\| T_K \|_p \leq C \varepsilon^{n-1} \left[ \| A \|_\infty \log \varepsilon^{-1} + \| B \|_\infty \right]
\]

when \( \alpha = 0 \);

(iii) if \( \alpha \in [0, n - 1) \), then \( T_K \) is a compact operator on \( L^2(\partial \Omega) \).

**Proof.** To prove (i), we apply Proposition 3.1.2. We shall prove the required result in the case where \( \alpha > 0 \). The case where \( \alpha = 0 \) is left to the reader. Observe that for every \( \delta > 0 \)

\[
\int_{\partial \Omega} |K(X, Y)| \, d\sigma(Y) \leq \| A \|_\infty \int_{\partial \Omega} |X - Y|^{-\alpha} \, d\sigma(Y)
\leq \| A \|_\infty C_1 \delta^{n-1-\alpha},
\]
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the last inequality being a consequence of Lemma \ref{lem:3.1.5}. Thus,

\[ \sup_{X \in \partial \Omega} \int_{\partial \Omega} |K(X, Y)| \, d\sigma(Y) < \infty. \]  \hspace{1cm} (3.1.14)

A similar argument, with the roles of $X$ and $Y$ interchanged, proves that

\[ \sup_{Y \in \partial \Omega} \int_{\partial \Omega} |K(X, Y)| \, d\sigma(X) < \infty, \]  \hspace{1cm} (3.1.15)

and (i) follows from this and Proposition \ref{prop:3.1.2}.

Now we prove (ii). We give full details in the case where $\alpha > 0$. The case $\alpha = 0$ is left to the interested reader. A close examination of the proof of (i) and of Lemma \ref{lem:3.1.5} shows that if $\varepsilon$ is small enough, then

\[ \| T_K \|_p \leq C \varepsilon^{n-1-\alpha}, \]

as required.

Finally we prove (iii). For each $\varepsilon > 0$ we set

\[ D_\varepsilon := \{(X, Y) \in \partial \Omega \times \partial \Omega : |X - Y| \leq \varepsilon\}. \]

Correspondingly, we define

\[ K_\varepsilon(X, Y) := K(X, Y) 1_{D_\varepsilon}(X, Y). \]

Clearly $K - K_\varepsilon$ is bounded on $\partial \Omega \times \partial \Omega$, so that $K - K_\varepsilon \in L^2(\partial \Omega \times \partial \Omega)$ for $\partial \Omega$ is compact. Therefore the corresponding operator $T_{K - K_\varepsilon}$ is a Hilbert–Schmidt operator, hence it is compact on $L^2(\partial \Omega)$.

We shall prove that

\[ \lim_{\varepsilon \downarrow 0} \| T_{K_\varepsilon} \|_2 = 0. \]  \hspace{1cm} (3.1.16)

The required conclusion then follows from the fact that the ideal of compact operators is closed in $\mathcal{L}(L^2(\partial \Omega))$ (see Exercise \ref{ex:2.5.4}).

To prove (3.1.16) observe that $T_{K_\varepsilon}$ is the integral operator whose kernel vanishes in $D_\varepsilon^c$ and is equal to $K(X, Y)$ in $D_\varepsilon$. By (ii),

\[ \| T_{K_\varepsilon} \|_2 \leq C \varepsilon^{n-1-\alpha}, \]

which tends to 0 as $\varepsilon$ tends to 0, as required. \hfill $\square$

So far, we have studied properties of operators associated to generic kernels of type $\alpha$, where $\alpha \in [0, n-1)$. Now we concentrate on operators associated to continuous kernels of order $\alpha$. 

Proposition 3.1.7 Suppose that $K$ is a continuous kernel of order $\alpha$, where $\alpha \in [0, n - 1)$. The following hold:

(i) the operator $T_K$ maps bounded functions on $\partial \Omega$ into continuous functions;

(ii) if $\lambda \neq 0$, $u \in L^2(\partial \Omega)$ and $\lambda u + T_K u \in C(\partial \Omega)$, then $u \in C(\partial \Omega)$.

Proof. First we prove (i). Suppose that $f$ is a bounded function on $\partial \Omega$. We shall prove that $T_K f$ is continuous on $\partial \Omega$. Fix $X \in \partial \Omega$, and suppose that $Y \in B_\delta(X) \cap \partial \Omega$. Then

$$
|T_K f(Y) - T_K f(X)| = \left| \int_{\partial \Omega} \left[ K(Y, Z) - K(X, Z) \right] f(Z) \, d\sigma(Y) \right|
\leq \|A\|_{\infty} \|f\|_{\infty} \int_{B_\delta(X)} \left[ |Y - Z|^{-\alpha} + |X - Z|^{-\alpha} \right] \, d\sigma(Y)
+ \|f\|_{\infty} \int_{\partial \Omega \setminus B_\delta(X)} \left| K(Y, Z) - K(X, Z) \right| \, d\sigma(Z).
$$

Notice that for each $\delta > 0$

$$
\lim_{Y \to X} \int_{\partial \Omega \setminus B_\delta(X)} \left| K(Y, Z) - K(X, Z) \right| \, d\sigma(Z) = 0 \quad (3.1.17)
$$

by the dominated convergence theorem. Indeed,

$$
|K(Y, Z) - K(X, Z)| \leq |K(Y, Z)| + |K(X, Z)|
\leq 2 \|A\|_{\infty} \delta^{-\alpha} \quad \forall Y \in B_\delta(X) \quad \forall Z \in \partial \Omega \setminus B_\delta(X),
$$

and

$$
\lim_{Y \to X} \left| K(Y, Z) - K(X, Z) \right| = 0 \quad \forall Z \in \partial \Omega \setminus B_\delta(X).
$$

Now, (3.1.17) and Lemma 3.1.5 imply that

$$
|T_K f(Y) - T_K f(X)|
\leq \|A\|_{\infty} \|f\|_{\infty} C_1 \delta^{n-1-\alpha} + \|f\|_{\infty} \int_{\partial \Omega \setminus B_\delta(X)} \left| K(Y, Z) - K(X, Z) \right| \, d\sigma(Z).
$$

By taking the limit of both sides as $Y$ tends to $X$, we see that

$$
\lim_{Y \to X} \left| T_K f(Y) - T_K f(X) \right| \leq \|A\|_{\infty} \|f\|_{\infty} C_1 \delta^{n-1-\alpha} \quad \forall \delta > 0.
$$
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By taking the infimum of both sides over all $\delta > 0$, we obtain that

$$\lim_{Y \to X} |T_K f(Y) - T_K f(X)| = 0,$$

as required to conclude the proof of (i).

Next we prove (ii). For each $\varepsilon > 0$ denote by $\phi_\varepsilon$ a continuous function on $\partial \Omega \times \partial \Omega$ that vanishes on

$$\{(X, Y) \in \partial \Omega \times \partial \Omega : |X - Y| \geq \varepsilon\}$$

and is equal to 1 in

$$\{(X, Y) \in \partial \Omega \times \partial \Omega : |X - Y| \leq \varepsilon/2\}.$$

Set $K_0^\varepsilon := \phi_\varepsilon K$ and $K_1^\varepsilon := (1 - \phi_\varepsilon) K$.

Note that $K_0^\varepsilon$ is supported near the diagonal of $\partial \Omega \times \partial \Omega$, whereas $K_1^\varepsilon$ is supported off the diagonal.

We claim that $T_{K_1^\varepsilon} u$ is continuous on $\partial \Omega$. Indeed,

$$|T_{K_1^\varepsilon} u(Y) - T_{K_1^\varepsilon} u(X)| \leq \int_{\partial \Omega} [K_1^\varepsilon(Y, Z) - K_1^\varepsilon(X, Z)] |u(Z)| \, d\sigma(Y)$$

(by Schwarz) \[\leq \|u\|_2 \left[ \int_{\partial \Omega} [K_1^\varepsilon(Y, Z) - K_1^\varepsilon(X, Z)]^2 |u(Z)| \, d\sigma(Y) \right]^{1/2}

(K_1^\varepsilon is continuous) \rightarrow 0 \quad \text{as } Y \text{ tends to } X,$$

as claimed.

Now, set $g := (\lambda u + T_K u) - T_{K_1^\varepsilon} u$.

Notice that $g$ is continuous, for $\lambda u + T_K u$ is continuous by assumption, and we have just proved that $T_{K_1^\varepsilon} u$ is continuous. Recall that, by Proposition 3.1.6 (iii),

$$\|T_{K_1^\varepsilon}\|_p \leq C \varepsilon^{a-1-\alpha}.$$

for all $p \in [1, \infty)$. Thus, we may choose $\varepsilon$ so small that

$$\|T_{K_1^\varepsilon}\|_p < |\lambda|$$

(recall that $\lambda \neq 0$ by assumption). Then the operator $\lambda I + T_{K_1^\varepsilon}$ is invertible both in $L^2(\partial \Omega)$ and in $L^\infty(\partial \Omega)$. By a classical result (see Exercise 3.1.8)

$$(\lambda I + T_{K_1^\varepsilon})^{-1} = \lambda^{-1} (I + \lambda^{-1} T_{K_1^\varepsilon})^{-1} = \lambda^{-1} \sum_{j=0}^{\infty} \left( \frac{T_{K_1^\varepsilon}}{\lambda} \right)^j,$$
so that

$$u = \lambda^{-1} \sum_{j=0}^{\infty} \left( \frac{T_{K^1_j}}{\lambda} \right)^j g.$$ 

Note that, by (i), each summand of the series above is a continuous function on $\partial \Omega$. Since

$$\sum_{j=0}^{\infty} \left\| \frac{T_{K^1_j}}{\lambda^j} g \right\| \leq \sum_{j=0}^{\infty} \frac{\|T_{K^1_j}\|}{\lambda^j} < \infty,$$

(the last inequality follows from (3.1.19)) the sum of the series above is continuous on $\partial \Omega$, whence $u$ is continuous on $\partial \Omega$, as required to conclude the proof of (ii), and of the proposition.

Exercise 3.1.8 Suppose that $A$ is a bounded linear operator on the Banach space $B$, and that $\|A\| < 1$. Then $I - A$ is invertible in the Banach algebra $L(B)$, and

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j,$$ 

the series being convergent in $L(B)$. Hint: prove that the series above is convergent in $L(B)$ by using the analogue of the classical Weierstrass test for series of elements in $B$. Then set $B := \sum_{j=0}^{\infty} A^j$, and show that $(I - A) B = I = B (I - A)$.

3.2 The double layer potential

Suppose that $\Omega$ is a bounded domain with $C^2$ boundary. In this section we study some properties of the double layer potential. First note that

$$\partial_{\nu(Y)} N(x - Y) = \frac{1}{\omega_n} \frac{(Y - x) \cdot \nu(Y)}{|x - Y|^n} \quad \forall Y \in \partial \Omega \quad \forall x \neq Y. \quad (3.2.1)$$

We already know that the Newtonian potential $x \mapsto N(x - Y)$ is harmonic (with respect to the $x$ variable) in $\mathbb{R}^n \setminus \{Y\}$. Since

$$\Delta_x \left[ \partial_{\nu(Y)} N(x - Y) \right] = \Delta_x \left[ \nabla_Y N(x - Y) \cdot \nu(Y) \right] = \nabla_Y \left[ \Delta_x N(x - Y) \right] \cdot \nu(Y) = 0,$$

we see that $x \mapsto \partial_{\nu(Y)} N(x - Y)$ is harmonic in $\mathbb{R}^n \setminus \{Y\}$. 
3.2. THE DOUBLE LAYER POTENTIAL

Definition 3.2.1 The **double layer potential** with density \( g \) is the function
\[
D_g(x) = \int_{\partial \Omega} g(Y) \, \partial_{\nu} N(x - Y) \, d\sigma(Y) \quad \forall x \in \mathbb{R}^n \setminus \partial \Omega.
\]
The function \((x, Y) \mapsto \partial_{\nu} N(x - Y)\) is called the **kernel** of the double layer potential.

Lemma 3.2.2 If \( g \in C(\partial \Omega) \), then \( D_g \) is harmonic in \( \mathbb{R}^n \setminus \partial \Omega \).

*Proof.* The required property follows from the remark above by differentiating under the integral sign. The reader should check that differentiation is permitted. \( \square \)

Now, we study \( D_g(X) \) when \( X \in \partial \Omega \). The difficulty here is that the kernel of the double layer potential has a singularity on \( \partial \Omega \), so that even the convergence of the integral
\[
\int_{\partial \Omega} g(Y) \, \partial_{\nu} N(X - Y) \, d\sigma(Y)
\]
must be proved for every \( X \in \partial \Omega \). As a first attempt, we examine the order of magnitude of the kernel. By (3.2.1),
\[
|\partial_{\nu(Y)} N(X - Y)| \leq C |X - Y|^{1-n} \quad \forall X \neq Y.
\]
Unfortunately, the function \( Y \mapsto |X - Y|^{1-n} \) is nonintegrable in a neighbourhood of \( X \) on every smooth hypersurface containing \( X \).

Exercise 3.2.3 Show that if \( S \) is a smooth hypersurface in \( \mathbb{R}^n \) and \( X \in S \), then the restriction of function \( Y \mapsto |X - Y|^{1-n} \) to \( S \) is nonintegrable with respect to the surface measure of \( S \).

However, for \( Y \) close to \( X \), the vector \( X - Y \) is almost orthogonal to \( \nu(Y) \), because \( \partial \Omega \) is assumed to be of class \( C^2 \). Thus, \( Y \mapsto |\partial_{\nu(Y)} N(X - Y)| \) is, in fact, much smaller than \( Y \mapsto |X - Y|^{1-n} \), as we shall show below. This will ensure the integrability of the former.

Lemma 3.2.4 There exists a constant \( C \) such that
\[
|(X - Y) \cdot \nu(Y)| \leq C |X - Y|^2 \quad \forall X, Y \in \partial \Omega.
\]
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Proof. This is clear if \(|X - Y| \geq \delta\) for some \(\delta > 0\). Indeed, by Schwarz’s inequality

\[
| (X - Y) \cdot \nu(Y) | \leq |(X - Y)| \leq \frac{|(X - Y)|^2}{\delta}.
\]

Thus, we may assume that \(|X - Y| \leq \delta\). Suppose that \(X\) is related to \(U_j\) (see the proof of Lemma 3.1.3 for the terminology), and that \(\delta \leq \delta_0\) (\(\delta_0\) is as in (3.1.8)). For the sake of simplicity, in the rest of the proof we shall write \(\Phi\) and \(\Psi\) instead of \(\Phi_j\) and \(\Psi_j\), and \(\kappa\) instead of \(\kappa_j\) (see (3.1.5) for the definition of \(\kappa_j\)). Set \(\eta := \Phi(Y)\) and \(\xi := \Phi(X)\). Moreover, we write \(\Psi = (\psi_1, \ldots, \psi_n)\), where \(\psi_j : U \rightarrow \mathbb{R}\) are functions of class \(C^2\). By Taylor’s formula for \(\psi_j\)

\[
\psi_j(\xi) - \psi_j(\eta) = \nabla \psi_j(\eta) \cdot (\xi - \eta) + \frac{1}{2} \langle H \psi_j(\zeta_j)(\xi - \eta), (\xi - \eta) \rangle,
\]

where \(\zeta_j\) are suitable points on the segment \((\eta, \xi)\), so that

\[
\Psi(\xi) - \Psi(\eta) = \sum_{\ell=1}^{n-1} \partial_\ell \Psi(\eta) (\xi_\ell - \eta_\ell) + \frac{1}{2} \sum_{k,\ell=1}^{n-1} \partial_{k,\ell}^2 \psi_j(\zeta_j) (\xi_k - \eta_k) (\xi_\ell - \eta_\ell),
\]

Observe that the vectors \(\partial_1 \Psi(\eta), \ldots, \partial_{n-1} \Psi(\eta)\) are tangent to \(\partial \Omega\) at the point \(Y\), whence they are orthogonal to \(\nu(Y)\). Therefore

\[
\left[ \sum_{\ell=1}^{n-1} \partial_\ell \Psi(\eta) (\xi_\ell - \eta_\ell) \right] \cdot \nu(Y) = 0, \tag{3.2.2}
\]

and

\[
|(X - Y) \cdot \nu(Y)|
\]

\[
= |(\Psi(\eta) - \Psi(0)) \cdot \nu(Y)|
\]

\[
\leq \frac{1}{2} \sum_{k,\ell=1}^{n-1} \left| \partial_{k,\ell}^2 \psi_1(\zeta_1), \ldots, \partial_{k,\ell}^2 \psi_n(\zeta_n) \right|^T \cdot \nu(Y) (\xi_k - \eta_k) (\xi_\ell - \eta_\ell)
\]

\[
\leq C \max_{h=1,\ldots,n} \max_{k,\ell=1,\ldots,n-1} \sup_{\omega \in \kappa} \left| \partial_{k,\ell}^2 \psi_h(\omega) \right| |\xi - \eta|^2
\]

\[
\leq C \max_{h=1,\ldots,n} \max_{k,\ell=1,\ldots,n-1} \sup_{\omega \in \kappa} \left| \partial_{k,\ell}^2 \psi_h(\omega) \right| C_0^2 |X - Y|^2.
\]

We have used (3.1.9) in the last inequality. Since \(\partial \Omega\) is covered by a finite number of neighbourhoods \(U_j\), the required estimate follows. \(\square\)
### Notation 3.2.5

The kernel of the double layer potential is denoted by $K$. Thus,

$$K(x, Y) := \frac{1}{\omega_n} \frac{Y - x}{|Y - x|^n} \cdot \nu(Y) \quad \forall Y \in \partial \Omega \quad \forall x \in \mathbb{R}^n \setminus \{Y\}.$$  

### Proposition 3.2.6

The kernel $K$ of the double layer potential is a continuous kernel of order $n - 2$.

**Proof.** Indeed, by Lemma 3.2.4,

$$|K(X, Y)| \leq C \frac{|X - Y|^2}{|X - Y|^n} = C |X - Y|^{2-n} \quad \forall X, Y \in \partial \Omega, \ X \neq Y,$$

as required. \qed

Next, we establish further properties of the double layer potential. We need a lemma which describes a special system of coordinates in a neighbourhood of the boundary of a $C^2$ domain.

### Lemma 3.2.7

Suppose that $S$ is a compact oriented hypersurface of class $C^2$. Then there exists a neighbourhood $V$ of $S$ in $\mathbb{R}^n$ and a number $\varepsilon > 0$ such that the map $F : S \times (-\varepsilon, \varepsilon) \to V$, defined by

$$F(X, t) = X + t\nu(X),$$

is a $C^1$ diffeomorphism onto $V$.

**Proof.** We give only a sketch of the proof. The details are left to the interested reader.

Clearly $F$ is of class $C^1$. Furthermore $dF(X, 0)$ is nonsingular for every $X \in S$. By the inverse function theorem, $F$ can be inverted in a neighbourhood $W_X$ of the point $(X, 0)$, yielding a map

$$F_X^{-1} : W_X \to (S \cap W_X) \times (-\varepsilon(X), \varepsilon(X))$$

for some $\varepsilon(X) > 0$. Since $S$ is compact, we can choose points $X_1, \ldots, X_N$ in $S$ such that $\bigcup_j W_{X_j}$ covers $S$, and the maps $F_{X_j}^{-1}$ patch together to yield a $C^1$ inverse of $F$ from a neighbourhood $V$ of $S$ to $S \times (-\varepsilon, \varepsilon)$, where $\varepsilon := \min_j \varepsilon_j(X)$. \qed

The neighbourhood $V$ of the lemma above is called a **tubular neighbourhood** of $S$. It is convenient to extend the definition of normal derivative from $S$ to the tubular neighbourhood $V$ as follows

$$\partial_\nu u(X + t\nu(X)) := \nabla u(X + t\nu(X)) \cdot \nu(X) \quad \forall t \in (-\varepsilon, \varepsilon). \quad (3.2.3)$$
**Proposition 3.2.8** Suppose that \( \Omega \) is a bounded domain with \( C^2 \) boundary. The kernel \( K \) of the double layer potential possesses the following properties:

(i) \[
\int_{\partial \Omega} K(x,Y) \, d\sigma(Y) = \begin{cases} 
1 & \forall x \in \Omega \\
1/2 & \forall x \in \partial \Omega \\
0 & \forall x \in \Omega^c;
\end{cases}
\]

(ii) \[
\sup_{x \in \mathbb{R}^n \setminus \partial \Omega} \int_{\partial \Omega} |K(x,Y)| \, d\sigma(Y) < \infty;
\]

(iii) for every \( \varphi \in C(\partial \Omega) \), and every \( X \in \partial \Omega \)
\[
\lim_{x \to X, x \in \Omega} \mathcal{D}\varphi(x) = \frac{1}{2} \varphi(X) + T_K \varphi(X),
\]
and
\[
\lim_{x \to X, x \in \Omega^c} \mathcal{D}\varphi(x) = -\frac{1}{2} \varphi(X) + T_K \varphi(X).
\]

Here \( T_K \) denotes the operator on \( L^2(\partial \Omega) \) whose kernel is \( K \).

**Proof.** First, we prove (i). Recall that the Newtonian potential with pole \( x \) is harmonic in \( \mathbb{R}^n \setminus \{x\} \). Suppose that \( x \in \Omega^c \). Then \( N(x - \cdot) \) is harmonic in a neighbourhood of \( \Omega \), and
\[
\int_{\partial \Omega} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = \int_{\Omega} \Delta_Y N(x - Y) \, dV(Y) = 0,
\]
as required.

Now suppose that \( x \in \Omega \). For \( \varepsilon \) small, consider the domain
\[
\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon(x)}.
\]
Then \( N(x - \cdot) \) is harmonic in a neighbourhood of \( \Omega \), and, by a standard consequence of the divergence theorem
\[
\int_{\partial \Omega_\varepsilon} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = \int_{\Omega_\varepsilon} \Delta_Y N(x - Y) \, dV(Y) = 0.
\]
Observe that
\[
\int_{\partial \Omega_\varepsilon} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y)
= \int_{\partial \Omega} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) - \int_{\partial B_\varepsilon(x)} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y).
\]
The last integral is equal to 1 (see Exercise 1.5.3). Hence
\[ \int_{\partial \Omega} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = 1, \]
as required.

Finally suppose that \( x \in \partial \Omega \). For \( \varepsilon > 0 \), we define
\[ S_\varepsilon := \partial \Omega \setminus (\partial \Omega \cap B_\varepsilon(x)), \quad \partial B_\varepsilon(x)' := \partial B_\varepsilon(x) \cap \Omega \]
and
\[ \partial B_\varepsilon(x)'' := \{ y \in \partial B_\varepsilon(x) : \nu(x) \cdot y < 0 \}. \]
Since \( Y \mapsto \partial_{\nu(Y)} N(x - Y) \) is integrable (see Lemma 3.1.5),
\[ \int_{\partial \Omega} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = \lim_{\varepsilon \downarrow 0} \int_{S_\varepsilon} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y). \]
Observe that \( Y \mapsto \partial_{\nu(Y)} N(x - Y) \) is harmonic in a neighbourhood of \( \Omega \setminus B_\varepsilon(x) \).
Therefore
\[ 0 = \int_{\partial \Omega \setminus B_\varepsilon(x)} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = \int_{S_\varepsilon} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) + \int_{\partial B_\varepsilon(x)'} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y). \]
We then deduce that
\[ \int_{S_\varepsilon} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = - \int_{\partial B_\varepsilon(x)'} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = \frac{\varepsilon^{1-n}}{\omega_n} \sigma(\partial B_\varepsilon(x)'), \]
It is straightforward to check that the assumption \( \partial \Omega \in C^2 \) implies that
\[ |\sigma(\partial B_\varepsilon(x)') - \sigma(\partial B_\varepsilon(x)'')| = O(\varepsilon^n) \]
as \( \varepsilon \) tends to 0. Thus, we see that
\[ \int_{\partial \Omega} \partial_{\nu(Y)} N(x - Y) \, d\sigma(Y) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^{1-n}}{\omega_n} \sigma(\partial B_\varepsilon(x)'), \]
\[ = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^{1-n}}{\omega_n} \sigma(\partial B_\varepsilon(x)'') \]
\[ = \frac{1}{2}. \]
as required to conclude the proof of (i).

Next we prove (ii). We choose \( \delta \leq \delta_0 \) (\( \delta_0 \) is as in (3.1.8)) so small that \( \delta < \varepsilon \), where \( \varepsilon \) is as in the statement of Lemma 3.2.7. We distinguish between two cases.

(I) If \( d(x, \partial \Omega) \geq \delta/2 \), then

\[
|\partial_{\nu(Y)} N(x - Y)| \leq \frac{1}{\omega_{n-1}} |x - Y|^{1-n} \leq \frac{2^{n-1}}{\omega_{n-1}} \delta^{1-n},
\]

whence

\[
\int_{\partial \Omega} |\partial_{\nu(Y)} N(x - Y)| \sigma(Y) \leq \frac{2^{n-1}}{\omega_{n-1}} \delta^{1-n} \sigma(\partial \Omega).
\]

(II) Suppose that \( d(x, \partial \Omega) < \delta/2 \). By Lemma 3.2.7 there exists a unique point \( X \in \partial \Omega \) such that

\[
x = X + |x - X| \nu(X). \tag{3.2.4}
\]

We write

\[
\int_{\partial \Omega} |\partial_{\nu(Y)} N(x - Y)| \sigma(Y)
= \int_{\partial \Omega \setminus B_\delta(X)} |\partial_{\nu(Y)} N(x - Y)| \sigma(Y) + \int_{B_\delta(X)} |\partial_{\nu(Y)} N(x - Y)| \sigma(Y).
\tag{3.2.5}
\]

We denote the integrals on the right hand side by \( J_1 \) and \( J_2 \), respectively, and estimate them separately.

To estimate the first, observe that if \( Y \notin (B_\delta(X) \cap \partial \Omega) \), then

\[
|x - Y| \geq |X - Y| - |X - x| \geq \delta/2,
\]

so that

\[
|\partial_{\nu(Y)} N(x - Y)| \leq \frac{2^{n-1}}{\omega_{n-1}} \delta^{1-n},
\]

and

\[
\int_{\partial \Omega \setminus B_\delta(X)} |\partial_{\nu(Y)} N(x - Y)| \sigma(Y) \leq \frac{2^{n-1}}{\omega_{n-1}} \delta^{1-n} \sigma(\partial \Omega).
\]

Thus, to conclude the proof of (ii), it remains to estimate

\[
\int_{B_\delta(X)} |\partial_{\nu(Y)} N(x - Y)| \sigma(Y). \tag{3.2.6}
\]
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Notice that, by triangle’s inequality, Schwarz’s inequality, and Lemma 3.2.4,
\[
\omega_n |\partial_\nu(Y) N(x - Y)| \leq \frac{|(x - X) \cdot \nu(Y)|}{|x - Y|^n} + \frac{|(Y - X) \cdot \nu(Y)|}{|x - Y|^n} \\
\leq \frac{|x - X|}{|x - Y|^n} + C \frac{|Y - X|^2}{|x - Y|^n},
\] (3.2.7)
where \(C\) does not depend on \(Y\) and \(X\) in \(\partial \Omega\) and of \(x \in \mathbb{R}^n \setminus \partial \Omega\).

We claim that if \(\delta\) is small enough (see (3.2.9) below) then
\[
|x - Y|^2 \geq \frac{1}{2} \left[ |Y - X|^2 + |x - X|^2 \right].
\] (3.2.8)

Given the claim, we show how to conclude the proof of (ii). The claim and (3.2.7) imply that there exists a constant \(C\) such that
\[
\omega_n |\partial_\nu(Y) N(x - Y)| \leq C \frac{|x - X|}{|Y - X|^2 + |x - X|^2}^{n/2} + C \frac{|Y - X|^2}{|Y - X|^2 + |x - X|^2}^{n/2} \\
\leq C \frac{|x - X|}{|Y - X|^2 + |x - X|^2}^{n/2} + C \frac{|Y - X|^2}{|Y - X|^2}^{2 - n}.
\]
Therefore the integral in (3.2.6) may be estimated by
\[
C \int_{B_\delta(X) \cap \partial \Omega} \frac{|x - X|}{|Y - X|^2 + |x - X|^2}^{n/2} d\sigma(Y) + C \int_{B_\delta(X) \setminus \partial \Omega} |Y - X|^{-n} d\sigma(Y).
\]
The second of the integrals above may be estimated from above by \(C_1 \delta\), by Lemma 3.1.5. To estimate the first, suppose that \(X\) is related to \(U_j\), where \((U_j, \Phi_j)\) is one of the local charts described in Lemma 3.1.3 (see its proof for the terminology). Write \(\xi = \Phi_j(X)\) and \(\eta = \Phi_j(Y)\), and \(a := |x - X|\). Observe that
\[
\int_{B_\delta(X) \cap \partial \Omega} \frac{|x - X|}{|Y - X|^2 + |x - X|^2}^{n/2} d\sigma(Y) \\
\leq \int_{B_\delta(\Phi_j(X))} \frac{a}{|\eta - \xi|^2 + a^2}^{n/2} \sqrt{1 + |\nabla f_j(\eta)|^2} d\eta \\
\leq C_0 \omega_{n-1} \int_0^\delta \frac{a r^{n-2}}{r^2 + a^2}^{n/2} dr,
\]
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where \( C_0 \) is defined in (3.1.6). The change of variables \( v = r/a \) transforms the last integral into

\[
\int_0^{\delta/a} \frac{v^{n-2}}{(1 + v^2)^{n/2}} \, dv,
\]

which is dominated by the convergent integral

\[
\int_0^\infty \frac{v^{n-2}}{(1 + v^2)^{n/2}} \, dv,
\]

which does not depend on \( a \).

Therefore the integral in (3.2.6) is finite, as required.

It remains to prove the claim (3.2.8). Observe that the vector \( (\nabla f_j(\eta), -1) \) is orthogonal to \( \partial \Omega \) at the point \( \Psi_j(\eta) \). Write

\[
x - Y = X - Y + |x - X| \nu(X)
\]

\[
= (\xi - \eta, f_j(\xi) - f_j(\eta)) + \frac{(\nabla f_j(\eta), -1)}{\sqrt{1 + |\nabla f_j(\xi)|^2}} |x - X|
\]

Then

\[
|x - Y|^2 = |(\xi - \eta, f_j(\xi) - f_j(\eta))|^2
\]

\[
+ \frac{2}{\sqrt{1 + |\nabla f_j(\xi)|^2}} |x - X| \left( \nabla f_j(\eta)(\xi - \eta) - f_j(\xi) + f_j(\eta) \right)
\]

\[
+ |x - X|^2
\]

Now,

\[
| (\xi - \eta, f_j(\xi) - f_j(\eta)) |^2 \geq |\xi - \eta|^2,
\]

and, by Taylor’s formula,

\[
| (\nabla f_j(\eta)(\xi - \eta) - f_j(\xi) + f_j(\eta)) | \leq C |\xi - \eta|^2.
\]

Therefore

\[
\frac{2}{\sqrt{1 + |\nabla f_j(\xi)|^2}} |x - X| \left( |(\nabla f_j(\eta)(\xi - \eta) - f_j(\xi) + f_j(\eta))| \right)
\]

\[
\leq |x - X| \sup_{\omega \in \kappa_j} \| Hf_j(\omega) \| |\xi - \eta|^2.
\]

We may choose \( \delta \) so that

\[
\frac{\delta}{2} \max_{j=1,\ldots,N} \sup_{\omega \in \kappa_j} \| Hf_j(\omega) \| \leq \frac{1}{2}.
\] (3.2.9)
Recall that we are assuming that $|x - X| < \delta/2$. Therefore
\[|x - Y|^2 \geq \frac{1}{2} |\xi - \eta|^2 + |x - X|^2 \geq \frac{1}{2} [\xi - \eta]^2 + |x - X|^2,\]
and the claim is proved. This concludes the proof of (ii).

Finally, we prove (iii). We shall prove the first formula. The proof of the second is similar and is omitted. Fix $X \in \partial \Omega$. Since
\[\int_{\partial \Omega} K(x, Y) \, d\sigma(Y) = 1 \quad \forall x \in \Omega,
\]
we have, for each $x \in \Omega$,
\[
D \varphi(x) = \int_{\partial \Omega} K(x, Y) \left[\varphi(Y) - \varphi(X)\right] \, d\sigma(Y) + \varphi(X)
= \int_{\partial \Omega} \left[K(x, Y) - K(X, Y)\right] \left[\varphi(Y) - \varphi(X)\right] \, d\sigma(Y)
+ \int_{\partial \Omega} K(X, Y) \left[\varphi(Y) - \varphi(X)\right] \, d\sigma(Y) + \varphi(X)
= \int_{\partial \Omega} \left[K(x, Y) - K(X, Y)\right] \left[\varphi(Y) - \varphi(X)\right] \, d\sigma(Y)
+ \frac{1}{2} \varphi(X) + T_K \varphi(X).
\]
Thus, to conclude the proof of (iii), it remains to show that
\[
\lim_{x \to X, x \in \Omega} \int_{\partial \Omega} \left[K(x, Y) - K(X, Y)\right] \left[\varphi(Y) - \varphi(X)\right] \, d\sigma(Y) = 0. \quad (3.2.10)
\]
We denote by $\mathcal{J}$ the last integral. Since, by assumption, $\varphi$ is continuous at $X$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that
\[|\varphi(Y) - \varphi(X)| < \varepsilon \quad \forall Y \in B_{2\delta}(X) \cap \partial \Omega.
\]
Then
\[
|\mathcal{J}| \leq \varepsilon \int_{B_{2\delta}(X) \cap \partial \Omega} |K(x, Y)| \, d\sigma(Y) + \varepsilon \int_{B_{2\delta}(X) \cap \partial \Omega} |K(X, Y)| \, d\sigma(Y)
+ 2 \|\varphi\|_\infty \int_{\partial \Omega \setminus B_{2\delta}(X)} \left|K(x, Y) - K(X, Y)\right| \, d\sigma(Y).
\]
By the Lebesgue dominated convergence theorem
\[
\lim_{x \to X, x \in \Omega} \int_{\partial \Omega \setminus B_{2\delta}(X)} \left|K(x, Y) - K(X, Y)\right| \, d\sigma(Y) = 0.
\]
Therefore
\[
\lim_{x \to X, x \in \Omega} |J| \leq 2\varepsilon \sup_{x \in \mathbb{R}^n} \int_{\partial \Omega} |K(x, Y)| \, d\sigma(Y) = C \varepsilon.
\]
We have used (ii) in the last equality. By taking the infimum of both sides with respect to \(\varepsilon > 0\), we obtain that
\[
\lim_{x \to X, x \in \Omega} |J| = 0,
\]
as required to conclude the proof of (iii), and of the proposition. \(\square\)

### 3.3 The single layer potential

Suppose that \(\Omega\) is a bounded domain with \(C^2\) boundary. In this section we study some properties of the single layer potential.

**Definition 3.3.1** The **single layer potential** with density \(g\) is the function
\[
Sg(x) := \int_{\partial \Omega} g(Y) N(x - Y) \, d\sigma(Y) \quad \forall x \in \mathbb{R}^n \setminus \partial \Omega.
\]
The function \((x, Y) \mapsto N(x - Y)\) is called the **kernel** of the single layer potential.

**Lemma 3.3.2** If \(g \in C(\partial \Omega)\), then \(Sg\) is harmonic in \(\mathbb{R}^n \setminus \partial \Omega\).

**Proof.** The required properties follow from the remark above by differentiating under the integral sign. The reader should check that differentiation is permitted. \(\square\)

Recall that \(\partial \Omega\) is assumed to be of class \(C^2\). Hence there is a tubular neighbourhood \(V\) of \(\partial \Omega\), where the operator \(\partial_\nu\) is well defined (see (3.2.3)). In particular, given \(\varphi \in C(\partial \Omega)\), the single layer potential \(S\varphi\), generated by \(\varphi\), is harmonic in \(V \setminus \partial \Omega\) by Lemma 3.3.2. Thus, we may compute \(\partial_\nu(S\varphi)(x)\) for every \(x \in V \setminus \partial \Omega\) (see (3.2.3) for the definition of \(\partial_\nu\)), and obtain
\[
\partial_\nu(S\varphi)(x) = \frac{1}{\omega_n} \int_{\partial \Omega} \frac{x - Y}{|x - Y|^n} \cdot \nu(X) \varphi(Y) \, d\sigma(Y) \quad \forall x \in V \setminus \partial \Omega. \quad (3.3.1)
\]
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**Definition 3.3.3** Define $K^* : \partial \Omega \times \partial \Omega \setminus \{(X, X) : X \in \partial \Omega\} \to \mathbb{R}$ by

$$K^*(X, Y) := \frac{1}{\omega_n} \frac{X - Y}{|X - Y|^n} \cdot \nu(X).$$

Notice that if $X$ and $Y$ are in $\partial \Omega$ with $X \neq Y$, then $K^*(X, Y) = K(Y, X)$. (3.3.2)

**Remark 3.3.4** Clearly, $K^*$ is a continuous kernel of type $n-2$, because $K$ is. Observe that the operator $T_{K^*}$ is the adjoint of $T_K$, when acting on $L^2(\partial \Omega)$. Indeed, suppose that $\varphi$ and $\psi$ are in $L^2(\partial \Omega)$. Then

$$(T_K \varphi, \psi) = \int_{\partial \Omega} T_K \varphi(X) \overline{\psi(X)} \, d\sigma(X)$$

$$= \int_{\partial \Omega} \nu(X) \overline{\psi(X)} \int_{\partial \Omega} K(X, Y) \varphi(Y) \, d\sigma(Y).$$

Since $K$ is a kernel of type $n-2$, the operator $T_K$ is bounded on $L^2(\partial \Omega)$ by Proposition 3.1.6 (i). Hence

$$\int_{\partial \Omega} \int_{\partial \Omega} |\varphi(Y)| |K(X, Y)| |\psi(X)| \, d\sigma(Y) \, d\sigma(X) \leq \|T_K\|_{L^2(\partial \Omega)} \|\varphi\|_2 \|\psi\|_2.$$

Thus, we may apply Fubini’s theorem, and conclude that

$$(T_K \varphi, \psi) = \int_{\partial \Omega} \nu(X) \varphi(Y) \int_{\partial \Omega} K(X, Y) \overline{\psi(X)} \, d\sigma(X)$$

$$(K \text{ is real}) = \int_{\partial \Omega} \nu(Y) \varphi(Y) \int_{\partial \Omega} K^*(Y, X) \overline{\psi(X)} \, d\sigma(X)$$

$$= (\varphi, T_{K^*} \psi),$$

as required.

For each $X \in \partial \Omega$ we define

$$\partial_{\nu^-}(S \varphi)(X) := \lim_{t \to 0^-} \partial_{\nu}(S \varphi)(X + t\nu(X))$$

and

$$\partial_{\nu^+}(S \varphi)(X) := \lim_{t \to 0^+} \partial_{\nu}(S \varphi)(X + t\nu(X)),$$

whenever the limits above exist.

Some of the main properties of the single layer potential are summarised in the next proposition.
**Proposition 3.3.5** Suppose that $\partial \Omega$ is of class $C^2$, and that $\varphi \in C(\partial \Omega)$. The following hold:

(i) $S\varphi$ is continuous in $\mathbb{R}^n$;

(ii) $\partial_{\nu^-}(S\varphi)(X)$ and $\partial_{\nu^+}(S\varphi)(X)$ exist for each $X \in \partial \Omega$, and

\[
\partial_{\nu^-}(S\varphi)(X) = -\frac{1}{2} \varphi(X) + T_{K^*} \varphi(X) \tag{3.3.5}
\]

and

\[
\partial_{\nu^+}(S\varphi)(X) = \frac{1}{2} \varphi(X) + T_{K^*} \varphi(X). \tag{3.3.6}
\]

Furthermore, the convergence of $\partial_{\nu}(S\varphi)(X + t\nu(X))$ to $\partial_{\nu^-}(S\varphi)(X)$ as $t$ tends to $0^-$ and of $\partial_{\nu}(S\varphi)(X + t\nu(X))$ to $\partial_{\nu^+}(S\varphi)(X)$ as $t$ tends to $0^+$ is uniform on $\partial \Omega$.

**Proof.** First we prove (i). Clearly $S\varphi$ is continuous on $\mathbb{R}^n \setminus \partial \Omega$. It remains to prove that $S\varphi$ is continuous at each $X \in \partial \Omega$. Suppose that $\delta > 0$ is small, and that $d(x,X) \leq \delta/10$. Then $d(x,\partial \Omega) \leq \delta/10$. Write

\[
\begin{align*}
|u(x) - u(X)| &\leq \frac{1}{(n-2)\omega_n} \int_{B_\delta(x) \setminus \partial \Omega} |x - Y|^{2-n} |\varphi(Y)| \, d\sigma(Y) \\
&\quad + \frac{1}{(n-2)\omega_n} \int_{B_\delta(X) \setminus \partial \Omega} |X - Y|^{2-n} |\varphi(Y)| \, d\sigma(Y) \\
&\quad + \frac{1}{(n-2)\omega_n} \int_{\delta \setminus B_\delta(X)} |X - Y|^{2-n} - |X - Y|^{2-n} |\varphi(Y)| \, d\sigma(Y).
\end{align*}
\]

Denote by $J_1$, $J_2$ and $J_3$ the integrals above.

By Lemma 3.1.5

\[
|J_2| \leq C_1 \|f\|_\infty \delta, \tag{3.3.7}
\]

with $C$ independent of $x$ and $X$. Furthermore

\[
\lim_{x \to X, x \in \Omega} J_3 = 0, \tag{3.3.8}
\]

by the Lebesgue dominated convergence theorem.

To estimate $J_1$ we argue as follows. Since $\delta$ is small, there exists a unique point $\zeta \in \partial \Omega$ such that

\[
x = \zeta - |x - \zeta| \nu(\zeta).
\]
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Then

\[ x - Y = \zeta - Y - |x - \zeta| \nu(\zeta). \]

We distinguish between two cases.

(I) If \( Y \in (B_{\delta}(X) \cap \partial \Omega) \setminus B_{\delta/5}(\zeta) \), then

\[ |Y - x| \geq |Y - \zeta| - |\zeta - x| \geq \delta/5 - \delta/10 = \delta/10. \]

Thus,

\[
\int_{B_{\delta}(X) \cap \partial \Omega} |x - Y|^{2-n} |\varphi(Y)| \, d\sigma(Y) \leq C \|\varphi\|_{\infty} \delta^{2-n} \sigma(B_{\delta}(X) \cap \partial \Omega)
\leq C \|\varphi\|_{\infty} \delta^{2-n} \delta^{-1}
\leq C \|\varphi\|_{\infty} \delta.
\]

(II) If \( Y \in B_{\delta/5}(\zeta) \cap \partial \Omega \) and \( \delta \) is small enough, then, by (3.2.8) with \( \zeta \) in place of \( X \), there exists a constant \( c > 0 \) such that

\[ |x - Y| \geq c |\zeta - Y|. \]

Therefore

\[
\int_{B_{\delta/5}(\zeta) \cap \partial \Omega} |x - Y|^{2-n} |\varphi(Y)| \, d\sigma(Y) \leq C \|\varphi\|_{\infty} \int_{B_{\delta/5}(\zeta) \cap \partial \Omega} |\zeta - Y|^{2-n} \, d\sigma(Y)
\leq C \|\varphi\|_{\infty} \delta.
\]

Thus,

\[ |J_3| \leq C \|f\|_{\infty} \delta. \quad (3.3.9) \]

By combining (3.3.9), (3.3.7), and (3.3.8), we see that for all \( \delta > 0 \) small enough

\[ \lim_{x \to X, x \in \Omega} |u(x) - u(X)| \leq C \|f\|_{\infty} \delta. \]

Thus, \( \lim_{x \to X, x \in \Omega} |u(x) - u(X)| = 0 \), as required to conclude the proof of (i).

Next we prove the first formula in (ii). The proof of the second formula is similar and is omitted. Set

\[ \mathcal{E}_t(X) := \partial_{\nu}(S \varphi)(X + t\nu(X)) + \frac{1}{2} \varphi(X) - T_{K^*} \varphi(X). \]

We must show that

\[ \lim_{t \to 0^-} \sup_{X \in \partial \Omega} |\mathcal{E}_t(X)| = 0. \quad (3.3.10) \]
By Proposition 3.2.8 (i) and (ii),

\[
\frac{1}{2} \varphi(X) = \frac{1}{\omega_n} \int_{\partial\Omega} \left[ \frac{X - Y}{|X - Y|^n} - \frac{X + t\nu(X) - Y}{|X + t\nu(X) - Y|^n} \right] \cdot \nu(Y) \varphi(X) \, d\sigma(Y)
\]

\[
= \frac{1}{\omega_n} \int_{\partial\Omega} \left[ \frac{X - Y}{|X - Y|^n} - \frac{X + t\nu(X) - Y}{|X + t\nu(X) - Y|^n} \right] \cdot \nu(Y) \left[ \varphi(X) - \varphi(Y) \right] \, d\sigma(Y)
\]

\[
+ \frac{1}{\omega_n} \int_{\partial\Omega} \left[ \frac{X - Y}{|X - Y|^n} - \frac{X + t\nu(X) - Y}{|X + t\nu(X) - Y|^n} \right] \cdot \nu(Y) \varphi(Y) \, d\sigma(Y)
\]

Furthermore

\[
T_K \varphi(X) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{X - Y}{|X - Y|^n} \cdot \nu(X) \varphi(Y) \, d\sigma(Y)
\]  

(3.3.11)

and

\[
\partial_n (S\varphi)(X + t\nu(X)) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{X + t\nu(X) - Y}{|X + t\nu(X) - Y|^n} \cdot \nu(X) \varphi(Y) \, d\sigma(Y)
\]  

(3.3.12)

Therefore

\[
E_t(X)
\]

\[
= \frac{1}{\omega_n} \int_{\partial\Omega} \left[ \frac{X - Y}{|X - Y|^n} - \frac{X + t\nu(X) - Y}{|X + t\nu(X) - Y|^n} \right] \cdot \nu(Y) \left[ \varphi(X) - \varphi(Y) \right] \, d\sigma(Y)
\]

\[
- \frac{1}{\omega_n} \int_{\partial\Omega} \left[ \frac{X - Y}{|X - Y|^n} - \frac{X + t\nu(X) - Y}{|X + t\nu(X) - Y|^n} \right] \cdot [\nu(X) - \nu(Y)] \varphi(Y) \, d\sigma(Y)
\]

(3.3.13)

We denote the two integrals above by \( I^1_t(X) \) and \( I^2_t(X) \), respectively.

Fix \( \varepsilon > 0 \), and choose \( \delta > 0 \) such that

\[
|\nu(X) - \nu(Y)| < \varepsilon \quad \text{and} \quad |\varphi(X) - \varphi(Y)| < \varepsilon \]  

(3.3.15)

for all \( X \) and \( Y \) in \( \partial\Omega \) such that \( |X - Y| < 2\delta \). This is possible for both \( \nu \) and \( \varphi \) are continuous, hence uniformly continuous, functions on the compact set \( \partial\Omega \).

We estimate \( I^1_t(X) \). We write

\[
I^1_t(X) = \int_{B_{2\delta}(X) \cap \partial\Omega} + \int_{\partial\Omega \setminus B_{2\delta}(X)} .
\]
Observe that
\[
\left| \int_{B_2(X) \cap \partial \Omega} \right| \leq \varepsilon \omega_n \int_{\partial \Omega} \left[ |K(X, Y)| + |K(X + t \nu(X), Y)| \right] d\sigma(Y)
\]
\[
\leq 2 \varepsilon \omega_n \sup_{x \in \mathbb{R}^n} \int_{\partial \Omega} |K(x, Y)| d\sigma(Y),
\]
where \( K(x, Y) \) is the kernel of the double layer potential. Recall that, by Proposition 3.2.8 (ii), the supremum above is finite.

Next, we apply the mean value theorem, and obtain that
\[
\sup_{Y \in \partial \Omega \setminus B_2(X)} \left| \frac{X - Y}{|X - Y|^n} - \frac{X + t \nu(X) - Y}{|X + t \nu(X) - Y|^n} \right| \leq C |t| \delta^{-n} \quad \forall t \in (-\delta, 0),
\]
where \( C \) is a constant, which depends only on the dimension \( n \). Thus,
\[
\left| \int_{\partial \Omega \setminus B_2(X)} \right| \leq 2 C |t| \delta^{-n} \| \varphi \|_{\infty} \sigma(\partial \Omega).
\]

By combining (3.3.16) and (3.3.17), we obtain that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\sup_{X \in \partial \Omega} |T_1^t(X)| \leq C (\varepsilon + |t| \delta^{-n}),
\]
whence
\[
\lim_{t \to 0^-} \sup_{X \in \partial \Omega} |T_1^t(X)| \leq C \varepsilon.
\]

Therefore
\[
\lim_{t \to 0^-} \sup_{X \in \partial \Omega} |T_1^t(X)| = 0.
\]

A similar reasoning shows that
\[
\lim_{t \to 0^-} \sup_{X \in \partial \Omega} |T_2^t(X)| = 0.
\]

The last two formulae imply (3.3.10), as required to conclude the proof of (ii), and of the proposition.

\[\square\]

3.4 Solvability of the Dirichlet problem

The purpose of this section is to prove Theorem 3.4.2 below. We shall need the following lemma.
Lemma 3.4.1 The operator $(1/2) I + T_{K^*}$ is injective.

Proof. Observe preliminarily that, since $\partial \Omega$ is a compact oriented hypersurface of class $C^2$, we may apply Lemma 3.2.7, which ensures the existence of a neighbourhood $V$ of $\partial \Omega$ and a $C^1$ diffeomorphism $F$ of $\partial \Omega \times (-\varepsilon, \varepsilon)$ onto $V$, defined by

$$F(X, t) = X + t\nu(X) \quad \forall X \in \partial \Omega, \forall t \in (-\varepsilon, \varepsilon),$$

where $\nu(X)$ denotes the exterior unit normal to $\Omega$. It is straightforward to check that for each $t \in (-\varepsilon, \varepsilon)$ the set of all points of the form $X + t\nu(X) \quad \forall X \in \partial \Omega$ is a compact hypersurface, denoted by $S_t$, which is contained in $\Omega^c$ for all $t \in (0, \varepsilon)$. Denote by $\Omega_t$ the domain

$$\{ \zeta \in \mathbb{R}^n : \zeta = X + s\nu(X), \text{ with } X \in \partial \Omega \text{ and } s \in [0, t] \}.$$ 

The boundary of $\Omega_t$ is $S_t$.

Since $\Omega_t$ is bounded, $\Omega_t$ is contained in $B_R(0)$ for $R$ large enough. For such $R$, consider the bounded domain

$$\Omega_{t,R} := B_R(0) \setminus \overline{S_t}.$$ 

Suppose that $f$ belongs to Ker$((1/2) I + T_{K^*})$. We must show that $f = 0$. Observe that $f$ is continuous on $\partial \Omega$, by Proposition 3.1.7(ii). We form the single layer potential $Sf$ of $f$. Since $Sf$ is harmonic on $\mathbb{R}^n \setminus \partial \Omega$,

$$I := \int_{\mathbb{R}^n \setminus \partial \Omega} |\nabla (Sf)(x)|^2 dV(x)$$

$$= \lim_{t \downarrow 0, R \uparrow \infty} \int_{\Omega_{t,R}} |\nabla (Sf)(x)|^2 dV(x)$$

by the Lebesgue monotone convergence theorem.

Now, $Sf$ is harmonic in a neighbourhood of $\Omega_{t,R}$, hence we may apply the first Green’s identity, and obtain that

$$I = \lim_{t \downarrow 0, R \uparrow \infty} \int_{\partial \Omega_{t,R}} Sf(X) \partial\nu(X)(Sf)(X) d\sigma(X) \quad (3.4.1)$$

The last integral is the sum of

$$J_t := -\int_{\partial S_t} Sf(X) \partial\nu(X)(Sf)(X) d\sigma(X)$$
and
\[ H_R := \int_{\partial B_R(0)} \mathcal{S} f(X) \partial_{\nu(X)}(\mathcal{S} f)(X) \, d\sigma(X). \]

We \textit{claim} that \( \lim_{R \to \infty} H_R = 0 \). Indeed, for all \( x \) such that \( d(x, \partial \Omega) \geq |x|/2 \)
\[ |\mathcal{S} f(x)| \leq \frac{\|f\|_\infty}{(n-2) \omega_n} \int_{\partial \Omega} |x - Y|^{2-n} \, d\sigma(Y) \]
\[ \leq \frac{\|f\|_\infty}{(n-2) \omega_n} d(x, \partial \Omega)^{2-n} \sigma(\partial \Omega) \]
\[ \leq 2^{n-2} \frac{\|f\|_\infty}{(n-2) \omega_n} |x|^{2-n} \sigma(\partial \Omega). \]

Similarly, for all \( x \) such that \( d(x, \partial \Omega) \geq |x|/2 \)
\[ |\partial_{\nu(X)}\mathcal{S} f(x)| \leq \frac{\|f\|_\infty}{\omega_n} \int_{\partial \Omega} |x - Y|^{1-n} \, d\sigma(Y) \]
\[ \leq 2^{n-2} \frac{\|f\|_\infty}{\omega_n} |x|^{1-n} \sigma(\partial \Omega). \]

Thus,
\[ |H_R| \leq 2^{2(n-2)} \frac{\|f\|^2_\infty}{(n-2) \omega_n^2} R^{3-2n} \sigma(\partial \Omega)^2 \omega_n R^{n-1} \]
\[ = 2^{2(n-2)} \frac{\|f\|^2_\infty}{(n-2) \omega_n^2} R^{2-n} \sigma(\partial \Omega)^2, \]

which tends to 0 as \( R \) tends to infinity (we are assuming \( n \geq 3 \)), as claimed.

We \textit{claim} that \( \lim_{t \to 0} J_t = 0 \). This follows from Proposition 3.3.5 (i)-(ii). We leave the verification of this fact to the reader.

The two claims above and (3.4.1) imply that \( I = 0 \). Therefore \( |\nabla(\mathcal{S} f)| = 0 \) a.e. on \( \mathbb{R}^n \setminus \overline{\Omega} \), hence \( |\nabla(\mathcal{S} f)| = 0 \) everywhere on \( \mathbb{R}^n \setminus \overline{\Omega} \), for \( |\nabla(\mathcal{S} f)| \) is continuous on \( \mathbb{R}^n \setminus \overline{\Omega} \). Since \( \mathbb{R}^n \setminus \overline{\Omega} \) is connected, \( \mathcal{S} f \) is constant on \( \mathbb{R}^n \setminus \overline{\Omega} \). We have already proved that \( \lim_{|x| \to \infty} \mathcal{S} f(x) = 0 \), whence \( \mathcal{S} f = 0 \) on \( \mathbb{R}^n \setminus \overline{\Omega} \). By (i), \( \mathcal{S} f \) is continuous on \( \mathbb{R}^n \), so that \( \mathcal{S} f = 0 \) on \( \partial \Omega \). By the maximum principle for \( \mathcal{S} f \) on \( \Omega \), it follows that \( \mathcal{S} f = 0 \) on \( \mathbb{R}^n \). Thus,
\[ \partial_{\nu^{-}}(\mathcal{S} f) = 0 \quad \text{on } \partial \Omega. \]

By (ii),
\[ \partial_{\nu^{-}}(\mathcal{S} f)(X) = -\frac{1}{2} f(X) + T_{K^*} f(X) \quad \forall X \in \partial \Omega. \]

Therefore
\[ f(X) = \frac{1}{2} f(X) + T_{K^*} f(X) - \left[ -\frac{1}{2} f(X) + T_{K^*} f(X) \right] = 0, \]
thereby proving the injectivity of $\frac{1}{2} I + T_{K^*}$, and concluding the proof of the lemma. \hfill \Box

Recall that $K$ denotes the kernel of the double layer potential.

**Theorem 3.4.2** Suppose that $\Omega$ is a bounded domain with $\partial \Omega$ of class $C^2$. For every $g \in C(\partial \Omega)$ there exists a unique function $u \in C(\overline{\Omega})$ which solves the Dirichlet problem
\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Omega \\
u|_{\partial \Omega} &= g.
\end{align*}
\]

Furthermore, the solution $u$ is given by
\[
u(x) = \int_{\partial \Omega} K(x,Y) \varphi(Y) \, d\sigma(Y) \quad \forall x \in \Omega,
\]
where $\varphi$ is the unique solution to the integral equation
\[
\frac{1}{2} \varphi(X) + \int_{\partial \Omega} K(X,Y) \varphi(Y) \, d\sigma(y) = g(X) \quad \forall X \in \partial \Omega.
\]

**Proof.** Recall that the operator $T_K$ is compact, because $K$ is a continuous kernel of order $n-2$ (see Proposition 3.2.6 and Proposition 3.1.6).

By Lemma 3.4.1 the operator $(1/2) I + T_K$ is injective (hence bijective by Theorem 2.6.4 (iv)). Therefore the equation $(1/2) \varphi + T_K \varphi = g$ has exactly one solution for every $g \in L^2(\partial \Omega)$. Since $g$ is continuous by assumption, the solution $\varphi$ is also continuous, by Proposition 3.1.7 (ii). By Lemma 3.2.2 the double layer potential $D\varphi$ generated by $\varphi$ is harmonic in $\Omega$ and it is continuous on $\overline{\Omega}$ by Proposition 3.2.8 (iii). \hfill \Box
Part III

The Dirichlet problem via $L^2$ methods
4.0.1 Introduction to Dirichlet’s principle

We have already seen how to prove the existence of a classical solution to the Dirichlet problem
\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = g
\end{cases}
\] (4.0.1)
via Perron’s method and via integral equations on the boundary \( \partial \Omega \). Here we study a different approach to the problem, based on the so-called Dirichlet principle. The latter is based on an idea of wide utility in mathematics and physics: to find the equilibrium state of a system one seeks to minimize an appropriate energy or action.

In the present case the role of this energy is played by the Dirichlet integral. Given a continuous function \( g \) on the boundary \( \partial \Omega \) of \( \Omega \), denote by \( \mathcal{Y}_g \) the affine space of all functions \( \varphi \in C^2(\Omega) \) with \( \varphi|_{\partial \Omega} = g \). Clearly \( \mathcal{Y}_0 \) is a vector space. The Dirichlet principle states that the solution to the Dirichlet problem above agrees with the solution to the following minimization problem
\[
\min_{u \in \mathcal{Y}_g} D(u),
\] (4.0.2)
where \( D(u) \) denotes the Dirichlet integral, defined by
\[
D(u) := \int_\Omega |\nabla u|^2 \, dV;
\] (4.0.3)
as usual, \( dV \) denotes the volume element in \( \mathbb{R}^n \).

There are two basic questions concerning Dirichlet’s principle:

(i) does a solution to the minimization problem (4.0.2) solve the Dirichlet problem (4.0.1)?

(ii) does the solution to the Dirichlet problem (4.0.1) solve the minimization problem (4.0.2)?

First we discuss the first question. The application of Dirichlet’s principle was thought to have been justified by the following proposition, whose proof we postpone to the next section.

**Proposition 4.0.1 (Dirichlet’s principle I)** Suppose that \( \Omega \) is a bounded domain with Lipschitz boundary and that \( g \in C(\partial \Omega) \), that \( \mathcal{Y}_g \) is nonempty and that \( u \in \mathcal{Y}_g \). The following are equivalent:
(i) $\Delta u = 0$ in $\Omega$ (in the classical sense);

(ii) $u$ is a critical point of the functional $D$ in the sense that

$$\frac{d}{d\varepsilon} D(u + \varepsilon \varphi)|_{\varepsilon=0} = 0 \quad \forall \varphi \in \mathcal{Y}_0;$$

(iii) $u$ minimizes $D$ in the sense that

$$D(u) \leq D(w) \quad \forall w \in \mathcal{Y}_g.$$

In fact, the proposition above does not establish the existence of a solution to the Dirichlet problem (4.0.1). Rather, it converts (4.0.1) into the problem of minimizing the Dirichlet integral, under the assumption that $g$ is the “trace” on the boundary of some functions in $C^2(\Omega)$.

In order to avoid interruptions in the flow of information, we postpone all the proofs of the results stated in this introduction to sections below (see Subsection 4.0.3 for details).

Weierstrass pointed out that it was not clear (and had not been proved) that a minimizing function for the Dirichlet integral exists, so there might simply be no winner to the implied competition in Proposition 4.0.1. He argued by analogy with a simpler one-dimensional problem: that of minimizing the integral

$$I(\varphi) := \int_{-1}^{1} |x\varphi'(x)|^2 \, dx$$

amongst all $C^1$ functions $\varphi$ on $[-1, 1]$ that satisfy $\varphi(-1) = -1$ and $\varphi(1) = 1$.

**Exercise 4.0.2** Prove that the infimum of $I$ is equal to 0. Prove that the infimum is not a minimum.

This suggests the possibility that the functional $D$ may reach its infimum in a set of competitors larger than $\mathcal{Y}_g$, but possibly not in $\mathcal{Y}_g$, and demands a proof that a minimizer, in fact, exists. After all, the problem

$$\min \{x^2 : x \in \mathbb{Q}, x^2 \geq 2\}$$

has no solutions, but it has indeed solutions in $\mathbb{R}$, the completion of $\mathbb{Q}$ with respect to the Euclidean distance.

To develop this line of reasoning further, let us provisionally assume that $\Omega$ is a bounded domain with Lipschitz boundary. This is a technical assumption, which guarantees that the Sobolev space $H^1(\Omega)$ may be
defined as the completion of $C^2(\Omega)$ with respect to a suitable norm. This is no longer true for more general domains. Observe that

$$\|\varphi\|_{H^1} := \left( \int_{\Omega} |\varphi|^2 \, dV + \int_{\Omega} |\nabla \varphi|^2 \, dV \right)^{1/2} \quad (4.0.4)$$

is a norm on $C^2(\Omega)$, where the latter is the space of all functions $\varphi$ in $C^2(\Omega)$ such that $D^\alpha \varphi$ admits a continuous extension to $\overline{\Omega}$ for each multiindex $\alpha$ with $|\alpha| \leq 2$.

**Definition 4.0.3** Assume that $\Omega$ is a bounded domain and that $\partial \Omega$ is Lipschitz. Define $H^1(\Omega)$ to be the completion of $C^2(\Omega)$ with respect to the norm (4.0.4) above.

It is tempting to enlarge the set of competitors in the minimizing problem (4.0.2) to the affine space of all $u \in H^1(\Omega)$ such that $u|_{\partial \Omega} = g$, but there is a problem here: what is the precise meaning of the relation $u|_{\partial \Omega} = g$? In particular, is it true that $C(\Omega)$ is included in $H^1(\Omega)$, so that the relation above may be intended in the obvious sense? The answer to the second question is positive if $n = 1$, and negative if $n \geq 2$ (see Exercise 5.5.15 for the case $n = 2$). The following result gives a partial answer to the first question.

**Proposition 4.0.4** Suppose that $\Omega$ is a bounded domain with Lipschitz boundary. The restriction map $u \mapsto u|_{\partial \Omega}$ from $C^2(\Omega)$ to $C(\partial \Omega)$ extends to a bounded operator $\gamma$ from $H^1(\Omega)$ to $L^2(\partial \Omega)$.

The operator $\gamma$ is called the trace operator. It is important to observe that $\gamma$ is not surjective, as a celebrated counterexample of Weierstrass shows (see Proposition 4.0.9 below). In fact, there are many continuous functions on $\partial \Omega$ that do not belong to $\gamma(H^1(\Omega))$. Loosely speaking, a function on $\partial \Omega$ is in $\gamma(H^1(\Omega))$ if and only if it possesses "half derivative". The space of all the traces on $\partial \Omega$ of functions in $H^1(\Omega)$ is often denoted by $H^{1/2}(\Omega)$. It is beyond the scope of these notes to develop the theory of such space. We refer the interested reader to [AF].

Now we may reformulate the minimization problem (4.0.2) as follows. **Given a function $g$ in $\gamma(H^1(\Omega))$, find $w \in H^1(\Omega)$ such that**

$$D(w) = \min \{ D(u) : u \in H^1(\Omega), \gamma(u) = g \} \quad (4.0.5)$$

For technical reasons, it may be more convenient to give a slightly different reformulation of the Dirichlet principle. To proceed further, we need to introduce another function space, which will play a key role in the sequel.
**Definition 4.0.5** We define $H^1_0(\Omega)$ to be the completion of $C^\infty_c(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$.

Since $C^\infty_c(\Omega)$ is a subspace of $C^2(\overline{\Omega})$, the space $H^1_0(\Omega)$ is a (closed) subspace of $H^1(\Omega)$. Since the restriction of the trace operator $\gamma$ to $C^\infty_c(\Omega)$ vanishes, the restriction of $\gamma$ to $H^1_0(\Omega)$ vanishes as well. Thus, we may say that elements of $H^1_0(\Omega)$ have “vanishing boundary values”. The converse statement is also true. For the proof, see [AF, Thm 5.37].

**Proposition 4.0.6** Suppose that $\Omega$ is a bounded domain with Lipschitz boundary. If $u \in H^1(\Omega)$ and $\gamma(u) = 0$, then $u \in H^1_0(\Omega)$.

Now we give another formulation of the minimization problem (4.0.5). Given a function $g$ in $\gamma(H^1(\Omega))$, denote by $G$ any function in $H^1(\Omega)$ such that $\gamma(G) = g$. The problem consists in finding $w \in G + H^1_0(\Omega)$ such that

$$D(w) = \min \{ D(u) : u \in H^1(\Omega), \ u - G \in H^1_0(\Omega) \}$$

$$= \min_{u \in G + H^1_0(\Omega)} D(u). \quad (4.0.6)$$

It is not hard to prove the following result, whose proof we postpone to Section 6.2.

**Theorem 4.0.7** Suppose that $\Omega$ is a bounded domain with Lipschitz boundary. Then the minimization problem (4.0.6) has a unique solution, which is harmonic in $\Omega$.

The following result states that, under some mild conditions, the classical solution and the solution to the minimization problem (4.0.6) agree.

**Theorem 4.0.8** Suppose that $\Omega$ is a bounded domain with Lipschitz boundary. Suppose that $g \in C(\partial\Omega)$, and that there exists $G \in H^1(\Omega)$ such that $\gamma(G) = g$. Then the solution to the minimization problem (4.0.6) is a classical solution.

We shall not prove this result. Its proof is a consequence of some deep estimates of the oscillation of solutions on balls close to the boundary $\partial\Omega$. The interested reader is referred to [GT, Ch. 8], especially Corollary 8.28 and the comments at the beginning of p. 206.

Now we discuss briefly the second question posed at the beginning of this subsection. We show that the assumption that the boundary datum $g$
is the trace of a function in $H^1(\Omega)$ is necessary in Theorem 4.0.8. Indeed, Weierstrass proved that solutions to the classical Dirichlet problem need not have finite Dirichlet integral.

**Proposition 4.0.9** There exists a continuous function $g$ on the boundary of the unit disc $B_1(0)$ in $\mathbb{R}^2$ such that the classical solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0) \\ u|_{\partial B_1(0)} = g \end{cases}$$

has infinite Dirichlet integral.

Thus, we cannot expect to solve all classical Dirichlet problems by $L^2$ methods. However, $L^2$ methods are quite flexible and may be adapted to a number of different problems, and they are most efficient in a great variety of situations.

### 4.0.2 Why weak solutions?

The discussion in Subsection 4.0.1 emphasises the role that the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ play in the approach to the Dirichlet problem via Dirichlet’s principle. It is straightforward to check that the following continuous inclusions hold

$$H_0^1(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega).$$

Thus, elements of $H^1(\Omega)$ and $H_0^1(\Omega)$ may be regarded as functions in $L^2(\Omega)$. It would be desirable to have simple and direct characterisations of those functions in $L^2(\Omega)$ that belong to each of these spaces. This will lead to the notion of weak derivatives of a function in $L^2(\Omega)$ (see Definition 5.5.5). Weak derivatives, in turn, are special instances of distributional derivatives (see Definition 5.3.1), and are best understood in the framework of the theory of distributions.

There is a further important reason to introduce weak (hence distributional) derivatives. We shall prove that solving the minimization problem (4.0.6) is equivalent to finding a weak solution of the Laplace equation (see Definition 6.4.1), which, in turn, is, via Weyl’s lemma (see Lemma 6.4.3), equivalent to finding a distributional solution (see Definition 5.4.8) of the Laplace equation. The definition of weak solution suggests that functions in $L^2(\Omega)$ with first order derivatives (in the sense of distributions) in $L^2(\Omega)$ may be an interesting space.
In conclusion, there are good reasons to introduce the so-called weak derivatives of \(L^2\) functions.

### 4.0.3 Plan of Part III

Here is an outline of Part III.

In Section 4.1, we prove some of the statements made in Subsection 4.0.1, specifically Proposition 4.0.1, the estimate for traces (Proposition 4.0.6), and Weierstrass counterexample (Proposition 4.0.9).

Distributions, weak derivatives and Sobolev spaces are treated in Chapter 5. In particular, in Section 5.1 we shall look at complex measures on \(\mathbb{R}^n\) and regard them as linear functionals on \(C_c(\mathbb{R}^n)\) that are continuous with respect to a certain topology \(\tau_{SI}\). Then we do the same for complex measures on a domain \(\Omega\) in \(\mathbb{R}^n\).

In Section 5.2, we modify the arguments of Section 5.1 to study the dual of the space \(C_c^\infty(\mathbb{R}^n)\) of test functions, endowed with the topology \(\tau_{SI}^\infty\), which generalises the topology \(\tau_{SI}\) on \(C_c(\mathbb{R}^n)\) studied before. The dual of test functions is the space of distributions.

It turns out that a distribution has derivatives of all orders: derivatives of a distribution are defined in Section 5.3, and applications to the solution of Poisson equation \(\Delta u = f\), where \(f\) is a given distribution, are discussed in Section 5.4.

In Section 5.5, we take up the study of Sobolev spaces. First we illustrate the subtleties that may arise when we try to substantiate the intent of defining derivatives in the \(L^2\) sense. Then we define the Sobolev spaces \(H^1(\Omega)\) and \(H^1_0(\Omega)\) and state their properties. The proofs thereof are quite lengthy, and would occupy a large portion of the course. Since Sobolev spaces are the subject of many books or chapters of treatises, we have made the choice of stating their properties without proofs, for which the reader is referred to [AF, Br, EG, GT, Fo2, Tr].

In Chapter 6, we develop the \(L^2\) theory for the Dirichlet problem. In particular, we study weak solutions of the Dirichlet problem, and prove Theorems 4.0.7 and 4.0.8.
4.1 Some details left behind

In this section we prove some of the results stated without proof in the introduction above.

Proof of Proposition 4.0.1. For every $\varphi$ in $\mathcal{Y}_0$

$$D(u + \varepsilon \varphi) = \int_{\Omega} \nabla (u + \varepsilon \varphi) \cdot \nabla (u + \varepsilon \varphi) \, dV$$

$$= D(u) + \varepsilon \int_{\Omega} (\nabla \varphi \cdot \nabla u + \nabla u \cdot \nabla \varphi) \, dV + \varepsilon^2 D(\varphi).$$

Hence

$$\frac{d}{d\varepsilon} D(u + \varepsilon \varphi)_{\varepsilon=0}$$

$$= \int_{\Omega} (\nabla \varphi \cdot \nabla u + \nabla u \cdot \nabla \varphi) \, dV$$

$$= - \int_{\Omega} (\varphi \, \text{div}(\nabla u) + \text{div}(\nabla \varphi)) \, dV + \int_{\partial \Omega} (\varphi \, \partial_n u + (\partial_n u) \varphi) \, d\sigma$$

$$= - \int_{\Omega} (\varphi \, \text{div}(\nabla u) + \text{div}(\nabla \varphi)) \, dV :$$

the penultimate equality is obtained by integrating by parts, and the last equality follows from the fact that $\varphi$ vanishes at the boundary $\partial \Omega$. We now rewrite this formula in the cases where $\varphi$ is real valued and where $\varphi$ is purely imaginary, $\varphi = i \psi$ with $\psi$ real valued say. We obtain the two formulae

$$\frac{d}{d\varepsilon} D(u + \varepsilon \varphi)_{\varepsilon=0} = -2 \int_{\Omega} \varphi \, \Delta (\text{Re} \, u) \, dV, \quad \forall \varphi \in \text{Re} \, \mathcal{Y}_0$$

and

$$\frac{d}{d\varepsilon} D(u + i \varepsilon \psi)_{\varepsilon=0} = 2i \int_{\Omega} \psi \, \Delta (\text{Im} \, u) \, dV \quad \forall \psi \in \text{Re} \, \mathcal{Y}_0.$$

Now, if (i) holds, then $\Delta (\text{Re} \, u) = 0 = \Delta (\text{Im} \, u)$, and the two formulae above imply that $u$ is a critical point of $D$, i.e. (ii) holds.

Conversely, if (ii) holds, then

$$\Delta (\text{Re} \, u) = 0 \quad \text{and} \quad \Delta (\text{Im} \, u) = 0$$

a.e., hence in the classical sense, for $u$ is of class $C^2(\overline{\Omega})$. Therefore (i) holds.
Next, trivially (iii) implies (ii). To conclude the proof of the proposition, we need to show that (ii) implies (iii). Observe that

\[ D(w) = D(u + w - u) \]
\[ = D(u) + \int_{\Omega} \left( \nabla u \cdot \nabla (w - u) + \nabla (w - u) \cdot \nabla \pi \right) \, dV + D(w - u). \]

(4.1.1)

The middle term of the right hand side vanishes. Indeed, and by integrating by parts

\[ \int_{\Omega} \left( \nabla u \cdot \nabla (w - u) + \nabla (w - u) \cdot \nabla \pi \right) \, dV \]
\[ = - \int_{\Omega} \left( \Delta u (w - u) + (w - u) \Delta \pi \right) \, dV \]
\[ = 0; \]

note that there are no boundary terms arising from the integration by parts, because \( w - u \) is in \( Y_0 \), hence it vanishes on the boundary of \( \Omega \). Furthermore, we already know that (ii) is equivalent to (i), whence \( \Delta u = 0 \), thereby justifying the last equality above.

Therefore, by (4.1.1),

\[ D(w) = D(u) + D(w - u) \geq D(u), \]

as required to complete the proof of the implication (ii) \( \Rightarrow \) (iii).

\[ \square \]

**Proof of Proposition 4.0.4.** We give a simple proof in the case where \( \partial \Omega \) is smooth. For a proof in the case where \( \partial \Omega \) is Lipschitz, see [EG, Thm 1, p. 133].

Extend the unit normal field \( \nu \) on \( \partial \Omega \) to a vector field on a neighbourhood of \( \Omega \) (for instance, use Lemma 3.2.7 to extend \( \nu \) to a vector field in a neighbourhood of \( \partial \Omega \), and then multiply it by a smooth cutoff function). By the divergence theorem

\[ \int_{\partial \Omega} |u|^2 \, d\sigma = \int_{\partial \Omega} (|u|^2 \nu) \cdot \nu \, d\sigma \]
\[ = \sum_{j=1}^{n} \int_{\Omega} \partial_j(|u|^2 \nu) \, dV \]
\[ \leq \sum_{j=1}^{n} \int_{\Omega} \left[ |u (\partial_j \pi) \nu_j| + |(\partial_j u) \pi \nu_j| + |u|^2 |\partial_j \nu_j| \right] \, dV \quad \forall u \in C^2(\Omega). \]
Now observe that $\nu$ and $\text{div} \nu$ are uniformly bounded in $\mathbb{R}^n$. Therefore there exists a constant $C$ such that
\[
\int_{\partial \Omega} |u|^2 \, d\sigma \leq C \sum_{j=1}^n \int_\Omega \left[ |u (\partial_j \overline{u})| + |(\partial_j u) \overline{u}| + |u|^2 \right] \, dV
\]
(by Schwartz’s inequality)
\[
\leq C \left[ 2 \sum_{j=1}^n \|u\|_{L^2(\Omega)} \|\partial_j u\|_{L^2(\Omega)} + n \|u\|^2_{L^2(\Omega)} \right]
\leq C \|u\|^2_{H^1(\Omega)} \quad \forall u \in C^2(\overline{\Omega}).
\]

Denote by $\gamma : C^2(\overline{\Omega}) \to L^2(\partial \Omega)$ the linear map, defined by
\[
\gamma(u) := u|_{\partial \Omega}.
\]
We have proved that there exists a constant $C$ such that
\[
\|\gamma(u)\|_{L^2(\partial \Omega)} \leq C \|u\|_{H^1(\Omega)} \quad \forall u \in C^2(\overline{\Omega}).
\]
By Exercise 2.2.4, the map $\gamma$ extends uniquely to a bounded linear map from $H^1(\Omega)$ to $L^2(\partial \Omega)$, for $C^2(\overline{\Omega})$ is dense in $H^1(\Omega)$ (see Theorem 5.5.12 below).

This concludes the proof of the proposition. \hfill \qed

**Exercise 4.1.1** Prove that if $u$ is a smooth function defined in the unit ball $B_1(0)$ of $\mathbb{R}^2$, then
\[
\int_\Omega |\nabla u|^2 \, dV = \int_{(0,1) \times (0,2\pi)} \left[ |\partial_r u|^2 + \frac{1}{r^2} |\partial_\theta u|^2 \right] r \, dr \, d\theta.
\]

**Proof of Proposition 4.0.9** We shall be a bit sketchy, leaving to the reader the task of providing full details. For $\alpha \in (0, 1]$, we consider the function $g_\alpha$, defined by
\[
g_\alpha(\theta) := \sum_{k=0}^\infty 2^{-\alpha k} e^{i2^k \theta} \quad \forall \theta \in [0, 2\pi].
\]
(4.1.2)
It is straightforward to check that the series is uniformly convergent in $[0, 2\pi]$, whence $g_\alpha$ is continuous therein. Furthermore, the function
\[
u_\alpha(r, \theta) := \sum_{k=0}^\infty r^{2^k} 2^{-\alpha k} e^{i2^k \theta} \quad \forall \theta \in [0, 2\pi] \quad \forall r \in [0, 1]
\]
agrees with $g_\alpha$ on the boundary of the unit disc in the plane, is harmonic in the (open) unit disc (the series can be differentiated term by term as long...
as $r < 1$) and it is continuous on the closure of the unit disc. Therefore $u$ solves the classical Dirichlet problem on the unit disc with boundary datum $g_\alpha$. By using polar coordinates, we see that

$$D(u_\alpha) = \int_{\Omega} |\nabla u_\alpha|^2 \, dV = \int_{(0,1) \times (0,2\pi)} \left[ |\partial_r u_\alpha|^2 + \frac{1}{r^2} |\partial_\theta u_\alpha|^2 \right] r \, dr \, d\theta = \lim_{\rho \uparrow 1} \int_{(\theta, \rho) \times (0,2\pi)} \left[ |\partial_r u_\alpha|^2 + \frac{1}{r^2} |\partial_\theta u_\alpha|^2 \right] r \, dr \, d\theta.$$

We may compute $\partial_r u$ and $\partial_\theta u$ in $\Omega_\rho$ by differentiating the series (4.1.2) term by term. We find

$$\partial_r u_\alpha = \sum_{k=0}^{\infty} r^{2k-1} 2^{(1-\alpha)k} e^{2^k \theta} \quad \text{and} \quad \partial_\theta u_\alpha = \sum_{k=0}^{\infty} r^{2k} 2^{(1-\alpha)k} i e^{2^k \theta}.$$

By using Plancherel’s identity, we see that

$$\lim_{\rho \uparrow 1} \int_{(\theta, \rho) \times (0,2\pi)} \left[ |\partial_r u_\alpha|^2 + \frac{1}{r^2} |\partial_\theta u_\alpha|^2 \right] r \, dr \, d\theta \sim \lim_{\rho \uparrow 1} \int_0^\rho \sum_{k} 2^{2k(1-\alpha)} r^{2k+1-1} \, dr.$$

An elementary computation shows that the right hand side is infinite for all $\alpha \leq 1/2$. Thus, for these values of $\alpha$ the classical solution to the Dirichlet problem with boundary datum $g_\alpha$ has **infinite Dirichlet integral**. □
Chapter 5

Distributions and Sobolev spaces

5.1 Continuous functions with compact support and measures

We aim at generalising the concept of complex valued function of several real variables. As a first step in this direction, we shall reinterpret such a function as a (continuous) linear functional on a suitable class of test functions. It will be convenient to treat from the beginning measures instead of functions.

Recall that a complex measure on $\mathbb{R}^n$ is a complex valued function $\mu$ on bounded Borel sets, with the property that if $E$ is a bounded Borel set and $\{E_j\}$ is a sequence of pairwise disjoint Borel sets such that

$$E = \bigcup_{j=1}^{\infty} E_j,$$

then

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E_j),$$

the series being absolutely convergent. Note that $\mu$ is not defined on unbounded Borel sets. It may be shown that, given a complex measure $\mu$, there exist four nonnegative measures $\mu_j$, $j = 1, \ldots, 4$, such that

$$\mu(E) = \sum_{j=1}^{4} \mu_j(E).$$
Clearly, we may integrate continuous functions \( \varphi \) with compact support with respect to each \( \mu_j \), and then define

\[
\int_{\mathbb{R}^n} \varphi \, d\mu = \sum_{j=1}^{4} \int_{\mathbb{R}^n} \varphi \, d\mu_j \quad \forall \varphi \in C_c(\mathbb{R}^n).
\]

Thus, for every complex measure \( \mu \) we have defined a complex linear functional \( T\mu \) on \( C_c(\mathbb{R}^n) \), defined by

\[
(T\mu)(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu \quad \forall \varphi \in C_c(\mathbb{R}^n).
\]

It is straightforward to check that this functional is linear. Thus, we have defined a map \( \mu \mapsto T\mu \) from complex measures to linear functionals on \( C_c(\mathbb{R}^n) \). Clearly, this map is linear. It is not hard to check that \( T \) is injective; in other words, if \( T\mu \) is the null functional on \( C_c(\mathbb{R}^n) \), then \( \mu = 0 \) on bounded Borel sets.

**Exercise 5.1.1** The map \( T \) is injective.

An important example of complex measure is the **delta measure** \( \delta_x \) at a point \( x \), defined by

\[
\delta_x(E) = \begin{cases} 
1 & \text{if } x \in E \\
0 & \text{if } x \notin E.
\end{cases}
\]

The linear functional \( T\delta_x \) is then given by

\[
T\delta_x(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\delta_x = \varphi(0) \quad \forall \varphi \in C_c(\mathbb{R}^n).
\]

We now briefly discuss continuity of the functionals \( T\mu \), and the continuity of the linear map \( T \). To do this, we need to define a topology on the space \( C_c(\mathbb{R}^n) \) and a topology on the space of complex measures. Remember that such measures are defined only on bounded Borel sets. As a first attempt, we may consider the sup norm on \( C_c(\mathbb{R}^n) \). But clearly this choice is not optimal. Consider, for instance, the measure

\[
\mu = \sum_{j=1}^{\infty} \delta_j
\]
on \( \mathbb{R} \), where \( \delta_j \) denotes the delta function at the point \( j \). Clearly \( \mu \) is a complex measure in the above sense. Note, however, that the linear functional \( T \mu \) is not continuous with respect to the sup norm. Indeed, let \( g \) a continuous function, which \textbf{vanishes at infinity}, i.e. such that

\[
\lim_{x \to \infty} g(x) = 0.
\]

It is straightforward to show that \( g \) is the limit in the uniform norm of a sequence of functions \( \{ \varphi_k \} \) in \( C_c(\mathbb{R}) \), with support contained in the interval \([-k, k]\). If \( T \mu \) were a continuous linear functional on \( C_c(\mathbb{R}) \) with respect to the sup norm, we should have

\[
\lim_{k \to \infty} (T \mu)(\varphi_k) = (T \mu)(g),
\]

and \((T \mu)(g)\) must be a complex number. But, observe that

\[
\lim_{k \to \infty} (T \mu)(\varphi_k) = \lim_{k \to \infty} \sum_{j \leq k} \varphi_k(j).
\]

If \( g \) is chosen so that \( \sum_j g(j) = \infty \), then the right hand side in the preceding formula is equal to \( +\infty \), thereby contradicting the fact that \((T \mu)(g)\) is a complex number.

Thus, a different topology on \( C_c(\mathbb{R}^n) \) is needed. A basic result we shall use is the following classical result of Riesz. The reader is referred to [Ru, Ch. 6] for the proof of a slight generalisation thereof. Recall that if \( X \) is a locally compact Hausdorff space, we denote by \( C_0(X) \) the completion of \( C_c(X) \) with respect to the uniform norm.

**Theorem 5.1.2** Let \( X \) be a locally compact Hausdorff space. Then to each bounded linear functional \( F \) on \( C_0(X) \) there corresponds a unique complex Borel measure \( \nu \) on \( X \) such that

\[
F(f) = \int_X f \, d\nu
\]

for each \( f \) in \( C_0(X) \). Moreover, \(||F|| = |\nu|(X)\), where \(|\nu|(X)\) denotes the total variation of the measure \( \nu \).

The total variation \(|\nu|\) of \( \nu \) is a positive measure such that

\[
|\nu(E)| \leq |\nu|(E)
\]
for every measurable set \( E \). In fact, \(|\nu|\) is the minimal solution to the problem of finding a positive measure \( \mu \) that dominates \( \nu \), in the sense that

\[
|\nu(E)| \leq \mu(E)
\]

for every measurable \( E \). The measure \(|\nu|\) may be defined as follows:

\[
|\nu|(E) := \sup_{\sum_{j=1}^{\infty} |\nu(E_j)|}
\]

where \( E = \bigcup_j E_j \), the measurable sets \( E_j \) are pairwise disjoint, and the supremum is taken with respect to all partitions of \( E \) as countable union of pairwise disjoint measurable subsets of \( E \). For these facts, and many more, the reader is referred to [Ru, Ch. 6].

Going one step further, for each \( K \subset\subset \mathbb{R}^n \) denote by \( \varrho_K \) the seminorm on \( C_c(\mathbb{R}^n) \), defined by

\[
\varrho_K(f) = \max_{x \in K} |f(x)|.
\]

The family \( \{\varrho_K\}_{K \subset\subset \mathbb{R}^n} \) induces a locally convex topology \( \tau_{UC} \) on \( C_c(\mathbb{R}^n) \), which is called the topology of uniform convergence on compact sets. We recall the definition \( \tau_{UC} \). For each \( N \)-tuple \( K_1, \ldots, K_N \) of compact sets in \( \mathbb{R}^n \) and for each positive \( \varepsilon \), consider the set

\[
U(\varrho_{K_1}, \ldots, \varrho_{K_N}; \varepsilon) := \{g \in C_c(\mathbb{R}^n) : \varrho_{K_j}(g) < \varepsilon, j = 1, \ldots, N\}. \quad (5.1.1)
\]

It is straightforward to check that the collection of all such sets (obtained by let \( \varepsilon, N \), and the compact subsets \( K_1, \ldots, K_N \) of \( \mathbb{R}^n \) chosen, vary) is a local base of convex neighbourhoods of the null function in \( C_c(\mathbb{R}^n) \). A base of neighbourhoods at \( f \) is obtained by simply translating the above neighbourhoods by \( f \), i.e., it consists of all the sets of the form

\[
f + U(\varrho_{K_1}, \ldots, \varrho_{K_N}; \varepsilon).
\]

It is not hard to see that \( \tau_{UC} \) is induced by a distance on \( C_c(\mathbb{R}^n) \). Indeed, first choose an exhaustion of \( \mathbb{R}^n \) by compact subsets. It is convenient to write

\[
\mathbb{R}^n = \bigcup_{j=1}^{\infty} B_j,
\]

where \( B_j := B(0, j) \), and then consider the countable family \( \varrho_{B_j} \) of the associated seminorms, which we shall denote by \( \varrho_j \) for the sake of simplicity.
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Note that a base of neighbourhoods at 0 is given by the collection of sets of the form
\[ \mathcal{U}(\varrho_j; \varepsilon) := \{ g \in C_c(\mathbb{R}^n) : \varrho_j(g) < \varepsilon \}. \] (5.1.2)
Consider the metric on \( C_c(\mathbb{R}^n) \) induced by the distance
\[ d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\varrho_j(f - g)}{1 + \varrho_j(f - g)}. \]

**Exercise 5.1.3** Prove that \( d \) is indeed a metric on \( C_c(\mathbb{R}^n) \), and that \( \tau_{UC} \) is the topology induced by \( d \).

It is straightforward to check that a sequence \( \{f_j\} \) of functions in \( C_c(\mathbb{R}^n) \) is convergent to \( f \) in this topology if and only if for every compact subset \( K \) of \( \mathbb{R}^n \) the sequence \( \{f_j\} \) is uniformly convergent to \( f \) in \( K \).

However, note that \( C_c(\mathbb{R}^n) \) is not complete with respect to the distance \( d \).

Indeed, observe preliminarily that \( \{f_j\} \) is a Cauchy sequence in \( C_c(\mathbb{R}^n) \) if and only if \( \{f_j\} \) is a Cauchy sequence with respect to each seminorm \( \varrho_k \).

Then set \( x_j = (j + (1/2))e_1 \) where \( e_1 \) denotes the first vector of the canonical basis of \( \mathbb{R}^n \). Clearly \( x_j \) belongs to \( B_{j+1} \setminus \overline{B}_j \) and the distance between \( x_j \) and \( \overline{B}_j \cup \overline{B}_{j+1} \) is clearly equal to 1/2. Pick a function \( \psi \) in \( C_c(\mathbb{R}^n) \), whose support is contained in \( \overline{B}(0, 1/4) \) and define
\[ \psi_j(x) = \psi(x - x_j) \quad \forall x \in \mathbb{R}^n. \]
Clearly, the support of \( \psi_j \) is contained in \( \overline{B}(x_j, 1/4) \), whence in \( \overline{B}_{j+1} \setminus \overline{B}_j \).

Now, define
\[ \varphi_j := \sum_{j=1}^{J} \psi_j. \]
Obviously \( \varrho_N(\varphi_j - \varphi_k) = 0 \) for every \( j, k > N \). Hence \( \{\varphi_j\} \) is a Cauchy sequence with respect to the metric \( d \). Furthermore, the sequence \( \{\varphi_j\} \) is convergent to the function \( \varphi := \sum_{j=1}^{\infty} \psi_j \), which, however, does not belong to \( C_c(\mathbb{R}^n) \).

We now define another topology on \( C_c(\mathbb{R}^n) \), strictly finer than \( \tau_{UC} \).

**Definition 5.1.4** For each \( \varphi \in C_c(\mathbb{R}^n) \), set
\[ \varrho_0(\varphi) := \max_{\mathbb{R}^n} |\varphi|, \quad \varrho_j(\varphi) := \max_{\mathbb{R}^n \setminus B_j} |\varphi|, \quad j = 1, 2, 3, \ldots \]
Note that the sequence \(\{\varrho_j'(\varphi)\}\) is decreasing. For each decreasing sequence \(\varepsilon := \{\varepsilon_j\}\) of positive real numbers we consider the set
\[
V_\varepsilon := \{f \in C_c(\mathbb{R}^n) : \varrho_j'(f) < \varepsilon_j, \ j = 1, 2, 3, \ldots\}. \tag{5.1.3}
\]
It is straightforward to check that the (uncountable) family of the sets
\[
V(0) := \{V_\varepsilon : \varepsilon\ as\ above\}
\]
is a fundamental system of convex neighbourhoods of 0 of a topology \(\tau_{SI}\) on \(C_c(\mathbb{R}^n)\), called the strict inductive limit topology of \(C_c(\mathbb{R}^n)\). A base of neighbourhoods of a function \(f\) in \(C_c(\mathbb{R}^n)\) is then
\[
\mathcal{V}(f) := f + V(0).
\]
We claim that \(\tau_{SI}\) is strictly finer than \(\tau_{UC}\). On the one hand, it is clear that for each \(j\) the neighbourhood \(U(\varrho_j; \varepsilon)\) of 0 in the \(\tau_{UC}\) topology (see (5.1.2)) contains the neighbourhood \(V_\varepsilon\) of 0 in the \(\tau_{SI}\) topology, where \(\varepsilon = \{2^{-j}\varepsilon : j = 0, 1, 2, \ldots\}\). Thus, \(\tau_{SI}\) is finer than \(\tau_{UC}\).

On the other hand, the neighbourhood
\[
\mathcal{V} := \{g \in C_c(\mathbb{R}^n) : \varrho_j'(g) < 2^{-j} \varepsilon, \ j = 0, 1, 2, \ldots\}
\]
of 0 does not contain any set of the form \(U(\varrho_j; \eta)\), whence \(\tau_{SI}\) is strictly finer than \(\tau_{UC}\), as claimed.

It will be important to understand \(\tau_{SI}\) better. In particular, it is interesting to characterise convergent sequences. We have the following result.

**Proposition 5.1.5** Let \(\{f_N\}\) be a sequence of functions in \(C_c(\mathbb{R}^n)\). The following are equivalent:

(i) \(\{f_N\}\) is convergent to a function \(f \in C_c(\mathbb{R}^n)\) in the strict inductive limit topology \(\tau_{SI}\);

(ii) there exists \(\ell\) such that the support of \(\{f_N\}\) is contained in \(\overline{B}_\ell\) and \(\{f_N\}\) is uniformly convergent to \(f\).

**Proof.** By replacing \(f_N\) by \(f_N - f\) we may assume that \(f\) is the function identically 0 in \(\mathbb{R}^n\).

Since \(\tau_{SI}\) is finer than \(\tau_{UC}\), (ii) implies (i).

To prove that (i) implies (ii), we argue by reductio ad absurdum. Suppose that \(\{f_N\}\) is not contained in \(C_c(\overline{B}_\ell)\) for any \(\ell\). Then there would exist two
increasing sequences of indices $N_j$ and $n_j$ such that $f_{N_j}$ belongs to $C_c(B_{n_j+1}) \setminus C_c(B_{n_j})$. Therefore there exist points $x_{n_j}$ in $B_{n_j+1} \setminus B_{n_j}$ such that

$$f_{N_j}(x_{n_j}) \neq 0, \quad j = 1, 2, 3, \ldots.$$ 

We may also assume that the sequence $\{f_{N_j}(x_{n_j})\}$ is decreasing. Then choose a decreasing sequence $\varepsilon := \{\varepsilon_k\}$ such that $\varepsilon_{N_j} = f_{N_j}(x_{n_j})$. Consider any tail $\{f_{h}, f_{h+1}, f_{h+2}, \ldots\}$ of the original sequence. We claim that this tail is not contained in $V_{\varepsilon}$, where $\varepsilon$ is the sequence chosen above. Indeed, the sequence $N_j$ tends to infinity as $j$ tends to infinity, so that at least one (in fact, infinitely many) function $f_{N_j}$ belongs to the tail, and

$$\varrho_{n_j}(f_{N_j}) = \sup_{x \in B_{n_j}} |f_{N_j}(x)|$$

$$\geq |f_{N_j}(x_{n_j})|$$

$$= \varepsilon_{n_j},$$

so that $f_{N_j}$ is not in $V_{\varepsilon}$, as claimed.

This contradicts the fact that $\{f_j\}$ tends to 0 as $j$ tends to infinity. Therefore, the supports of the functions $\{f_j\}$ must be contained in $B_{\ell}$ for some $\ell$, as required.

**Corollary 5.1.6** Every Cauchy sequence for the topology $\tau_{SI}$ on $C_c(\mathbb{R}^n)$ is convergent to an element of $C_c(\mathbb{R}^n)$.

**Proof.** Let $\{f_j\}$ be a Cauchy sequence with respect to $\tau_{SI}$. Since $\tau_{SI}$ is finer than $\tau_{UC}$, $\{f_j\}$ is a Cauchy sequence with respect to the topology $\tau_{UC}$. Therefore $\{f_j\}$ is a Cauchy sequence with respect to each of the seminorms $g_k$. In particular, the restrictions of the functions $f_j$ to $\overline{B}_k$ is a Cauchy sequence in $C(\overline{B}_k)$ for every $k$. Since $C(\overline{B}_k)$ is complete with respect to the uniform norm, there exists a function $g_k$ in $C(\overline{B}_k)$ such that

$$\sup_{x \in \overline{B}_k} |f_j(x) - g_k(x)| \to 0 \quad \text{as } j \text{ tends to } \infty.$$

It is clear that the restriction of $g_{k+1}$ to $\overline{B}_k$ agrees with $g_k$. Therefore, we may define a continuous function $g$ on $\mathbb{R}^n$ by

$$g(x) = g_k(x) \quad \forall x \in \overline{B}_k.$$

This function is the uniform limit of $\{f_j\}$ on each ball $\overline{B}_k$. 

It remains to show that the support of \( g \) is compact and that \( \{ f_j \} \) is convergent to \( g \) with respect to \( \tau_{SI} \).

We argue by contradiction. Suppose that the support of \( g \) is not compact. Denote by \( n_j \) an increasing sequence of indices such that the support of \( f_{N_j} \) is contained in \( C_c(B_{n_j}) \). Denote by \( x_{n_j} \) a sequence of points in \( B_{n_{j+1}} \setminus B_{n_j} \) for which \( g(x_{n_j}) \) does not vanish, \( j = 1, 2, 3, \ldots \). Then, choose positive real numbers \( \eta_{n_j} > 1 \) so that

\[
\varepsilon_{n_j} := \frac{|g(x_{n_j})|}{\eta_{n_j}} \text{ form a decreasing sequence.}
\]

We claim that for each positive integer \( h \) the tail \( \{ f_h - g, f_{h+1} - g, \ldots \} \) is not contained in \( V_{\varepsilon} \), where \( \varepsilon \) is the sequence \( \{ \varepsilon_k \} \), defined above. Indeed, the tail contains an element of the form \( f_{N_j} - g \), and

\[
\varrho_{n_j} (f_{N_j} - g) \geq |(f_{N_j} - g)(x_{n_j})| > |(f_{N_j} - g)(x_{n_j})|/\eta_{n_j} = \varepsilon_{n_j},
\]

as claimed.

This contradicts the fact that \( \{ f_N \} \) is convergent to \( g \). Hence we have proved that the support of \( g \) is compact.

The proof that \( \{ f_N \} \) is convergent to \( g \) in the \( \tau_{SI} \) topology is straightforward and is left to the reader. \( \square \)

Now we show that, given a complex measure \( \mu \) on \( \mathbb{R}^n \), the linear functional \( T\mu \), defined by

\[
(T\mu)(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu \quad \forall \varphi \in C_c(\mathbb{R}^n),
\]

is continuous on \( (C_c(\mathbb{R}^n), \tau_{SI}) \).

We need to show that for every \( \varepsilon > 0 \), the set

\[
(T\mu)^{-1}(B_\varepsilon(0)) = \{ \varphi \in C_c(\mathbb{R}^n) : \left| \int_{\mathbb{R}^n} \varphi \, d\mu \right| < \varepsilon \}
\]

(here \( B_\varepsilon(0) \) denotes the open disc in the complex plane with centre 0 and radius \( \varepsilon \) ) contains a neighbourhood of the origin in the strict inductive limit topology, i.e., there exists a decreasing sequence \( \eta_j := \{ \eta_j \} \) such that

\[
|(T\mu)(\varphi)| < \varepsilon \quad \forall \varphi \in V_{\eta_j}.
\]
5.1. CONTINUOUS FUNCTIONS AND MEASURES

Set $\eta_j := 2^{-j-1} \varepsilon / |\mu|(\overline{B}_{j+1})$. Then, given $\varphi \in \mathcal{V}_\eta$, we write

\[
\left| (T\mu)(\varphi) \right| \leq \left| \int_{B_1} \varphi \, d\mu \right| + \sum_{j=1}^{\infty} \left| \int_{B_{j+1} \setminus B_j} \varphi \, d\mu \right| \\
\leq \left( \sup_{\overline{B}_1} |\varphi| \right) |\mu|(B_1) + \sum_{j=1}^{\infty} \left( \sup_{(B_j)^c} |\varphi| \right) |\mu|(\overline{B}_{j+1} \setminus B_j) \\
\leq \eta_0 |\mu|(B_1) + \sum_{j=1}^{\infty} \eta_j |\mu|(B_{j+1}) \\
< \varepsilon,
\]

as required.

It is an interesting fact that the converse of this result holds.

**Theorem 5.1.7** To each continuous linear functional $F$ on $(C_c(\mathbb{R}^n), \tau_{SI})$ there corresponds a unique complex Borel measure $\nu$ on $X$ such that

\[
F(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\nu \quad \forall \varphi \in C_c(\mathbb{R}^n).
\]

**Proof.** We give a sketch of the proof, leaving the details to the reader. Since $F$ is continuous on $(C_c(\mathbb{R}^n), \tau_{SI})$, for every $\varepsilon > 0$ there exists a sequence $\eta := \{\eta_j\}$ such that

\[
\left| F(\varphi) \right| < \varepsilon \quad \forall \varphi \in \mathcal{V}_\eta.
\] (5.1.4)

Recall that $\varphi \in \mathcal{V}_\eta$ if and only if

\[
\sup_{\mathbb{R}^n} |\varphi| < \eta_0 \quad \text{and} \quad \sup_{B_j^c} |\varphi| < \eta_j \quad j = 1, 2, 3, \ldots
\]

We prove that for every positive integer $j$ the restriction of $F$ to $C_0(B_j)$ is a continuous linear functional on $C_0(B_j)$ (with respect to the uniform norm). Indeed, suppose that $\varphi$ is a function in $C_0(B_j)$ such that $\|\varphi\|_\infty \leq \eta_j$. Then clearly $\varphi$ belongs to $\mathcal{V}_{\eta_j}$. Therefore $|F(\varphi)| < \varepsilon$, as required. By the Riesz representation theorem [5.1.2] there exists a unique finite complex measure $\nu_j$ such that

\[
F(\varphi) = \int_{B_j} \varphi \, d\nu_j \quad \forall \varphi \in C_0(B_j).
\]

Since $C_0(B_j)$ is continuously included in $C_0(B_{j+1})$, the restriction of $\nu_{j+1}$ to $C_0(B_j)$ agrees with $\nu_j$. Thus, there is a uniquely defined Radon measure $\nu$
on $C_c(\mathbb{R}^n)$ such that its restriction to $C_0(B_j)$ agrees with $\nu_j$, and

$$F(\varphi) = \int_{B_j} \varphi \, d\nu_j \quad \forall \varphi \in C_c(\mathbb{R}^n),$$

as required. \hfill \Box

Finally, we describe a mild generalisation of the theory developed so far, in which a fixed open set $\Omega$ plays the role of $\mathbb{R}^n$. Recall the following classical result.

**Theorem 5.1.8** Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$. The following hold:

(i) there exists a sequence $\{V_N\}$ of precompact sets in $\Omega$ such that $\overline{V}_N \subset V_{N+1}$ for every positive integer $N$ and $\bigcup_{N=1}^{\infty} \overline{V}_N = \Omega$;

(ii) there exists a sequence $\{f_N\}$ of functions in $C_c(\Omega)$ such that $\text{supp} (f_N) \subset V_{N+1}$ and $f_N = 1$ on $\overline{V}_N$.

**Proof.** [Fo1, Prop. 4.31, Prop. 4.32, Lemma 8.18]. \hfill \Box

The sequence of sets $\{V_N\}$ is called an **exhaustion** of $\Omega$. There are many exhaustions of the same set $\Omega$. The topologies we shall introduce below will be independent of the chosen exhaustion.

The role played above by the space $C_c(\mathbb{R}^n)$ will now be played by the space $C_c(\Omega)$. The relevant topology on $C_c(\Omega)$ will be the **strict inductive limit topology** $\tau_{SI}(\Omega)$, which is defined similarly to the strict inductive limit topology $\tau_{SI}$ on $C_c(\mathbb{R}^n)$, with the role of the seminorms $\varrho'_j$ played now by the seminorms $\rho'_j$, defined by

$$\rho'_0(f) := \sup_{x \in \Omega} |f(x)| \quad \text{and} \quad \rho'_j(f) := \sup_{x \in \Omega \setminus V_j} |f(x)| \quad j = 1, 2, 3, \ldots$$

By arguing as in Proposition 5.1.5, it may be shown that a sequence $\{f_N\}$ of functions in $C_c(\Omega)$ is convergent to $f$ in the strict inductive limit topology if and only if there exists $\ell$ such that the supports of all the functions $f_N$ are contained in $\overline{V}_\ell$, and $\{f_N\}$ tends uniformly to $f$ on $V_\ell$.

Furthermore, every Cauchy sequence with respect to $\tau_{SI}(\Omega)$ is convergent to an element of $C_c(\Omega)$. We express this fact by saying that $C_c(\Omega)$, endowed with the strict inductive limit topology, is **complete**.
5.2 SMOOTH FUNCTIONS AND DISTRIBUTIONS

Now, we attribute to the Lebesgue measure $\lambda$ a distinguished role. For every locally integrable function $f$ (with respect to the Lebesgue measure), we consider the measure

$$d\mu_f = f \, d\lambda$$

absolutely continuous with respect to the Lebesgue measure and with density $f$. Thus, to every locally integrable function with respect to $\lambda$, we may associate the linear functional $T\mu_f$ on $C_c(\mathbb{R}^n)$, defined by

$$(T\mu_f)(\varphi) = \int_{\mathbb{R}^n} \varphi f \, d\lambda \quad \forall \varphi \in C_c(\mathbb{R}^n).$$

5.2 Smooth functions and distributions

There are physical situations which lead naturally to notions more complicated than that of masses, and require mathematical tools more refined and general than measures.

For instance, consider a positive charge $1/\varepsilon$ placed at the point $\varepsilon$ on the real line, and a negative charge $-1/\varepsilon$ at the point $-\varepsilon$. Does the “limit” of such a configuration make sense?

If we proceed naively, by just taking the

$$\lim_{\varepsilon \to 0} \frac{\delta_{\varepsilon}(E) - \delta_{-\varepsilon}(E)}{\varepsilon},$$

i.e., the limit of the measures, as additive set functions, we get into trouble. Indeed, the limit vanishes over all intervals, except over those with an end at the origin, for which it is undetermined.

If, instead, we adopt the point of view that measures correspond to continuous linear functionals on $C_c(\mathbb{R}^n)$, and define $T\varepsilon$ to be the linear functional

$$T\varepsilon(\varphi) := \lim_{\varepsilon \to 0} \frac{\delta_{\varepsilon}(\varphi) - \delta_{-\varepsilon}(\varphi)}{\varepsilon} \quad \forall \varphi \in C_c(\mathbb{R}^n),$$

then we see that

$$\lim_{\varepsilon \to 0} T\varepsilon(\varphi) = \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(\varepsilon)}{\varepsilon} = \varphi'(0),$$

at least for every function $\varphi$ in $C^1_c(\mathbb{R}^n)$. Thus, the linear form

$$T(\varphi) := \lim_{\varepsilon \to 0} T\varepsilon(\varphi)$$
is defined only on a proper linear subspace of $C_c(\mathbb{R}^n)$, namely $C^1(\mathbb{R}^n)$. Moreover, $T$ is discontinuous (with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}^n$). Indeed, given a sequence $\{\varphi_j\}$ of functions in $C^1(\mathbb{R}^n)$, uniformly convergent to 0 on compact subsets of $\mathbb{R}^n$, the sequence of their derivatives $\{\varphi'_j\}$ need not converge uniformly to 0, as simple examples show. Thus, in order to consider multipoles of any order, we are led to consider linear functionals on spaces of infinitely differentiable functions endowed with a finer topology than the topology $\tau_{SI}$ considered in the last section.

**Definition 5.2.1** Let $K$ be a compact subset of $\Omega$. Denote by $\mathcal{D}_K(\Omega)$ the space of all functions in $C^\infty(\Omega)$ with support contained in $K$, endowed with the family of seminorms

$$\varrho_{m,K}(f) := \max_{|\alpha| \leq m} \max_{x \in K} |D^\alpha f(x)|,$$

where $m$ is a nonnegative integer.

Note that $\mathcal{D}_K(\Omega)$ and $\mathcal{D}_K(\mathbb{R}^n)$ are just the same space.

**Exercise 5.2.2** Produce an explicit example of a function $\varphi$ in $\mathcal{D}_B(0,1)(\mathbb{R}^n)$. Then produce an example of a smooth function on $\mathbb{R}^n$ with support contained in $B(0,1)$ that is equal to 1 on $B(0,1/2)$.

**Exercise 5.2.3** Suppose that $K$ is a compact set in $\mathbb{R}^n$, and $g$ is a continuous function on $K$. Prove that there exists a sequence $\{G_j\}$ of smooth functions on $\mathbb{R}^n$ such that $G_j \to g$ uniformly on $K$, by completing the steps below:

(i) if $K_0$ and $K_1$ are two disjoint compact sets in $\mathbb{R}^n$, show that the continuous function

$$\eta(x) := \frac{d(x,K_0)}{d(x,K_0) + d(x,K_1)} \quad \forall x \in \mathbb{R}^n$$

vanishes on $K_0$, it is equal to 1 on $K_1$, and satisfies $0 \leq \eta \leq 1$;

(ii) show that it suffices to prove the result for functions $g$ satisfying $0 \leq g \leq 1$;
(iii) define $K_0 := \{ x \in K : 2/3 \leq f(x) \leq 1 \}$ and $K_1 := \{ x \in K : 0 \leq g(x) \leq 1/3 \}$, and denote by $F_1$ a continuous function on $\mathbb{R}^n$ which is equal to $1/3$ on $K_0$, to $0$ on $K_1$, and satisfies $0 \leq F_1 \leq 1/3$ on $\mathbb{R}^n$. Show that 
\[
0 \leq g(x) - F_1(x) \leq 2/3 \quad \forall x \in K;
\]

(iv) by iterating the procedure in (iii), find continuous functions $F_2, \ldots, F_j$ on $\mathbb{R}^n$ such that 
\[
0 \leq g(x) - F_1(x) - \cdots - F_j \leq (2/3)^j \quad \forall x \in K,
\]

and $0 \leq F_j \leq (1/3)(2/3)^{j-1}$ on $\mathbb{R}^n$;

(v) set $F = \sum_j F_j$. Prove that $F$ is continuous on $\mathbb{R}^n$ and $F|_K = g$;

(vi) set $H_{\epsilon} := F * \varphi_{\epsilon}$, where $\varphi \in C_c^\infty(B(0,1))$, $\int \varphi \, dV = 1$, and $\varphi_{\epsilon}(x) = \epsilon^{-n} \varphi(x/\epsilon)$. Prove that 
\[
\|H_{\epsilon}(x) - F(x)\| \leq \sup_{\|x-y\| \leq \epsilon} \|F(x) - F(y)\|
\]

for every $x \in K$. Show that $\{H_{1/j}\}$ is the required sequence.

Observe that $D_K(\Omega)$ is metrizable. Indeed, its topology is induced by the distance 
\[
d_K(f, g) := \sum_{m=0}^{\infty} 2^{-m} \frac{\varrho_{m,K}(f-g)}{1 + \varrho_{m,K}(f-g)} \quad \forall f, g \in D_K(\Omega).
\]

A sequence $\{f_N\}$ is convergent to $f$ in $D_K(\Omega)$ if and only if for every multi-index $\alpha$, the sequence $\{D^\alpha f_N\}$ tends to $D^\alpha f$ uniformly in $K$.

**Definition 5.2.4** Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$, and let $\{V_N\}$ be an exhaustion of $\Omega$ as in Theorem 5.1.8. We denote by $D(\Omega)$ the vector space $C_c^\infty(\Omega)$, endowed with the strict inductive limit topology $\tau_{\infty}^S(\Omega)$. Explicitly, a fundamental system of neighbourhoods of $0$ in $\tau_{\infty}^S(\Omega)$ is given by the sets 
\[
V_{\pm m} := \{ f \in C_c^\infty(\Omega) : \varrho'_{m_0,\Omega}(f) < \epsilon_0, \varrho'_{m_j,\Omega \setminus V_j}(f) < \epsilon_j, j = 1, 2, \ldots \},
\]
as $\underline{\epsilon} := \{\epsilon_j\}$ and $m := \{m_j\}$ vary among all possible decreasing sequences $\underline{\epsilon}$ that converge to $0$ and all possible increasing sequences of integers that tend to infinity. Here 
\[
\varrho'_{m_0,\Omega}(f) := \max_{|\alpha| \leq m_0} \max_\Omega |D^\alpha f|, \quad \varrho'_{m_j,\Omega \setminus V_j}(f) := \max_{|\alpha| \leq m_j} \max_{\Omega \setminus V_j} |D^\alpha f|. \quad (5.2.1)
\]
Observe that $\tau_{SI}^\infty(\Omega)$ is not the metrizable topology induced by the countable family of seminorms $\varrho_{m,K_N}$ (here $K_N = \overline{V}_N$). It is a finer topology, which induces on $D_{K_N}(\Omega)$ the topology introduced in Definition 5.2.1.

We might argue that $\tau_{SI}^\infty(\Omega)$ may depend on the chosen exhaustion $\{K_N\}$ of $\Omega$, but, in fact, it is straightforward to prove that two different exhaustion give rise to the same topology. Henceforth, for every domain $\Omega$ we fix once and for all an exhaustion, and all seminorms are associated to that particular exhaustion.

**Proposition 5.2.5** Let $\{f_N\}$ be a sequence of functions in $D(\Omega)$. The following are equivalent:

(i) $\{f_N\}$ is convergent to the function $f \in D(\Omega)$ in the strict inductive limit topology $\tau_{SI}^\infty(\Omega)$;

(ii) there exists $\ell$ such that the support of $f_N$ is contained in $\overline{V}_\ell$ for every $N$ and for every multi-index $\alpha$ the sequence $\{D^\alpha f_N\}$ is uniformly convergent to $D^\alpha f$ on $\overline{V}_\ell$.

**Proof.** The proof is similar to the proof of Proposition 5.1.5 and it is left as an exercise.

**Corollary 5.2.6** Every Cauchy sequence for the topology $\tau_{SI}^\infty(\Omega)$ on $D(\Omega)$ is convergent to an element of $D(\Omega)$.

**Proof.** The proof is similar to the proof of Corollary 5.1.6 and it is left as an exercise.

**Definition 5.2.7** A distribution on $\Omega$ is a continuous linear functional on $D(\Omega)$. The linear space of all distributions on $\Omega$ is denoted by $D'(\Omega)$. We write $D$ and $D'$ instead of $D(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$, respectively.

It is a natural question to ask whether there are simple criteria which imply that a linear functional on $D(\Omega)$ is also continuous (with respect to the $\tau_{SI}^\infty$ topology). The following theorem gives a handy characterisation of distributions.

**Theorem 5.2.8** Suppose that $T$ is a linear functional on $D(\Omega)$. The following are equivalent:
5.2. SMOOTH FUNCTIONS AND DISTRIBUTIONS

(i) $T$ is a distribution (i.e., $T$ is continuous with respect to the $\tau^{\infty}_{SI}$ topology);

(ii) for every compact set $K \subset \Omega$, there exist a constant $C_K$ and a number $m_K$ such that

$$|\langle \varphi, T \rangle| \leq C_K \varrho_{m_K,K}(\varphi) \quad \forall \varphi \in \mathcal{D}_K(\Omega)$$

(see Definition 5.2.1 for the meaning of $\varrho_{j,K}$).

**Proof.** Suppose first that $T \in \mathcal{D}'(\Omega)$. Then for every $\varepsilon > 0$ there exist sequences $\eta$ and $m$ such that

$$|\langle \varphi, T \rangle| < \varepsilon \quad \forall \varphi \in \mathcal{V}_{\eta,m}.$$  

In particular, if $K$ is a compact subset of $\Omega$, $\varphi \in \mathcal{D}_K(\Omega)$, and $K \subset V_\ell$, then

$$\varrho'_{h,\Omega \setminus V_j}(\varphi) = 0 \quad \forall j \geq \ell \quad \forall h \in \mathbb{N},$$

because $\varphi$ vanishes identically in $\Omega \setminus V_j$. Since $\eta$ is decreasing and $m$ is increasing, the set

$$\{\varphi \in \mathcal{D}_K(\Omega) : \varrho_{m,K}(\varphi) < \eta_\ell\}$$

is contained in $\mathcal{V}_{\eta,m}$ (check the details). Now, suppose that $\psi$ is any function in $\mathcal{D}_K(\Omega)$ not identically equal to 0. We claim that the function

$$\tilde{\psi} := \eta_\ell \frac{\psi}{\varrho_{m,K}(\psi)}$$

belongs to $\mathcal{V}_{\eta,m}$. Indeed, $\varrho'_{h,\Omega \setminus V_j}(\tilde{\psi}) = 0$ for every $j \geq \ell$, and, if $j < \ell$, then

$$\varrho'_{m_j,\Omega \setminus V_j}(\tilde{\psi}) = \max_{|\alpha| \leq m_j} \max_{\Omega \setminus V_j} |D^\alpha \tilde{\psi}|$$

$$\leq \max_{|\alpha| \leq m_j} \max_K |D^\alpha \tilde{\psi}|$$

$$= \eta_\ell \max_{|\alpha| \leq m_j} \max_K \frac{|D^\alpha \psi|}{\varrho_{m,K}(\psi)}$$

$$< \eta_\ell$$

$$< \eta_j,$$

the last inequality being a consequence of the facts that $\eta$ is decreasing and $j < \ell$. Therefore

$$\varepsilon > |\langle \tilde{\psi}, T \rangle| = \eta_\ell \frac{|\langle \psi, T \rangle|}{\varrho_{m,K}(\psi)},$$
whence
\[ |\langle \tilde{\psi}, T \rangle| < \frac{\varepsilon}{\eta} \varrho_{m,K}(\psi), \]
as required, with \( C_K := \varepsilon/\eta \).

Conversely, suppose that for every compact set \( K \subset \Omega \) there exist a constant \( C_K \) and an integer \( j_K \) such that
\[ |\langle \psi, T \rangle| \leq C_K \varrho_{j_K,K}(\psi) \quad \forall \psi \in \mathcal{D}_K(\Omega). \tag{5.2.2} \]
We must show that \( T \) is a distribution on \( \Omega \), i.e., that for every \( \varepsilon > 0 \) there exist two sequences \( \eta \) and \( m \) such that
\[ |\langle \varphi, T \rangle| < \varepsilon \quad \forall \varphi \in \mathcal{V}_{\eta,m}. \]
Choose as \( m \) any increasing sequence \( \{m_\ell\} \) such that \( m_\ell \geq j_\ell \). There exist test functions \( \{\omega_\ell\} \) such that
\[ \text{supp} \left( \omega_0 \right) \subset V_2 \quad \text{and} \quad \text{supp} \left( \omega_\ell \right) \subset V_{\ell+2} \setminus \overline{V_\ell} \quad \ell = 1, 2, \ldots \]
and
\[ 1 = \sum_{\ell=0}^{\infty} \omega_\ell(x) \quad \forall x \in \Omega. \]
Given a test function \( \varphi \) on \( \Omega \), write \( \varphi = \sum_{\ell=0}^{\infty} \omega_\ell \varphi \). Observe that only finitely many summands in the series above do not vanish identically on \( \Omega \), for \( \varphi \) has compact support.

For the sake of simplicity we write \( \varrho_{\ell+2} \) instead of \( \varrho_{\sigma_{\ell+2},\tau_{\ell+2}} \). It is straightforward to show that there exists a constant \( C_{\ell+2} \) such that
\[ \varrho_{\ell+2}(\omega_\ell \varphi) \leq C_{\ell+2} \varrho'_{\sigma_{\ell+2},\tau_{\ell+2}}(\Omega \setminus \overline{V_\ell})(\varphi) \]
(expand the derivatives of \( \omega_\ell \varphi \) with Leibnitz’s formula). Denote by \( \eta \) any decreasing sequence of positive numbers such that
\[ \eta_\ell < \frac{2^{-\ell-1} \varepsilon}{C_{\tau_{\ell+2}} C_{\ell+2}}. \]
Note that for every \( \varphi \in \mathcal{V}_{\eta, m} \)
\[
|\langle \varphi, T \rangle| \leq \sum_{\ell=0}^{\infty} |\langle \omega_\ell \varphi, T \rangle| \\
\leq \sum_{\ell=0}^{\infty} C_{\tau_{\ell+2}} g_{\ell+2}(\omega_\ell \varphi) \\
\leq \sum_{\ell=0}^{\infty} C_{\tau_{\ell+2}} C_{\ell+2} g_{\tau_{\ell+2}} \Omega_{\eta, \ell}(\varphi) \\
< \sum_{\ell=0}^{\infty} 2^{-\ell-1} \\
\leq \varepsilon,
\] (5.2.3)
as required.

The proof of the theorem is complete. \( \square \)

A consequence of the characterisation above is that each locally integrable function and, more generally, every complex measure (in the sense of Section 5.1) on \( \Omega \), is a distribution on \( \Omega \). This partially justifies the term *generalised function* often used to denote a distribution.

**Proposition 5.2.9** Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \). The following hold:

(i) \( L^p(\Omega) \) is included in \( \mathcal{D}'(\Omega) \) for each \( p \in [1, \infty] \);

(ii) every locally integrable function \( f \) on \( \Omega \) “is” a distribution on \( \Omega \).

**Proof.** First we prove (i). Suppose that \( f \) is in \( L^p(\Omega) \), and consider the linear functional \( T_f \) on \( \mathcal{D}(\Omega) \), given by
\[
\langle \varphi, T_f \rangle = \int_{\Omega} \varphi f \, d\lambda \quad \forall \varphi \in \mathcal{D}(\Omega),
\]
where \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}^n \). We prove that \( T_f \) is continuous with respect to the \( \tau_{\mathcal{S}_f} \) topology. Suppose that \( K \) is a compact subset of \( \Omega \). Observe that for every \( \varphi \in \mathcal{D}_K(\Omega) \)
\[
|\langle \varphi, T_f \rangle| \leq \sup_{K} |\varphi| \left( \int_{K} |f| \, d\lambda \right) \\
\leq \sup_{K} |\varphi| \|f\|_{L^p(K)} \lambda(K)^{1/p'}
\]
and the continuity of $T_f$ follows from Theorem 5.2.8.

The proof of (ii) is left as an exercise. □

Note that two locally integrable functions which are equal a.e. represent the same distribution.

**Exercise 5.2.10** Prove that every complex measure is a distribution.

**Exercise 5.2.11** Prove that the linear functional $T$ on $\mathcal{D}(\mathbb{R})$, defined by
\[ \langle \varphi, T \rangle := \varphi'(0), \]
is a distribution. More generally, prove that for every multiindex $\alpha$, and for every $x \in \Omega$, the linear functional $T_{x,\alpha}$ on $\mathcal{D}(\Omega)$, defined by
\[ \langle \varphi, T_{x,\alpha} \rangle := D^\alpha \varphi(x), \]
is a distribution.

### 5.3 Derivatives of distributions

One of the motivations for the introduction of distributions is that every distribution may be differentiated infinitely many times. More precisely, for every multi-index $\alpha$, we may define a continuous linear map $D^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ with the property that if $T$ is a distribution which may be represented by a function $f_T$ in $C^{1|\alpha}(\Omega)$, then $D^\alpha T$ is represented by the usual derivative $D^\alpha f_T$. Thus, the notion of derivative of distributions generalises that of derivative of smooth functions.

If $f$ is in $C^1(\Omega)$, then for every test function $\varphi$ in $\mathcal{D}(\Omega)$ we may write
\[ \langle \varphi, \partial_k f \rangle = \int_\Omega \varphi(x) \partial_k f(x) \, dx \]
(by integrating by parts)
\[ = -\int_\Omega \partial_k \varphi(x) f(x) \, dx \]
\[ = -\langle \partial_k \varphi, f \rangle; \]

note that there are no boundary terms in the integration by parts above, for $\varphi$ vanishes in a neighbourhood of the boundary of $\Omega$. Thus, for any continuously differentiable function $f$ in $\Omega$, we have the formula
\[ \langle \varphi, \partial_k f \rangle = -\langle \partial_k \varphi, f \rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \]

The key observation is that the right hand side makes sense for every distribution $f$. This leads to the following definition.
### Definition 5.3.1

The $j^{th}$ **partial derivative** of the distribution $T$ in $\mathcal{D}'(\Omega)$ is the distribution $\partial_j T$, defined by

$$
\langle \varphi, \partial_j T \rangle = -\langle \partial_j \varphi, T \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).
$$

Of course we need to check that the right hand side of the formula above defines a distribution, i.e., the linear functional $\varphi \mapsto -\langle \partial_j \varphi, T \rangle$ is continuous for the strict inductive limit topology of $\mathcal{D}(\Omega)$. By Theorem 5.2.8, $\partial_j T$ is continuous if and only if for every compact set $K \subset \Omega$, there exist a constant $C_K$ and a nonnegative integer $m_K$ such that

$$
|\langle \psi, \partial_j T \rangle| \leq C_K \varrho^{m_K} \langle \varphi, T \rangle \quad \forall \psi \in \mathcal{D}_K(\Omega).
$$

(5.3.1)

Since $T$ is a distribution, for every compact set $K \subset \Omega$, there exists a constant $C_K$ and an integer $m'_K$ such that

$$
|\langle \varphi, T \rangle| \leq C_K \varrho^{m'_K} \langle \varphi, T \rangle \quad \forall \varphi \in \mathcal{D}_K(\Omega).
$$

Therefore

$$
|\langle \varphi, \partial_j T \rangle| = |\langle \partial_j \varphi, T \rangle| \leq C_K \varrho^{m'_K} \langle \partial_j \varphi, T \rangle \leq C_K \varrho^{m'_K+1} \langle \varphi, T \rangle \quad \forall \varphi \in \mathcal{D}_K(\Omega),
$$

so that (5.3.1) is satisfied with $m_K = m'_K + 1$, as required.

It is clear that we can iterate the reasoning above, and define $D^\alpha T$ for every distribution $T$. This will be the distribution defined by

$$
\langle \varphi, D^\alpha T \rangle = (-1)^{\alpha} \langle D^\alpha \varphi, T \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).
$$

(5.3.2)

Note, in particular, that $\partial^2_{j,k} T = \partial^2_{k,j} T$ for every pair of indices.

### Example 5.3.2

Consider the absolute value function $T$ on the real line

$$
T(x) = |x| \quad \forall x \in \mathbb{R}.
$$

Clearly $T$ is differentiable in the classical sense off the origin with derivative

$$
T'(x) = \text{sgn}(x),
$$

where $\text{sgn}$ denotes the signum function, which vanishes at 0, is equal to $-1$ on the negative real line and to $+1$ on the positive real line. We claim that
$T' = \text{sgn}$ in the sense of distributions. Indeed, by the definition of derivative of a distribution

$$
\langle \varphi, T' \rangle = - \langle \varphi', T \rangle
$$

($T$ is a loc. int. function)

$$
= - \int_{-\infty}^{0} \varphi'(x) |x| \, dx
$$

(by parts)

$$
= - \int_{-\infty}^{0} \varphi(x) \, dx + \int_{0}^{\infty} \varphi(x) \, dx
$$

for all $\varphi \in \mathcal{D}$, as claimed.

**Example 5.3.3** We show that $H' = \delta_0$ in the sense of distributions. Here $H$ is the **Heaviside function**, which vanishes in $(-\infty, 0]$ and is equal to 1 in $(0, \infty)$, and $\delta_0$ denotes the **Dirac mass at the origin**. Indeed,

$$
\langle \varphi, H' \rangle = - \langle \varphi', H \rangle
$$

($H$ is a loc. int. function)

$$
= - \int_{0}^{\infty} \varphi'(x) \, dx
$$

$$
= \varphi(0)
$$

$$
= \langle \varphi, \delta_0 \rangle \quad \forall \varphi \in \mathcal{D},
$$

as required. Similarly, for every positive integer $k \geq 2$ one can prove that

$$
\langle \varphi, \delta_0^{(k)} \rangle = (-1)^k \varphi^{(k)}(0) \quad \forall \varphi \in \mathcal{D}.
$$

**Example 5.3.4** Consider the distribution $T$, which agrees with the locally integrable function $x \mapsto x^{-1/2} 1_{(0, \infty)}(x)$. We want to compute $T'$ (in the sense of distributions). Clearly, $T'$ is not the function $x \mapsto -(1/2) x^{-3/2} 1_{(0, \infty)}(x)$,
for this function is not locally integrable. Observe that
\[
\langle \varphi, T' \rangle = -\langle \varphi', T \rangle
\]
(T is a loc. int. function) 
\[
= -\int_0^\infty \varphi'(x) x^{-1/2} \, dx \\
= -\lim_{\varepsilon \downarrow 0} \int_0^\varepsilon \varphi'(x) x^{-1/2} \, dx \\
\text{(by parts)} \\
= -\lim_{\varepsilon \downarrow 0} \left[ -\varphi(\varepsilon) \varepsilon^{-1/2} - \frac{1}{2} \int_\varepsilon^\infty \varphi(x) x^{-3/2} \, dx \right] \\
= -\lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_\varepsilon^\infty \frac{\varphi(x) - \varphi(\varepsilon)}{x^{3/2}} \, dx \\
= -\frac{1}{2} \int_0^\infty \frac{\varphi(x) - \varphi(0)}{x^{3/2}} \, dx \\
\forall \varphi \in \mathcal{D}.
\]
The last inequality is justified by an application of the Lebesgue dominated convergence theorem. We leave the details to the reader. We already know that the linear functional
\[
\varphi \mapsto -\frac{1}{2} \int_0^\infty \frac{\varphi(x) - \varphi(0)}{x^{3/2}} \, dx
\]
is continuous on \( \mathcal{D} \), because it is the distributional derivative of a distribution. Prove directly the continuity of the above functional. Note also that “the restriction” of \( T' \) to \((0, \infty)\) “agrees” with the function \(-\frac{1}{2} x^{-3/2}\) in the following sense: for every \( \varphi \) in \( \mathcal{D} \) with support contained in \((0, \infty)\), then
\[
\langle \varphi, T' \rangle = -\frac{1}{2} \int_0^\infty \frac{\varphi(x)}{x^{3/2}} \, dx.
\]

**Example 5.3.5 (Cauchy’s principal value)** Denote by \( T \) the linear functional on \( \mathcal{D}(\mathbb{R}) \), defined by
\[
\langle \varphi, T \rangle = \lim_{\varepsilon \to 0^+} \int_{I_\varepsilon} \frac{\varphi(x)}{x} \, dx \\
\forall \varphi \in \mathcal{D}(\mathbb{R}),
\]
where \( I_\varepsilon = [-\varepsilon, \varepsilon] \). We show that \( T \) is a distribution. It suffices to prove that the restriction of \( T \) to \( \mathcal{D}_{[-R,R]}(\mathbb{R}) \) is a continuous linear functional (with respect to the topology of \( \mathcal{D}_{[-R,R]}(\mathbb{R}) \)). Observe that if \( 0 < \varepsilon < 1 \), then we may write
\[
\langle \varphi, T \rangle = \lim_{\varepsilon \to 0^+} \int_{I_\varepsilon} \frac{\varphi(x)}{x} \, dx + \int_{I_{1/\varepsilon}} \frac{\varphi(x)}{x} \, dx.
\]
and that
\[ \left| \int_{I_{\varepsilon}} \frac{\varphi(x)}{x} \, dx \right| \leq \|x \varphi\|_{\infty} \int_{I_{\varepsilon}} \frac{1}{x^2} \, dx \leq 2R \|\varphi\|_{\infty}. \]
Furthermore
\[ \left| \lim_{\varepsilon \to 0^+} \int_{I_1 \setminus I_{\varepsilon}} \frac{\varphi(x)}{x} \, dx \right| = \left| \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{\varphi(x) - \varphi(-x)}{x} \, dx \right| \]
\[ = \left| \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x} \int_{-x}^{x} \varphi'(s) \, ds \, dx \right| \leq 2 \|\varphi'\|_{\infty} \forall \varphi \in D_{[-R,R]}(\mathbb{R}). \]
Hence
\[ \langle \varphi, T \rangle \leq C_{R} (\|\varphi'\|_{\infty} + \|\varphi\|_{\infty}) \forall \varphi \in D_{[-R,R]}(\mathbb{R}). \]
Therefore the restriction of \(T\) to \(D_{[-R,R]}(\mathbb{R})\) is continuous, as required.

**Exercise 5.3.6** Prove that \(\delta_0'\) is not a measure.

**Exercise 5.3.7** Compute the derivative (in the sense of distributions) of the distribution \(T\), which agrees with the locally integrable function \(x \mapsto |x|^{-1/2}\).

**Exercise 5.3.8** Compute the second derivative (in the sense of distributions) of the distribution \(T\), which agrees with the locally integrable function \(x \mapsto x^{-1/2} \mathbf{1}_{(0,\infty)}(x)\).

**Exercise 5.3.9** Compute the derivative of the distribution \(T\), which agrees with the locally integrable function \(x \mapsto \log |x|\). (Answer: \(T' = \text{p.v.}(1/x)\).)

**Exercise 5.3.10** Compute the derivative of the distribution \(T\), which agrees with the locally integrable function \(x \mapsto 1_{R^+}(x) \log |x|\).

**Exercise 5.3.11** Compute the derivative of the distribution “Cauchy principal value”.

**Exercise 5.3.12** Compute \(T''\), where \(T\) is the distribution on the real line that agrees with the locally integrable function \(x \mapsto \max(1 - |x|, 0)\).
Exercise 5.3.13 Denote by $T$ the linear functional on $\mathcal{D}(\mathbb{R}^2)$, defined by

$$\langle \varphi, T \rangle = \int_{\mathbb{R}} \varphi(x,x) \, dx.$$ 

Prove that $T$ is a distribution, and compute $\partial_{x_1} T + \partial_{x_2} T$.

Exercise 5.3.14 Denote by $\sigma$ the normalized surface measure of the unit sphere in $\mathbb{R}^2$, and let $f$ and $g$ be the functions on $\mathbb{R}^2$, defined by

$$f(x,y) = \max(1-\sqrt{x^2+y^2},0) \quad \text{and} \quad g(x,y) = (x^2+y^2)^{-1/2} 1_{B(0,1)}(x,y).$$

Prove that $\Delta f = \sigma - g$ in the sense of distributions.

Exercise 5.3.15 For each $r$ in $\mathbb{R}^+$ denote by $\sigma_r$ the surface measure of the sphere with centre 0 and radius $r$ in $\mathbb{R}^n$. Prove that

$$\lim_{r \to 0^+} \frac{2n}{r^2} \left[ \frac{1}{s_n r^{n-1}} \sigma_r - \delta_0 \right] = \Delta \delta_0$$

in the sense of distributions. Here $s_n$ denotes the surface measure of the unit sphere.

Exercise 5.3.16 Denote by $f$ the distribution which agrees on $\mathbb{R}^2 \setminus \{(0,0)\}$ with the function $(x,y) \mapsto \sqrt{x^2+y^2}$. Compute the partial derivatives of $f$ up to the second order, and, then the Laplacian of $f$.

Exercise 5.3.17 Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $\{f_N\}$ a sequence in $L^1_{\text{loc}}(\Omega)$ such that

$$\lim_{N \to \infty} \int_{\Omega} |f_N| \, d\lambda = 0$$

for every $K \subset \subset \Omega$. Show that the sequence $\{D^\alpha f_N\}$ is convergent to 0 in $\mathcal{D}'(\Omega)$ for every multi-index $\alpha$.

5.4 Fundamental solutions

In this section we introduce the concept of fundamental solution of a differential operator, and illustrate its application to the solution of the Poisson equation.
Definition 5.4.1 Suppose that $J$ is a nonnegative integer and that

$$P(x) := \sum_{|\alpha| \leq J} a_{\alpha} x^{\alpha}.$$ 

We denote by $P(D)$ the differential operator, defined by

$$P(D)f = \sum_{|\alpha| \leq J} a_{\alpha} D^{\alpha} f.$$ 

Definition 5.4.2 A distribution $E$ is a fundamental solution of the differential operator $P(D)$ if

$$P(D)E = \delta_0.$$ 

The following remarkable result is due to Malgrange and Ehrenpreis.

Theorem 5.4.3 Every differential operator $P(D)$ admits a fundamental solution.

We do not prove this result. The interested reader is referred to [Fo2, Thm 1.56, p. 62] or [Me, Thm 3.3.20].

Proposition 5.4.4 The Newtonian potential is a fundamental solution of the Laplace operator.

Proof. We already know that $N$ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Thus,

$$\langle \varphi, \Delta N \rangle = 0 = \langle \varphi, \delta_0 \rangle \quad (5.4.1)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that 0 does not belong to the support of $\varphi$.

Now suppose that 0 belongs to the support of $\varphi$. Then

$$\langle \varphi, \Delta N \rangle = \langle \Delta \varphi, N \rangle$$

$$= \int_{\mathbb{R}^n} \Delta \varphi \ N \ dV$$

$$= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta \varphi \ N \ dV.$$

The last equality follows from the local integrability of $N$ near the origin and the Lebesgue dominated convergence theorem. Suppose that the support of $\varphi$
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is contained in $B_R(0)$. Then, by the second Green’s identity the harmonicity of $N$ in $\mathbb{R}^n \setminus B_\varepsilon(0)$ and the fact that $\varphi$ vanishes on $\partial B_R(0)$,

$$
\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta \varphi \ N \, dV = \int_{\partial B_\varepsilon(0)} \left[ \varphi \partial_\nu N - \partial_\nu \varphi \ N \right] \, d\sigma,
$$

where $\nu$ denotes the outward unit normal to $\partial B_\varepsilon(0)$. By arguing as in the proof of Theorem 1.5.5, we see that

$$
\int_{\partial B_\varepsilon(0)} \varphi \partial_\nu N \, d\sigma \to \varphi(0)
$$

and that

$$
\int_{\partial B_\varepsilon(0)} \partial_\nu \varphi \ N \, d\sigma \to 0
$$
as $\varepsilon \downarrow 0$. Thus, we have proved that

$$
\langle \varphi, \Delta N \rangle = \varphi(0) = \langle \varphi, \delta_0 \rangle
$$

for all $\varphi$, whose support contains $0$. This, together with (5.4.1), proves the required result.

\[\square\]

**Exercise 5.4.5** Prove that the function $E(x) := \frac{1}{2} \frac{|x|}{2}$ is a fundamental solution of the operator $d^2/dx^2$ (with pole $0$).

**Exercise 5.4.6** Prove that the function

$$
u(x_1, x_2) := \begin{cases} 
1 & \text{for } x_1 > \xi_1, x_2 > \xi_2 \\
0 & \text{elsewhere}
\end{cases}
$$
is a fundamental solution of the operator $\partial^2/\partial x_1 \partial x_2$ with pole $(\xi_1, \xi_2)$.

**Exercise 5.4.7** Prove that for every complex number $c$ the function

$$
u_c(x) := e^{-c|x|} \frac{1}{4\pi|x|} \quad \forall x \in \mathbb{R}^3 \setminus \{0\}
$$
is a fundamental solution of the operator $-\Delta + c^2$ with pole $0$.

We shall apply Proposition 5.4.4 to the solution of the Poisson equation.

**Definition 5.4.8** Suppose that $P(D)$ is a differential operator with constant coefficients and that $f$ is a distribution on $\Omega$. We say that a distribution $T$ on $\Omega$ is a **distributional solution** of the equation $P(D)u = f$ if

$$
\langle \varphi, P(D)T \rangle = \langle \varphi, f \rangle \quad \forall \varphi \in C_c^\infty(\Omega).
$$
Note that no a priori regularity assumptions are made on the solution $u$.

Observe that if $T$ is a classical solution to the Poisson equation (then $f$ must be continuous), then a repeated integration by parts shows that $T$ is a distributional solution.

Now we show how the fundamental solution of the Laplacian can be used to solve Poisson’s equation. We need the definition of convolution between a distribution with compact support and a test function.

**Definition 5.4.9** Suppose that $T$ is a distribution and that $\varphi$ is in $\mathcal{D}$. Then the convolution $T * \varphi$ is the distribution, defined by

$$\langle \psi, T * \varphi \rangle := \langle \psi * \tilde{\varphi}, T \rangle \quad \forall \psi \in \mathcal{D}. \quad (5.4.2)$$

**Exercise 5.4.10** Prove that $T * \varphi$ defines indeed a distribution on $\mathbb{R}^n$, i.e., show that the linear functional on $\mathcal{D}$, defined by (5.4.10), is continuous with respect to the $\tau^\infty_{\mathcal{S}}$ topology.

**Exercise 5.4.11** Suppose that $T$ is a distribution with compact support. This means that there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\langle \varphi, T \rangle = 0 \quad \forall \varphi : \text{supp} (\varphi) \cap K = \emptyset.$$

Prove that the linear functional $T * N$ on $\mathcal{D}$, defined by

$$\langle \varphi, T * N \rangle = \langle \varphi * \tilde{N}, T \rangle \quad \forall \varphi \in \mathcal{D},$$

is a distribution on $\mathbb{R}^n$. **Hint:** Observe that $\varphi * \tilde{N}$ is a smooth function, and that its behaviour outside $K$ is “irrelevant”, for the support of $T$ is contained in $K$.

**Proposition 5.4.12** Suppose that $f$ is a distribution on $\mathbb{R}^n$ with compact support. The following hold:

(i) the distribution $N * f$ is a distributional solution of the equation $\Delta u = f$;

(ii) if $f \in C^2_0 (\mathbb{R}^n)$, then $N * f$ is a classical solution (i.e., a $C^2 (\mathbb{R}^n)$ solution) of the equation $\Delta u = f$.

**Proof.** To prove (i), observe that the convolution $N * f$ is well defined, because $f$ has compact support. Then, by the definition of distributional Laplacian and of convolution of distributions,

$$\langle \varphi, \Delta (N * f) \rangle = \langle \Delta \varphi, N * f \rangle$$

$$= \langle (\Delta \varphi) * \tilde{N}, f \rangle. \quad (5.4.3)$$
Recall that $\tilde{N} = N$, and observe that

$$\int_{\mathbb{R}^n} \Delta \varphi(y) N(x - y) \, dV(y) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \Delta \varphi(y) N(x - y) \, dV(y),$$

the last equality being a consequence of the dominated convergence theorem. Now we apply the second Green’s identity to the domain $B_R(0) \setminus B_\varepsilon(x)$, where $R$ is chosen so large that $B_R(0)$ contains the support of $\varphi$. We see that

$$\int_{\partial B_\varepsilon(x)} \varphi(Y) \partial_\nu N(x - Y) \, d\sigma(Y).$$

By arguing much as in the proof of Theorem 1.5.5 (i), we see that

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(x)} \varphi(Y) \partial_\nu N(x - Y) \, d\sigma(Y) = \varphi(x)$$

and

$$\lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(x)} \partial_\nu \varphi(Y) N(x - Y) \, d\sigma(Y) = 0.$$

Therefore

$$\langle (\Delta \varphi) \ast N, f \rangle = \langle \varphi, f \rangle.$$

The formula above, together with (5.4.3), shows that $\Delta(\ast f) = f$ in the sense of distributions, as required.

Next we prove (ii). We already know that $N \ast f$ is a smooth function on $\mathbb{R}^n$. It remains to prove that

$$\Delta(\ast f)(x) = f(x) \quad \forall x \in \mathbb{R}^n.$$

By (i), $\Delta(\ast f) = f$ in the sense of distributions. Since both $\Delta(\ast f)$ and $f$ are locally integrable functions (for they are continuous), they must be equal a.e. Since both $\Delta(\ast f)$ and $f$ are continuous, they must agree at every point, as required.

### 5.5 The Sobolev spaces $H^k(\Omega)$

Suppose that $I$ is an open interval of the real line. Various notions of “regularity” of a distribution $u \in \mathcal{D}'(I)$ may be expressed by requiring that its distributional derivative $u'$ belongs to a certain function space. Here are some examples:
(i) \( u \) is in \( C^1(I) \) if and only if \( u' \) is in \( C(I) \);

(ii) \( u \) is in \( \text{Lip}(I) \) if and only if \( u' \) is in \( L^\infty(I) \);

(iii) \( u \) is absolutely continuous in \( I \) if and only if \( u' \) is in \( L^1(I) \);

(iv) \( u \) is of bounded variation in \( I \) if and only if \( u' \) is in \( \mathcal{M}(I) \), the class of all finite (complex) Borel measures on \( I \).

Note that if a distribution \( u \) "is" a function, then any function \( v \) which is equal to \( u \) a.e. represents the same distribution \( u \). Thus, for instance the statement (i) above should be interpreted in the following way: \( u \) is equal a.e. to a \( C^1(I) \) function if and only if \( u' \) (in the distributional sense) is equal a.e. to a continuous functions. Similar comments apply to the other statements above.

We prove statement (i) above in full details to illustrate the sort of subtleties which arise when distributional derivatives are considered. The proofs of statements (ii)-(iv) are similar, and are omitted.

Suppose first that \( u \) is a distribution in \( C^1(I) \). This means that there exists a function \( v \) in \( C^1(I) \) which agrees pointwise a.e. with \( u \). We want to show that the distributional derivative \( u' \) of \( u \) agrees pointwise a.e. with a continuous function \( C(I) \). For every \( \varphi \) in \( \mathcal{D}(I) \), we have

\[
\langle \varphi, u' \rangle = - \langle \varphi', u \rangle \\
= - \int_I \varphi'(x) u(x) \, dx \\
= - \int_I \varphi'(x) v(x) \, dx \\
= \int_I \varphi(x) v'(x) \, dx \\
= \langle \varphi, v' \rangle,
\]

so that \( u' = v' \) in the sense of distributions. Hence \( u' \) is a function, which agrees a.e. with the continuous function \( v' \).

Conversely, assume that \( u' \) agrees (as a distribution) with a function \( w \) in \( C(I) \). Fix a point \( x_0 \) in \( I \) and define

\[
v(x) = \int_{x_0}^x w(t) \, dt \quad \forall x \in I.
\]

Clearly, \( v \) is a distribution in \( C^1(I) \) and \( v' = w \). Any other distribution of the form \( v + c \), where \( c \) is a constant, satisfies \( (v + c)' = w \). Let \( U \) be
a distribution such that \( U' = w \) (in the sense of distributions). May we assert that \( U \) agrees with \( v + c \) for some constant \( c \)? In other words, if the distributional derivative \( T' \) of a distribution \( T \) vanishes, may we conclude that \( T \) is constant (as a distribution). The answer is “yes”, as the following result shows.

**Proposition 5.5.1** Suppose that \( I \) is an open interval of \( \mathbb{R} \) and that \( u \) is in \( \mathcal{D}'(I) \). If \( u' = 0 \) in the sense of distributions, then \( u \) is constant (as a distribution).

**Proof.** The assumption \( u' = 0 \) is equivalent to \( \langle \varphi', u \rangle = 0 \) for every \( \varphi \) in \( \mathcal{D}(I) \).

First we prove that \( \langle \psi, u \rangle = 0 \) for every \( \psi \) in \( \mathcal{D}(I) \) with vanishing integral. Indeed, if \( \int_I \psi(x) \, dx = 0 \), then its primitive function

\[
\varphi(x) := \int_{-\infty}^x \psi(s) \, ds \quad \forall x \in \mathbb{R}
\]

is a function in \( \mathcal{D}(I) \), and \( \varphi' = \psi \). Hence \( \langle \psi, u \rangle = 0 \) by assumption.

Now, suppose that \( \psi \) is a generic function in \( \mathcal{D}(I) \). Denote by \( \psi_0 \) a function in \( \mathcal{D}(I) \) such that \( \int_I \psi_0(x) \, dx = 1 \). The function \( \psi - \left( \int_I \psi(x) \, dx \right) \psi_0 \) is in \( \mathcal{D}(I) \) and its integral vanishes. By the first part of the proof, it follows that

\[
\left\langle \psi - \left( \int_I \psi(x) \, dx \right) \psi_0, u \right\rangle = 0,
\]

i.e.

\[
\langle \psi, u \rangle = \left( \int_I \psi(x) \, dx \right) \langle \psi_0, u \rangle.
\]

Therefore \( u = \langle \psi_0, u \rangle \, 1_I \), as required.

Coming back to the point, we may now assert that \( u = v + c \) (in the sense of distributions) for some constant \( c \), hence \( u \in C^1(I) \), as required.

**Exercise 5.5.2** Prove that the function \( f \), defined by

\[
f(x) := \begin{cases} 
x^2 \sin(1/x^2) & \text{if } x \neq 0 \\
0 & \text{if } x = 0,
\end{cases}
\]

is differentiable (in the classical sense) at every point \( x \in \mathbb{R} \), and compute its derivative. Show that the distributional derivative \( f' \) of \( f \) does not agree...
with its classical derivative. Prove that
\[ \langle \varphi, f' \rangle = \int_{\mathbb{R}} \varphi(x) 2x \sin(1/x^2) \, dx - 2 \int_{-1}^{1} \frac{\varphi(x) - \varphi(0)}{x} \cos(1/x^2) \, dx \]
\[ - 2 \int_{\mathbb{R} \setminus [-1,1]} \frac{\varphi(x)}{x} \cos(1/x^2) \, dx. \]

An important notion of “regularity” of a distribution \( u \) is obtained by requiring that one or more derivatives of \( u \) belong to some \( L^p \) space. We shall mainly concerned with the case where \( p = 2 \), but the other values of \( p \) are important as well. The spaces thus obtained are referred to as Sobolev spaces. Before going into the details, it may be helpful to discuss the main motivation behind the introduction of such spaces.

There are many ways of defining the notion of function with derivative in \( L^2(I) \). Most of them are equivalent and useful. There is one which is definitely not good, i.e., that there exists a function \( g \in L^2(I) \) such that
\[ \frac{f(x+h) - f(x)}{h} \rightarrow g(x) \]
pointwise a.e.. The following example illustrate this statement.

**Example 5.5.3** Denote by \( F : [0,1] \rightarrow [0,1] \) the Cantor function. Extend \( F \) to a continuous function on \( \mathbb{R} \), still denoted by \( F \), which is equal to 0 when \( x < 0 \) and equal to 1 when \( x > 1 \). We compute the distributional derivative of \( F \).

It is well known that \( F \) is differentiable in the classical sense in \( \mathbb{R} \setminus C_{1/3} \), where \( C_{1/3} \) denotes Cantor’s ternary set, and that the derivative is equal to 0 therein. Denote by \( \mu^F \) the Lebesgue–Stieltjes measure on \( \mathbb{R} \) associated to \( F \). In particular,
\[ \mu^F((a, b]) = F(b) - F(a) \quad \forall a, b \in \mathbb{R} : a \leq b. \]

Now, for every \( \varphi \) in \( D \)
\[ \langle \varphi, F' \rangle = - \langle \varphi', F \rangle \]
\[ = - \int_{\mathbb{R}} \varphi'(x) F(x) \, dx \]
\[ = - \int_{\mathbb{R}} \, dx \varphi'(x) \int_{-\infty}^{x} \, d\mu^F(y) \]
By Fubini’s theorem,
\[ \langle \varphi, F' \rangle = -\int_{\mathbb{R}} d\mu^F(y) \int_{y}^\infty \varphi'(x) \, dx \]
\[ = \int_{\mathbb{R}} \varphi(y) \, d\mu^F(y) \]
\[ = \langle \varphi, \mu^F \rangle. \]

Therefore \( F' = \mu^F \) in the sense of distributions. Note that if \( \varphi \in \mathcal{D} \) with support contained in \( \mathbb{R} \setminus C_{1/3} \), then \( \langle \varphi, F' \rangle = 0 \). Therefore, the support of \( \mu^F \) is contained in the Cantor ternary set.

The following result shows that there are at least five equivalent definitions of \( L^2 \)-derivative of a function in \( L^2(\mathbb{R}^n) \).

**Proposition 5.5.4** Suppose that \( f \in L^2(\mathbb{R}^n) \). The following are equivalent:

(i) the distributional derivative \( \partial_1 f \) is in \( L^2(\mathbb{R}^n) \);

(ii) \( \xi_1 \hat{f}(\xi_1, \ldots, \xi_n) \) is in \( L^2(\mathbb{R}^n) \);

(iii) \( h^{-1} \left[ f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n) \right] \) is convergent in \( L^2(\mathbb{R}^n) \) as \( h \) tends to 0;

(iv) there exists a sequence \( \{ f_n \} \) of test functions such that
\[ \| f - f_n \|_{L^2(\mathbb{R}^n)} \to 0 \]
and \( \{ \partial_1 f_n \} \) is a Cauchy sequence in \( L^2(\mathbb{R}^n) \);

(v) there exists \( g \) in \( L^2(\mathbb{R}^n) \) such that
\[ -\int_{\mathbb{R}^n} \partial_1 \varphi \, f \, dV = \int_{\mathbb{R}^n} \varphi \, g \, dV \quad \forall \varphi \in C^1_c(\Omega). \]

**Proof.** We prove only that (i) and (v) are equivalent. The rest of the proof is left to the interested reader.

Suppose that (i) holds. Then
\[ \int_{\mathbb{R}^n} \varphi \, \partial_1 f \, dV = -\int_{\mathbb{R}^n} \partial_1 \varphi \, f \, dV \quad \forall \varphi \in \mathcal{D}. \quad (5.5.1) \]
We must show that (5.5.1) holds for all \( \varphi \in C^1(\mathbb{R}^n) \). For \( \varphi \in C^1(\mathbb{R}^n) \), consider \( \varphi * \psi_\varepsilon \), where \( \psi \in D_{B_1(0)}(\mathbb{R}^n) \) is such that \( \int_{\mathbb{R}^n} \psi \, dV = 1 \), and
\[
\psi_\varepsilon(x) := \varepsilon^{-n} \psi(x/\varepsilon) \quad \forall x \in \mathbb{R}^n.
\]
Since the support of both \( \varphi \) and \( \psi \) is compact, so is the support of \( \varphi * \psi_\varepsilon \). Furthermore, \( \varphi * \psi_\varepsilon \) is smooth. Thus, we may rewrite (5.5.1) with \( \varphi * \psi_\varepsilon \) in place of \( \varphi \), and get
\[
\int_{\mathbb{R}^n} (\varphi * \psi_\varepsilon) \, \partial_1 f \, dV = -\int_{\mathbb{R}^n} \partial_1 (\varphi * \psi_\varepsilon) \, f \, dV = -\int_{\mathbb{R}^n} [(\partial_1 \varphi) * \psi_\varepsilon] \, f \, dV. \tag{5.5.2}
\]
Note that
\[
\| \varphi * \psi_\varepsilon - \varphi \|_2 \to 0 \quad \text{and} \quad \| (\partial_1 \varphi) * \psi_\varepsilon - \partial_1 \varphi \|_2 \to 0
\]
as \( \varepsilon \) tends to 0. Therefore, by taking the limit of both sides of (5.5.2), we obtain
\[
\int_{\mathbb{R}^n} \varphi \, \partial_1 f \, dV = -\int_{\mathbb{R}^n} \partial_1 \varphi \, f \, dV.
\]
Since this holds for every \( \varphi \in C^1_\text{c}(\mathbb{R}^n) \), (v) is proved, with \( g = \partial_1 f \). Conversely, if (v) holds, then, in particular,
\[
-\langle \partial_1 \varphi, f \rangle = \langle \varphi, g \rangle \quad \forall \varphi \in \mathcal{D}.
\]
Thus, \( g \) is the distributional derivative \( \partial_1 f \) of \( f \), whence \( \partial_1 f \in L^2(\mathbb{R}^n) \), as required.

**Definition 5.5.5** Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \) and that \( \alpha \) is a multindex. We say that a function \( u \in L^2(\Omega) \) admits a weak \( \alpha \)-derivative, if there exists a function \( g \in L^2(\Omega) \) such that
\[
(-1)^{|\alpha|} \int_{\mathbb{R}^n} (D^\alpha \varphi) \, u \, dV = \int_{\mathbb{R}^n} \varphi \, g \, dV \quad \forall \varphi \in C^1_\text{c}(\alpha)(\Omega).
\]
The function \( g \) is then called the *weak* \( \alpha \)-derivative of \( u \).

A slight generalisation to \( \alpha \)-derivatives on \( \Omega \) of the equivalence between (i) and (v) in Proposition 5.5.4 shows that for a function \( u \in L^2(\Omega) \) the following are equivalent:
5.5. SOBOLEV SPACES

(i) $u$ admits weak $\alpha$-derivative;

(ii) the distributional derivative $D^\alpha u$ of $u$ belongs to $L^2(\Omega)$.

We shall use this equivalence without any further comment in the sequel.

If $u \in C^k(\Omega)$, then an integration by parts shows that for every multi-index $\alpha$ with $|\alpha| \leq k$ the weak $\alpha$-derivative of $u$ agrees with the classical derivative $D^\alpha u$.

Exercise 5.5.6 Prove directly that the function $\text{sgn}$ on the real line does not admit a weak first derivative.

Definition 5.5.7 Suppose that $\Omega$ is a bounded open set in $\mathbb{R}^n$. Denote by $C^k(\bar{\Omega})$ the vector space of all functions $u$ in $C^k(\Omega)$ such that for every multi-index $\alpha$ such that $|\alpha| \leq k$, $D^\alpha u$ extends to a continuous function on $\bar{\Omega}$.

Definition 5.5.8 Suppose that $\Omega$ is a possibly unbounded open subset of $\mathbb{R}^n$. We denote by $H^k(\Omega)$ the space of all complex valued functions $u$ in $L^2(\Omega)$ whose distributional derivatives up to the order $k$ belong to $L^2(\Omega)$, endowed with the norm

\[
\|u\|_{H^k(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 \, d\lambda \right)^{1/2}.
\]

Note that this norm is associated to the following natural inner product in $H^k(\Omega)$

\[
(u, v) := \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u \overline{D^\alpha v} \, d\lambda.
\]

More generally, for each $p$ in $[1, \infty]$ we denote by $W^{k,p}(\Omega)$ the space of all complex valued functions $u$ in $L^p(\Omega)$ whose distributional derivatives up to the order $k$ belong to $L^p(\Omega)$, endowed with the norm

\[
\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, d\lambda \right)^{1/p}
\]

in the case where $p < \infty$, and with the norm

\[
\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\infty}
\]

in the case where $p = \infty$.

Clearly $W^{k,2}(\Omega) = H^k(\Omega)$. In the rest of these notes we focus on the case where $p = 2$. 

**Exercise 5.5.9** Suppose that Ω is a bounded domain in \( \mathbb{R}^n \) containing the origin. Prove that the function \( r_\alpha(x) := |x|^{-\alpha} \) is in \( H^k(\Omega) \) if and only if \( 2k + \alpha < n/2 \).

**Exercise 5.5.10** Consider the function
\[
u_\alpha(x) := (1 + x^2)^{-\alpha/2} \left( \log(2 + x^2) \right)^{-1} \quad \forall x \in \mathbb{R}.
\]
Prove that \( \nu_\alpha \in W^{1,p}(\mathbb{R}) \) for all \( p \in [1/\alpha, \infty] \) and that \( \nu_\alpha \notin L^q(\mathbb{R}) \) for all \( q \in [1, 1/\alpha) \).

The following results summarises some relevant properties of Sobolev spaces. For the proof, the reader is referred to [AF, Br, EG, GT, Fo2, Tr].

**Proposition 5.5.11** Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \). The following hold:

(i) \( H^k(\Omega) \) is an Hilbert space;

(ii) \( H^k(\Omega) \cap C^\infty(\Omega) \) is dense in \( H^k(\Omega) \).

To state the second result we need the following definition. Suppose that \( \alpha \) is a number in \((0, 1] \). We say that a function \( u \in C^k(\Omega) \) is in \( C^{k,\alpha}(\Omega) \) if for every multiindex \( \beta \) of order \( k \), the derivatives \( D^\beta u \) are Hölder continuous of order \( \alpha \) on \( \Omega \).

**Theorem 5.5.12** Suppose that \( \Omega \) is a bounded domain with Lipschitz boundary. The following hold:

(i) \( H^k(\Omega) \cap C^\infty(\overline{\Omega}) \) is dense in \( H^k(\Omega) \);

(ii) there exists a continuous linear map (called extension operator) \( E : H^1(\Omega) \to H^1(\mathbb{R}^n) \) such that \( Eu \) has compact support and \( Eu = u \) on \( \Omega \);

(iii) the imbedding \( H^1(\Omega) \to L^2(\Omega) \) is a compact operator;

(iv) if \( m > n/2 \), then \( H^m(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega}) \); here \( k := m - \lfloor n/2 \rfloor - 1 \), and, if \( n \) is odd, then \( \alpha := m - n/2 - k \), whereas, if \( n \) is even, then \( \alpha \) is any number in \((0, 1)\).
Remark 5.5.13 By Theorem 5.5.12 (i), the Sobolev space $H^m(\Omega)$ may be equivalently defined as the completion of $C^\infty(\Omega)$ with respect to the norm $\| \cdot \|_m$.

Remark 5.5.14 For a generic domain $\Omega$, the space $C^\infty(\Omega)$ is not dense in $H^k(\Omega)$, as the following example in one dimension shows. Denote by $\Omega$ the set $(-1/2, 0) \cup (0, 1/2)$ in $\mathbb{R}$. We argue by contradiction. Suppose that $C^\infty(\Omega)$ is dense in $H^1(\Omega)$. Denote by $f$ the function, which is equal to $-1$ on $(-1/2, 0)$ and equal to $1$ on $(0, 1/2)$. Clearly $f$ belongs to $H^1(\Omega)$ and $\| f \|_{H^1(\Omega)} = \| f \|_{L^2(\Omega)}$, because the distributional derivative of $f$ vanishes. Suppose that $\{ \varphi_n \}$ is a sequence of functions in $C^\infty([-1/2, 1/2])$ that is convergent to $f$ in $H^1(\Omega)$. Then $\{ \varphi_n \}$ is convergent to $f$ in $L^2(\Omega)$, hence

$$\lim_{n \to \infty} \| \varphi_n \|_{L^2(\Omega)} = \| f \|_{L^2(\Omega)} = 1.$$  

By possibly taking a subsequence, which we still denote by $\{ \varphi_n \}$, $\varphi_n$ is convergent to $f$ a.e. Therefore, there exists at least a point $x_n$ in $(-1/2, 1/2)$ such that $\varphi_n(x_n) = 0$. By the fundamental theorem of calculus,

$$\varphi_n(x) = \varphi_n(x) - \varphi_n(x_n) = \int_{x_n}^x \varphi'_n \, d\lambda,$$

whence

$$|\varphi_n(x)| \leq \int_{-1/2}^{1/2} |\varphi'_n| \, d\lambda \leq \| \varphi'_n \|_{L^2(\Omega)} \quad \forall x \in [-1/2, 1/2]$$

(the last inequality follows from Schwarz’s inequality). Thus, by taking the supremum over all the points $x$ in $[-1/2, 1/2]$, we obtain

$$\| \varphi_n \|_\infty \leq \| \varphi'_n \|_2 \quad \forall n \in \mathbb{N}.$$  

But $\| \varphi_n \|_2 \leq \| \varphi_n \|_\infty$, so that

$$\| \varphi_n \|_2 \leq \| \varphi'_n \|_2 \quad \forall n \in \mathbb{N}.$$  

Therefore

$$\| f \|_{H^1(\Omega)} = \lim_{n \to \infty} \| \varphi_n \|_{H^1(\Omega)} \geq 2 \lim_{n \to \infty} \| \varphi_n \|_2 = 2 \| f \|_{H^1(\Omega)}.$$

As a consequence, $\| f \|_{H^1(\Omega)} = 0$, which is clearly impossible.
Exercise 5.5.15  Prove that the function $u$, defined by

$$u(x) = \log \log \frac{1}{|x|} \quad \forall x \in \mathbb{R} \setminus \{0\},$$

is in $H^1(B_{1/2}(0))$. Therefore $H^1(B_{1/2}(0)) \not\subset \{\text{bounded functions}\}$.

Exercise 5.5.16  Prove directly that $H^1((-1, 1)) \subset C([-1, 1])$. Hint: use the fundamental theorem of calculus (for the Lebesgue integral).
Chapter 6

The Dirichlet problem with $L^2$ methods

6.1 Poincare’s inequality

One of the cornerstones of the $L^2$ methods for the Dirichlet problem is the so-called Poincaré inequality, which relates the Dirichlet integral and the $L^2$ norm of a function in $H^1_0(\Omega)$. We need to introduce the following notation.

Suppose that $\Omega$ is a (possibly unbounded) open set in $\mathbb{R}^n$. Given a unit vector $\omega$ in $\mathbb{R}^n$, we denote by $\delta_\Omega(\omega)$ “the diameter of $\Omega$ in the direction $\omega$”, namely the length of the projection of $\Omega$ in the direction of $\omega$. Then define

$$\delta_\Omega := \inf_{|\omega|=1} \delta_\Omega(\omega). \quad (6.1.1)$$

Note that $\delta_\Omega$ may be finite even if $\Omega$ is unbounded. For instance, a strip $S$ of width $a$ in the plane is obviously unbounded, but $\delta_S = a$. Notice that in the following result no assumptions are made on the boundary of the open set $\Omega$.

**Lemma 6.1.1** Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$. Then

$$\|\varphi\|_{L^2(\Omega)} \leq \delta_\Omega \|\nabla \varphi\|_{L^2(\Omega)} \quad \forall \varphi \in H^1_0(\Omega).$$

**Proof.** Since $H^1_0(\Omega)$ is the completion of $C^\infty_c(\Omega)$, it suffices to show that the required estimate holds for all $\varphi \in C^\infty_c(\Omega)$.

The one dimensional case has already been proved in Proposition 2.2.6.
Suppose that \( n \geq 2 \). Choose a unit vector \( \omega \) in \( \mathbb{R}^n \) so that \( \delta_{\Omega}(\omega) \) is finite, and then choose an orthonormal basis of \( \mathbb{R}^n \) whose last vector is \( \omega \). For each \( x \) fixed, \( \varphi(x', \cdot) \) vanishes outside an interval \( I(x') \) of length at most \( \delta_{\Omega}(\omega) \). By the one dimensional result

\[
\int_{I(x')} |\varphi(x', t)|^2 \, dt \leq |I(x')|^2 \int_{I(x')} |\partial_{x_n} \varphi(x', t)|^2 \, dt.
\]

Recall that \( |I(x')| \leq \delta_{\Omega}(\omega) \). By integrating both sides with respect to \( x' \), we obtain

\[
\int_{\Omega} |\varphi|^2 \, dV \leq \delta_{\Omega}(\omega)^2 \int_{\Omega} |\partial_{x_n} \varphi|^2 \, dV \\
\leq \delta_{\Omega}(\omega)^2 \int_{\Omega} |\nabla \varphi|^2 \, dV.
\]

By taking the infimum of both sides with respect to \( \omega \), we obtain the required estimate. \( \square \)

Note that we cannot hope to control the \( L^2 \) norm of a generic smooth function with the \( L^2 \) norm of its gradient (just take a nonzero constant function on an interval of the real line). Thus, the assumption that \( \varphi \) vanish at some point in \( \Omega \) is really needed.

Observe that \( \sqrt{D} \) is a norm on \( H^1_0(\Omega) \). Indeed, observe that if \( u \in H^1_0(\Omega) \), then

\[
D(u) = 0 \quad \Rightarrow \quad u \text{ is constant a.e. on each connected component of } \Omega \\
\Rightarrow \quad u = 0 \text{ a.e. on } \Omega,
\]

the last implication being justified by the fact that \( \gamma(u) \) must vanish, for \( u \) is in \( H^1_0(\Omega) \). The next result shows that \( \sqrt{D} \) is a norm on \( H^1_0(\Omega) \) equivalent to the \( H^1(\Omega) \) norm.

**Corollary 6.1.2** Suppose that \( \Omega \) is an open subset of \( \mathbb{R}^n \). Then

\[
\sqrt{D(u)} \leq \|u\|_{H^1_0(\Omega)} \leq \sqrt{1 + \frac{\delta^2_{\Omega}}{\delta_{\Omega}} \sqrt{D(u)}} \quad \forall u \in H^1_0(\Omega).
\]

**Proof.** The left hand inequality follows directly from the definition of \( \|u\|_{H^1(\Omega)} \); the right hand inequality is a consequence of Poincaré’s inequality in the preceding lemma. \( \square \)
6.2. SOLUTION TO THE MINIMIZATION PROBLEM

Remark 6.1.3 A consequence of Corollary 6.1.2 is that the inner product induced on \( H_0^1(\Omega) \) by restricting that of \( H^1(\Omega) \) is equivalent to the following

\[
(u, v)_{H_0^1} := \int_{\Omega} \nabla u \cdot \nabla v \, dV \quad \forall u, v \in H_0^1(\Omega).
\] (6.1.2)

Henceforth, we shall endow \( H_0^1(\Omega) \) with the inner product (6.1.2), and with the corresponding Hilbertian norm \( \sqrt{D(u)} \), which, by abuse of notation, will still be denoted by \( \|u\|_{H_0^1(\Omega)} \).

6.2 Solution to the minimization problem

In this section we prove (a slight generalisation of) Theorem 4.0.7. In order to formulate the new minimization problem, we need a few preliminary considerations.

If \( \mathcal{X} \) is a complex Banach space of functions on an open set \( \Omega \) (possibly \( \mathbb{R}^n \)), we denote by \( \text{Re} \mathcal{X} \) the real Banach space of all real valued functions in \( \mathcal{X} \).

Suppose that \( f \) and \( g \) are real valued functions. We aim at convincing the reader that any solution of the Dirichlet problem problem

\[
\begin{aligned}
\Delta u &= f & \text{in} & \Omega \\
u|_{\partial \Omega} &= g
\end{aligned}
\]

is real valued. Indeed, if \( u = u_1 + iu_2 \) is a solution, with \( u_1 \) and \( u_2 \) real valued, then \( u_1 \) and \( u_2 \) are solutions to the Dirichlet problems

\[
\begin{aligned}
\Delta u_1 &= f & \text{in} & \Omega \\
u_1|_{\partial \Omega} &= g
\end{aligned}
\]

and

\[
\begin{aligned}
\Delta u_2 &= 0 & \text{in} & \Omega \\
u_2|_{\partial \Omega} &= 0
\end{aligned}
\]

Clearly, \( u_2 = 0 \), thereby showing that \( u \) is real valued.

Thus, when trying to solve the Dirichlet problem with real data, it suffices to look for solutions in a real Banach space. Note that a fundamental role in the reasoning below is played by the fact that \( \Delta \) preserves the class of real valued functions, This may fail for more general operators.
We consider the following minimization problem. Given $f \in \text{Re} L^2(\Omega)$ and $G \in \text{Re} H^1(\Omega)$, find $u$ in $\text{Re} H^1(\Omega)$ that minimizes the functional

$$J(w) := \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dV - \int_{\Omega} f \, w \, dV$$

subject to the condition

$$w - G \in \text{Re} H^1_0(\Omega).$$

We follow a recent expository article of N.H. Friedel [Fr]. Note that there are no restrictions on $\partial\Omega$ in the next statement. An estimate of the unique solution to the minimization problem (6.0.6) is obtained by just setting $f = 0$ in the next statement.

**Theorem 6.2.1** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$. The minimization problem above has a unique solution $u$, which satisfies the estimate

$$\|u\|_{H^1} \leq \sqrt{\delta^2_{\Omega} + 1} \left( \delta_{\Omega} \|f\|_{L^2} + \sqrt{D(G)} \right) + \|G\|_{H^1},$$

where $\delta_{\Omega}$ is the constant in Poincaré’s inequality for the domain $\Omega$.

**Proof.** We make the following change of variables:

$$v := w - G.$$

Observe that

$$J(v + G) = \frac{1}{2} D(v) + \int_{\Omega} \nabla v \cdot \nabla G \, dV + \frac{1}{2} D(G) - (v, f)_{L^2} - (G, f)_{L^2}$$

$$= \tilde{J}(v) + \text{const},$$

where

$$\tilde{J}(v) := \frac{1}{2} D(v) + \int_{\Omega} \nabla v \cdot \nabla G \, dV - (v, f)_{L^2}$$

(6.2.3)

and $\text{const} = \frac{1}{2} D(G) - (f, G)_{L^2}$. Thus, $w$ solves the above minimization problem if and only if $v$ minimizes the functional $\tilde{J}$, subject to the condition

$$v \in \text{Re} H^1_0(\Omega).$$

Denote by $\Lambda_{f,G}$ the linear functional on $H^1_0(\Omega)$ defined by

$$\Lambda_{f,G}(v) := \int_{\Omega} \nabla v \cdot \nabla G \, dV - (v, f)_{L^2},$$

(6.2.4)
Observe that $\Lambda_{f,G}$ is continuous on $H^1_0(\Omega)$. Indeed, by Schwarz’s inequality,

\[
|\Lambda_{f,G}(v)| \leq \|\nabla G\|_{L^2} \|\nabla v\|_{L^2} + \|f\|_{L^2} \|v\|_{L^2} \\
\leq (\|G\|_{H^1_0} + \delta_\Omega \|f\|_{L^2}) \|v\|_{H^1_0}, \tag{6.2.5}
\]

we have used Poincaré’s inequality for the domain $\Omega$ in the second inequality above. Then, by Riesz’s representation theorem, there exists a unique element $v_{\Lambda_{f,G}} \in H^1_0(\Omega)$, with $\|\Lambda_{f,G}\|_{H^1_0} = \|v_{\Lambda_{f,G}}\|_{H^1_0}$, that represents the functional $\Lambda_{f,G}$, i.e.

$$
\Lambda_{f,G}(v) = (v, v_{\Lambda_{f,G}})_{H^1_0} \quad \forall v \in H^1_0(\Omega).
$$

Since $\Lambda_{f,G}$ is real on $\text{Re } H^1_0(\Omega)$, the element $v_{\Lambda_{f,G}}$ must be real valued, i.e., $v_{\Lambda_{f,G}}$ is in $\text{Re } H^1_0(\Omega)$.

Now, the functional $\tilde{J}$ on $\text{Re } H^1_0(\Omega)$ may be rewritten as follows

$$
\tilde{J}(v) = \frac{1}{2} (v, v)_{H^1_0} + (v, v_{\Lambda_{f,G}})_{H^1_0}.
$$

The advantage of this rewriting is that the existence and uniqueness of a minimizer for $J$ is now apparent. Indeed, the minimizer is just the same as the minimizer for the functional

$$
v \mapsto \frac{1}{2} (v, v)_{H^1_0} + (v, v_{\Lambda_{f,G}})_{H^1_0} + \frac{1}{2} (v_{\Lambda_{f,G}}, v_{\Lambda_{f,G}})_{H^1_0} = \frac{1}{2} \|v + v_{\Lambda_{f,G}}\|^2_{H^1_0},
$$

which is obviously equal to $-v_{\Lambda_{f,G}}$. Thus, the minimizer of the original functional $J$ is $u := G - v_{\Lambda_{f,G}}$. To conclude the proof of the theorem, it remains to estimate the norm of $u$ in terms of the data. The triangle inequality and (6.2.5) imply that

\[
\|u\|_{H^1} \leq \|G\|_{H^1} + \|v_{\Lambda_{f,G}}\|_{H^1} \\
\leq \|G\|_{H^1} + (\|v_{\Lambda_{f,G}}\|^2_{L^2} + \|v_{\Lambda_{f,G}}\|^2_{H^1_0})^{1/2} \\
\leq \|G\|_{H^1} + \sqrt{1 + \delta_\Omega^2} \|v_{\Lambda_{f,G}}\|_{H^1_0} \\
\leq \|G\|_{H^1} + \sqrt{1 + \delta_\Omega^2} (\|G\|_{H^1_0} + \delta_\Omega \|f\|_{L^2}),
\]

as required.

In view of Theorem 6.2.1, we have a solution map

$$
S : \text{Re } L^2(\Omega) \times \text{Re } H^1(\Omega) \to \text{Re } H^1(\Omega)
$$
which associates to the pair \((f, G)\) the unique solution \(S(f, G)\) of the minimization problem with data \((f, G)\). The proof of Theorem 6.2.1 above shows that
\[
S(f, G) = G - v_{\Lambda_f, G},
\]
where \(v_{\Lambda_f, G}\) is the unique function in \(\text{Re } H^1_0(\Omega)\) that represents the continuous linear functional
\[
\Lambda_{f,G}(v) = \int_{\Omega} \nabla v \cdot \nabla G \, dV - (v, f)_{L^2}.
\]
Note that if \(H\) is a function in \(\text{Re } H^1(\Omega)\) such that \(G - H\) is in \(\text{Re } H^1_0(\Omega)\), then
\[
S(f, G) = S(f, H).
\]
Indeed,
\[
S(f, G) - S(f, H) = G - v_{\Lambda_f, G} + v_{\Lambda_f, H} - H
= G - H - v_{\Lambda_{0,G-H}}
= S(0, G - H).
\]

Thus, (6.2.7) is equivalent to the statement that \(S(0, G - H) = 0\). Since \(\Lambda_{0,G-H}\) is clearly represented as an inner product in \(\text{Re } H^1_0(\Omega)\) by the function \(G - H\) (which belongs to \(\text{Re } H^1_0(\Omega)\)), we have
\[
G - H - v_{\Lambda_{0,G-H}} = G - H - (G - H) = 0,
\]
as required.

Thus, if fact, the solution map \((f, G) \mapsto S(f, G)\) does not depend on the element \(G\) in \(\text{Re } H^1(\Omega)\), but on the coset \(G + \text{Re } H^1_0(\Omega)\) in the coset space \(\text{Re } H^1(\Omega)/\text{Re } H^1_0(\Omega)\). Therefore there exists a unique map \(\tilde{S} : \text{Re } L^2(\Omega) \times \text{Re } H^1(\Omega)/\text{Re } H^1_0(\Omega) \to \text{Re } H^1(\Omega)\) such that the following diagram

\[
\begin{array}{ccc}
\text{Re } L^2(\Omega) \times \text{Re } H^1(\Omega) & \xrightarrow{I \times \pi} & \text{Re } L^2(\Omega) \times \text{Re } H^1(\Omega)/\text{Re } H^1_0(\Omega) \\
& & \downarrow S \\
& & \text{Re } H^1(\Omega) \\
\end{array}
\]

is commutative. Here \(\pi\) denotes the canonical projection of \(\text{Re } H^1(\Omega)\) onto \(\text{Re } H^1(\Omega)/\text{Re } H^1_0(\Omega)\). Thus, \(\text{Re } H^1(\Omega)/\text{Re } H^1_0(\Omega)\) may be interpreted as the space of all possible boundary data of the minimization problem (6.2.1).
6.2. SOLUTION TO THE MINIMIZATION PROBLEM

Now, we know that the minimization problem (6.2.1) subject to the condition (6.2.2) has a unique solution. We would like to relate the minimizing function of the functional $J$ to some sort of solution to the original Dirichlet problem.

In the proof of Theorem 6.2.1 it is shown that proving the existence of a minimizing function for the problem (6.2.1) subject to the condition (6.2.2) is equivalent to proving the existence of a minimizing function for the problem (6.2.3) subject to the condition $v \in \text{Re } H^1_0(\Omega)$.

Furthermore, a close examination of the proof of Theorem 6.2.1 shows that the assumption that $f$ belong to $\text{Re } L^2(\Omega)$ is used only to ensure that the linear functional $v \mapsto (v, f)_{L^2}$ extends to a bounded linear functional on $H^1_0(\Omega)$. It is straightforward to check that slight modifications of the proof of Theorem 6.2.1 also prove the following.

**Theorem 6.2.2** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, and that $F$ is in the dual space $\text{Re } H^1_0(\Omega)^*$. The minimization problem

$$\min_{v \in \text{Re } H^1_0(\Omega)} \tilde{J}(v) := \min_{v \in \text{Re } H^1_0(\Omega)} \left[ \frac{1}{2} D(v) - F(v) \right]$$

(6.2.8)

has a unique solution $v$, which satisfies the estimate

$$\|v\|_{H^1(\Omega)} \leq \|F\|_{\text{Re } H^1_0(\Omega)^*}.$$  

This result suggest to study the Banach dual of $H^1_0(\Omega)$ in some detail. This will be done in Section 6.3.

We conclude this section with the following result, which goes in the direction of interpreting the minimizing function of the problem (6.2.8) as a solution of a differential problem associated to the Laplace operator.

**Theorem 6.2.3** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, that $F$ is in the dual space $\text{Re } H^1_0(\Omega)^*$ and that $v$ is in $\text{Re } H^1_0(\Omega)$. The following are equivalent:

(i) $v$ is a solution to the minimization problem (6.2.8);

(ii) for each $\varphi \in \text{Re } H^1_0(\Omega)$, the function $v$ satisfies the equation

$$\int_{\Omega} \nabla \varphi \cdot \nabla v \, dV = F(\varphi).$$  

(6.2.9)
This result may be regarded as a modified form of the Dirichlet principle (compare with Proposition [4.0.1]). Note that here we know that the minimization problem has always a solution. The equation in (ii) is called the Euler equation associated to the functional $\tilde{J}$.

**Proof.** Suppose that $\varepsilon > 0$, and observe that

$$
\tilde{J}(v + \varepsilon \varphi) - \tilde{J}(v) = \varepsilon \left[ \int_{\Omega} \nabla \varphi \cdot \nabla v \, dV - F(\varphi) \right] + \frac{\varepsilon^2}{2} D(\varphi).
$$

It is clear that $v$ is a minimizing function of $\tilde{J}$ if and only if the left hand side is nonnegative for every $\varepsilon > 0$ and every $\varphi \in \text{Re} H_0^1(\Omega)$, and this holds if and only if the coefficient of $\varepsilon$ on the right hand side vanishes for every $\varphi \in \text{Re} H_0^1(\Omega)$, i.e., if and only if $v$ satisfies the equation in (ii). \qed

### 6.3 The dual of $H_0^1(\Omega)$

In this section we characterise the Banach dual of $H_0^1(\Omega)$. Note that, by the Riesz representation theorem, every continuous linear functional on $H_0^1(\Omega)$ is of the form

$$
u \mapsto (\nu, w)_{H_0^1}
$$

for a suitable $w \in H_0^1(\Omega)$. For reasons which will become clear later, it is important to give another representation of elements in the dual of $H_0^1(\Omega)$.

First, observe that the dual of $H_0^1(\Omega)$ may be viewed as a subspace of the space of distributions $\mathcal{D}'(\Omega)$. Indeed, if $\Lambda$ is a continuous linear functional on $H_0^1(\Omega)$, then its restriction to $\mathcal{D}(\Omega)$ satisfies the following

$$
|\Lambda(\varphi)| \leq \|\Lambda\| \|\varphi\|_{H_0^1} \quad \forall \varphi \in \mathcal{D}(\Omega).
$$

Thus, given a compact subset $K$ of $\Omega$,

$$
|\Lambda(\varphi)| \leq \|\Lambda\| \lambda(K)^{1/2} \mathcal{G}_1 K(\varphi) \quad \forall \varphi \in \mathcal{D}_K(\Omega).
$$

Then $\Lambda$ is a distribution on $\Omega$, by Theorem [5.2.8].

Furthermore, the restriction of $\Lambda$ to $\mathcal{D}(\Omega)$ identifies $\Lambda$. Indeed, suppose that $\Lambda$ is a continuous linear functionals on $H_0^1(\Omega)$ such that $\Lambda(\varphi) = 0$ for all $\varphi$ in $\mathcal{D}(\Omega)$. Then $\Lambda = 0$ on $H_0^1(\Omega)$. Indeed, for every $u \in H_0^1(\Omega)$ there exists a sequence $\{\varphi_k\}$ of test functions in $\Omega$ such that $\|u - \varphi_k\|_{H_0^1} \to 0$ as $k$ tends to $\infty$. Hence

$$
\Lambda(u) = \lim_{k \to \infty} \Lambda(\varphi_k) = 0,
$$

as required.
6.3. THE DUAL OF $H_0^1(\Omega)$

**Theorem 6.3.1** Suppose that $\Omega$ is a bounded open subset of $\mathbb{R}^n$. The following hold:

(i) every distribution in $\Omega$ of the form $f + \sum_{j=0}^n \partial_j g_j$, where $f$ and $g_j$ are in $L^2(\Omega)$, $j \in \{1, \ldots, n\}$, extends to a bounded linear functional on $H_0^1(\Omega)$;

(ii) for every bounded linear functional $\Lambda$ on $H_0^1(\Omega)$ there exist $f$ and $g_j$ in $L^2(\Omega)$, $j \in \{1, \ldots, n\}$, such that

$$\Lambda(u) = \int_{\Omega} u \, f \, dV + \int_{\Omega} u \, \text{div} \, \mathbf{G} \, dV \quad \forall u \in H_0^1(\Omega).$$

**Proof.** First we prove (i). Define

$$\Lambda(\varphi) = \left\langle \varphi, f + \sum_{j=0}^n \partial_j g_j \right\rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Observe that

$$\left| \left\langle \varphi, f + \sum_{j=0}^n \partial_j g_j \right\rangle \right| \leq |\langle \varphi, f \rangle| + \sum_{j=0}^n |\langle \partial_j \varphi, g_j \rangle|$$

$$\leq \|\varphi\|_{L^2} \|f\|_{L^2} + \sum_{j=0}^n \|\partial_j \varphi\|_{L^2} \|g_j\|_{L^2}$$

$$\leq C \|\varphi\|_{H_0^1} \left[ \|f\|_{L^2} + \sum_{j=0}^n \|g_j\|_{L^2} \right] \quad \forall \varphi \in \mathcal{D}(\Omega).$$

We have used Poincaré’s inequality in the last inequality above. We have proved that

$$|\Lambda(\varphi)| \leq C \|\varphi\|_{H_0^1} \left[ \|f\|_{L^2} + \sum_{j=0}^n \|g_j\|_{L^2} \right] \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Thus, $\Lambda$ extends to a bounded linear functional on $H_0^1(\Omega)$ and

$$\|\Lambda\| \leq C \left[ \|f\|_{L^2} + \sum_{j=0}^n \|g_j\|_{L^2} \right],$$

as required.
Next we prove (ii). Suppose that \( \Lambda \) is a continuous linear functional on \( H^1_0(\Omega) \). By the observation just above the statement of the theorem, \( \Lambda \) may be interpreted as a distribution. By the Riesz representation theorem there exists a unique function \( v_\Lambda \) in \( H^1_0(\Omega) \) such that
\[
\langle \varphi, \Lambda \rangle = (\varphi, v_\Lambda)_{H^1_0} \quad \forall \varphi \in \mathcal{D}(\Omega).
\]
Observe that
\[
(\varphi, v_\Lambda)_{H^1_0} = (\nabla \varphi, \nabla v_\Lambda)_{L^2} = -\langle \varphi, \text{div} \nabla v_\Lambda \rangle = -\langle \varphi, \Delta v_\Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).
\]
We have proved that
\[
\Lambda(\varphi) = -\langle \varphi, \Delta v_\Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{6.3.1}
\]
Thus, \( \Lambda \), viewed as a distribution, is just \(-\Delta v_\Lambda\). Hence, \( \Lambda \) has the desired representation (take \( f = 0 \) and \( G = \nabla v_\Lambda \)).

**Definition 6.3.2** The dual of \( H^1_0(\Omega) \), endowed with the operator norm, will be denoted by \( H^{-1}(\Omega) \).

Note that the Dirac delta at 0 is in \( H^{-1}((−1, 1)) \), for it is the distributional derivative of the Heaviside function.

**Exercise 6.3.3** Prove that \( \delta_0 \) is not in \( H^{-1}(B_1(0)) \) for \( n \geq 2 \).

**Exercise 6.3.4** Prove that \( \nabla 1_{B_1(0)} \) is in \( H^{-1}(B_1(0)) \) for all \( n \). Which is the support of \( \nabla 1_{B_1(0)} \)?

We conclude this section with the following result, which we shall use later.

**Lemma 6.3.5** The operator \( \Delta \) is an isometric isomorphism between \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \).

**Proof.** First we show that \( \Delta \) is a continuous map from \( H^1_0(\Omega) \) to \( H^{-1}(\Omega) \). This is implicit in the proof of Theorem \( \ref{6.3.1} \)(ii). However, for the sake of clarity, we repeat the argument here. Since \( H^1_0(\Omega) \) is included in \( \mathcal{D}'(\Omega) \), \( \Delta u \) is a distribution on \( \Omega \). For every \( \varphi \in \mathcal{D}(\Omega) \)
\[
\langle \varphi, \Delta u \rangle = \langle \varphi, \text{div} \nabla u \rangle = -\langle \nabla \varphi, \nabla u \rangle.
\]
6.3. THE DUAL OF $H^1_0(\Omega)$

Since $u$ is in $H^1_0(\Omega)$, the components of $\nabla u$ are in $L^2(\Omega)$. Therefore, by Schwarz’s inequality,

$$\left| \langle \varphi, \Delta u \rangle \right| \leq \| \nabla \varphi \|_{L^2(\Omega)} \| \nabla u \|_{L^2(\Omega)}$$

$$= \| \varphi \|_{H^1_0(\Omega)} \| u \|_{H^1_0(\Omega)}.$$

Here the space $H^1_0(\Omega)$ is endowed with the norm $\| u \|_{H^1_0(\Omega)} := \| \nabla u \|_{L^2(\Omega)}$, which is equivalent to the $H^1(\Omega)$-norm, as already mentioned. We now take the supremum of both sides with respect to all test functions $\varphi$ such that $\| \varphi \|_{H^1_0(\Omega)} \leq 1$, and obtain

$$\| \Delta u \|_{H^{-1}(\Omega)} = \sup_{\| \varphi \|_{H^1_0(\Omega)} \leq 1} \left| \langle \varphi, \Delta u \rangle \right| \leq \| u \|_{H^1_0(\Omega)} \| \Delta u \|_{H^{-1}(\Omega)} \quad (6.3.2)$$

Hence $\Delta : H^1_0(\Omega) \to H^{-1}_0(\Omega)$ with operator norm at most one.

Next we prove that $\Delta$ is injective. Note that

$$\| u \|_{H^1_0(\Omega)}^2 = \| \nabla u \|_{L^2(\Omega)}^2$$

$$= \langle u, -\Delta u \rangle$$

$$\leq \| u \|_{H^1_0(\Omega)} \| \Delta u \|_{H^{-1}(\Omega)} \quad \forall u \in H^1_0(\Omega), \quad (6.3.3)$$

which obviously implies

$$\| u \|_{H^1_0(\Omega)} \leq \| \Delta u \|_{H^{-1}(\Omega)} \quad \forall u \in H^1_0(\Omega). \quad (6.3.4)$$

The injectivity of $\Delta$ follows directly from this inequality, which, together with (6.3.2), shows that $\Delta$ is an isometry.

We claim that $\Delta$ has closed range. Indeed, suppose that $\{g_k\}$ is a Cauchy sequence in $\text{Ran}(\Delta)$. Denote by $u_k$ the unique (for $\Delta$ is injective) function in $H^1_0(\Omega)$ such that $\Delta u_k = g_k$. By (6.3.4) we have that

$$\| u_j - u_k \|_{H^1_0(\Omega)} \leq C \| g_j - g_k \|_{H^{-1}(\Omega)}$$

for all $j$ and $k$. Hence $\{u_k\}$ is a Cauchy sequence in $H^1_0(\Omega)$. Since $H^1_0(\Omega)$ is complete, there exists $u$ in $H^1_0(\Omega)$ such that $\| u_k - u \|_{H^1_0(\Omega)}$ tends to 0 as $j$ and $k$ tend to infinity. Since $\Delta$ is continuous from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$, it follows that

$$\Delta u = \lim_{k \to \infty} g_k,$$

whence the range of $\Delta$ is closed.
Finally, observe that $\Delta$ is onto. Indeed, we have already proved (see (6.3.1)), that, given $\Lambda \in H^{-1}(\Omega)$, there exists $v_\Lambda \in H^1_0(\Omega)$ such that
\[ \Lambda(\varphi) = -\langle \varphi, \Delta v_\Lambda \rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \]
Since $\mathcal{D}(\Omega)$ is dense in $H^1_0(\Omega)$, the restriction of $\Lambda$ to $\mathcal{D}(\Omega)$ identifies $\Lambda$ completely. Thus, $\Lambda$, viewed as a distribution, is just $\Delta(-v_\Lambda)$, as required.

By a standard consequence of the open mapping theorem, $\Delta$ is a Banach space isomorphism between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$, as required to conclude the proof of the lemma.

Here is the main result of this section.

**Theorem 6.3.6** Suppose that $\Omega$ is a bounded $C^2$ domain. For each $F$ in $H^{-1}(\Omega)$, there is a unique function $u$ in $H^1_0(\Omega)$ such that
\[ \Delta u = F. \]
Furthermore,
\[ \|u\|_{H^1_0(\Omega)} = \|F\|_{H^{-1}(\Omega)}. \]

**Proof.** Denote by $\Delta^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$ its inverse (which, by Lemma 6.3.5, is a continuous linear map). Then, given $F$ in $H^{-1}(\Omega)$, then $u := \Delta^{-1}F$ in $H^1_0(\Omega)$ and $\Delta u = F$. Thus, $u$ is the unique solution in $H^1_0(\Omega)$ of the Dirichlet problem associated to the Poisson equation.

The required norm estimate is a consequence of the fact, established in Lemma 6.3.5, that $\Delta$ is an isometric isomorphism between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. \qed

### 6.4 Weak solutions and regularity

In Section 6.2, we have proved that the minimization problem (6.2.1)-(6.2.2) has a unique solution $u$ in $H^1(\Omega)$. Suppose that $f = 0$ in (6.2.1). It is natural to speculate whether $u$ solves the Dirichlet equation $\Delta u = 0$ in the classical sense. In other words, we ask whether a solution of the minimization problem (6.2.1)-(6.2.2) is an harmonic function in $\Omega$. In this section, we shall prove that this is indeed the case.

Recall the definition of distributional solution to the equation
\[ \Delta u = f \quad (6.4.1) \]
6.4. WEAK SOLUTIONS AND REGULARITY

(see Definition 5.4.8). The following argument leads to an \textit{a priori} slightly different definition of solution. Suppose for a moment that \( f \) is a continuous function and that \( u \) is a (classical) solution of (6.4.1). Then multiply both sides of (6.4.1) by a function \( \varphi \) in \( C^1_c(\Omega) \) and integrate on \( \Omega \). We obtain

\[
\int_{\Omega} \varphi(x) \, \Delta u(x) \, dx = \int_{\Omega} \varphi(x) \, f(x) \, dx.
\]

Notice that both integrals above are absolutely convergent, because \( \varphi \) vanishes near the boundary of \( \Omega \). We can integrate by parts in the left hand side and get

\[
-\int_{\Omega} \nabla \varphi(x) \cdot \nabla u(x) \, dx = \int_{\Omega} \varphi(x) \, f(x) \, dx.
\]

(6.4.2)

The key observation here is that we do not need to assume that \( u \) is in \( C^2(\Omega) \) to give meaning to the left hand side of (6.4.2). For instance, it suffices to assume that \( u \) and its distributional derivatives \( \partial_j u \) are in \( L^1_{\text{loc}}(\Omega) \). Obviously this is a much weaker requirement than \( u \) belong to \( C^2(\Omega) \).

**Definition 6.4.1** Suppose that \( F \in H^{-1}(\Omega) \). A locally integrable function \( u \) with locally integrable first derivatives is a \textbf{weak solution} of the equation \( \Delta u = F \) if

\[
-\int_{\Omega} \nabla \varphi(x) \cdot \nabla u(x) \, dV = F(\varphi) \quad \forall \varphi \in C^1_c(\Omega).
\]

(6.4.3)

**Exercise 6.4.2** Suppose that \( F \in H^{-1}(\Omega) \). Prove that a weak solution of the equation \( \Delta u = F \) is a distributional solution.

In principle, there might be distributional solutions of the Poisson equation (6.4.1) with datum \( F \in D'(\Omega) \) that are not weak solutions. In fact, this is not the case. Indeed, if \( u_1 \) and \( u_2 \) are distributional solutions of the Poisson equation with datum \( f \in D'(\Omega) \), then \( u_1 - u_2 \) is a distributional solution to the Laplace equation. Weyl’s lemma, Lemma 6.4.3 below, shows that \( u_1 - u_2 \) is harmonic in \( \Omega \). Therefore, if \( u_1 \) is a weak solution of the Poisson equation with datum \( F \in D'(\Omega) \), so is \( u_2 \).

**Lemma 6.4.3 (Weyl’s lemma)** Suppose that \( u \) is a distributional solution of the Laplace equation \( \Delta u = 0 \). Then \( u \) is a classical solution (i.e., \( u \) is harmonic in \( \Omega \)).
Proof. We prove this result under the additional assumption that the distributitional solution $u$ be locally integrable. For the proof of the general case, see [Me, Thm 3.3.14 and Corollary 3.3.15].

Choose a function $\varphi \in C^\infty_c(B_1(0))$ such that $\int_{\mathbb{R}^n} \varphi \, dV = 1$, and for every $\varepsilon > 0$ denote by $\varphi_\varepsilon$ the function, defined by

$$
\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi(x/\varepsilon) \quad \forall x \in \mathbb{R}^n.
$$

Set

$$
\Omega_\varepsilon := \{ x \in \mathbb{R}^n : B_\varepsilon(x) \subset \Omega \}.
$$

Clearly $\Omega_\varepsilon \subset \Omega$. Observe that $\varphi_\varepsilon \ast u(x)$ makes sense for every $x \in \Omega_\varepsilon$, and that $\varphi_\varepsilon \ast u$ is smooth in $\Omega_\varepsilon$. Furthermore, if $\psi \in D(\Omega_\varepsilon)$, then $\psi \ast \varphi_\varepsilon \in D(\Omega)$.

We compute

$$
\langle \psi, \Delta(\varphi_\varepsilon) \rangle = \langle \Delta \psi, u \ast \varphi_\varepsilon \rangle
$$

$$
= \langle (\Delta \psi) \ast \check{\varphi}_\varepsilon, u \rangle
$$

$$
= \langle \Delta(\psi \ast \check{\varphi}_\varepsilon), u \rangle
$$

$$
= \langle \psi \ast \check{\varphi}_\varepsilon, \Delta u \rangle
$$

$$
= 0.
$$

Thus, $\Delta(u \ast \varphi_\varepsilon) = 0$ on $\Omega_\varepsilon$, i.e., $u \ast \varphi_\varepsilon$ is harmonic in $\Omega_\varepsilon$. Fix $\Omega' \subset \Omega$ with compact closure in $\Omega$. By the mean value theorem, for every $r$ small enough

$$
u \ast \varphi_\varepsilon(x) = \frac{n}{r^n \omega_n} \int_{B_r(0)} (u \ast \varphi_\varepsilon)(x + y) \, dV(y) \quad \forall x \in \Omega'.
$$

(6.4.4)

Now, by standard properties of convolution, for every compact subset $K$ of $\Omega$

$$
\lim_{\varepsilon \to 0} \|u \ast \varphi_\varepsilon - u\|_{L^1(K)} = 0.
$$

(6.4.5)

Therefore

$$
\sup_{x \in \Omega'} \left| u \ast \varphi_\varepsilon(x) - u \ast \varphi_{\varepsilon'}(x) \right|
$$

$$
\leq \frac{n}{r^n \omega_n} \sup_{x \in \Omega'} \int_{B_r(0)} \left| (u \ast \varphi_\varepsilon)(x + y) - (u \ast \varphi_{\varepsilon'})(x + y) \right| \, dV(y)
$$

$$
\leq \frac{n}{r^n \omega_n} \sup_{x \in \Omega'} \left\| (u \ast \varphi_\varepsilon)(x + \cdot) - (u \ast \varphi_{\varepsilon'})(x + \cdot) \right\|_{L^1(B_r(0))}
$$

$$
\leq \frac{n}{r^n \omega_n} \left\| u \ast \varphi_\varepsilon - u \ast \varphi_{\varepsilon'} \right\|_{L^1(\Omega + B_r(0))}
$$

$$
\to 0
$$
as $\varepsilon$ and $\varepsilon'$ tend to 0. Thus, $\{u * \varphi_\varepsilon\}$ is convergent uniformly in $\overline{\Omega}$ as $\varepsilon$ tends to 0. By (6.4.5), its uniform limit must be $u$. Hence $u$ is continuous. We then deduce from (6.4.4) that for $r$ small enough

$$u(x) = \frac{n}{r^n \omega_n} \int_{B_r(0)} u(x + y) \, dV(y) \quad \forall x \in \Omega'. $$

Therefore $u$ is harmonic in $\Omega'$ by Theorem 1.3.9. This holds for every $\Omega' \subset \Omega_\varepsilon$ with compact closure in $\Omega_\varepsilon$, hence it holds in $\Omega_\varepsilon$ for every $\varepsilon > 0$. Thus, it holds in $\Omega$, as required. \hfill \square

As a consequence of Weyl’s lemma, we obtain the following regularity result for the minimizer of the problem (6.2.1)-(6.2.2) in the case where $f = 0$.

**Theorem 6.4.4 (Interior regularity)** Suppose that $G \in H^1(\Omega)$, and that $u$ is the unique solution to the minimization problem

$$\min_{w \in G + H^1_0(\Omega)} \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dV.$$

Then $u$ is harmonic in $\Omega$.

**Proof.** Recall that $u$ is the solution of the minimization problem above if and only if $v := u - G$ solves the problem

$$\min_{w \in H^1_0(\Omega)} \left[ \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dV - F(w) \right],$$

where $F$ denotes the element of $H^{-1}(\Omega)$, defined by

$$F(w) := -\int_{\Omega} \nabla w \cdot \nabla G \, dV \quad \forall w \in H^1_0(\Omega).$$

Then $v$ is a weak solution, hence a distributional solution, of the equation

$$-\Delta w = F,$$

i.e., $v$ satisfies

$$\int_{\Omega} \nabla \varphi \cdot \nabla v \, dV = F(\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Since $H^{-1}(\Omega) = \Delta H^1_0(\Omega)$, there exists a unique $G_1 \in H^1_0(\Omega)$ such that $F = \Delta G_1$. Hence

$$\Delta (v + G_1) = 0 \quad \text{in } \Omega.$$
By Weyl’s lemma, there exists a function $h_1$, which is harmonic in $\Omega$, such that
\[ v + G_1 = h_1. \]  
(6.4.6)
Observe that $h_1$ is in $H^1_0(\Omega)$, for it is the sum of two functions in that space. Furthermore,
\[ \int_{\Omega} \nabla w \cdot \nabla G \, dV = \int_{\Omega} \nabla w \cdot \nabla G_1 \, dV \quad \forall w \in \mathcal{D}(\Omega), \]
whence
\[ \Delta(G - G_1) = 0 \]
in the sense of distributions. We may use Weyl’s lemma once more to conclude that there exists a function $h_2$, which is harmonic in $\Omega$, such that
\[ G = G_1 + h_2. \]  
(6.4.7)
Observe that $h_2$ is in $H^1(\Omega)$, for it is the sum of two functions in that space. Thus,
\[
\begin{align*}
    u &= G + v \\
    \text{(by (6.4.6))} &\quad = G - G_1 + h_1 \\
    \text{(by (6.4.7))} &\quad = h_2 + h_1,
\end{align*}
\]
thereby proving that $u$ is harmonic in $\Omega$, as required. $\square$

It is natural to speculate whether the solution to the minimization problem in Theorem 6.4.4 above is a classical solution to the Dirichlet problem
\[
\begin{aligned}
    \Delta u &= 0 & \text{in } \Omega \\
    u|_{\partial \Omega} &= g,
\end{aligned}
\]
where $g = \gamma(G)$. More generally, it is natural to ask the following. Suppose that $G \in H^k(\Omega)$ for some $k \geq 2$. Does it follow that $u$ is in $H^k(\Omega)$? The next theorem gives a partial answer to this question.

**Theorem 6.4.5 (Global regularity)** Suppose that $k \geq 2$, that $\Omega$ is a domain of class $C^k$, that $G \in H^k(\Omega)$, and that $u$ is the unique solution to the minimization problem
\[
\min_{w \in G + H^1_0(\Omega)} \frac{1}{2} \int_{\Omega} |
\nabla w |^2 \, dV.
\]
Then $u$ is in $H^k(\Omega)$. Furthermore, there exists a constant $C$, which depends only on $n$ and $\Omega$, such that

$$\|u\|_{H^k(\Omega)} \leq C \|G\|_{H^k(\Omega)}.$$

Therefore, if $k > n/2$, then $u$ belongs to $C^{k-[n/2]-1}(\Omega)$. In particular, if the domain is of class $C^\infty$ and $G \in C^\infty(\overline{\Omega})$, then $u$ belongs to $C^\infty(\overline{\Omega})$.

We do not prove the first statement and the norm estimate above. We refer the reader to any of the books [Fo2, GT, R, Tr]. The other statements are direct consequences of Sobolev’s imbedding theorem (see Theorem 5.5.12 (iv)).
Bibliography


